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CENTERS OF PLANAR POLYNOMIAL SYSTEMS. A REVIEW

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Contents.

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1. Introduction.

A planar polynomial system is a pair of two ordinary differential equations

$$\dot{x} = P(x, y), \dot{y} = Q(x, y) \tag{1.1}$$

where $\dot{x} = dx/dt$, $\dot{y} = dy/dt$, $t \in \mathbb{R}$, and $P(x, y)$, $Q(x, y)$ are polynomials in $(x, y) \in \mathbb{R}^2$ with real coefficients.

(1.1) is a system of degree n if the integer n is the maximum of the degrees of $P(x, y)$ and $Q(x, y)$.

We shall assume that $P(x, y)$, $Q(x, y)$ are relatively prime so that (1.1) has n^2 singular points at most. □

Only recently Dulac's Theorem asserting that planar polynomial systems have finitely many limit cycles, was proved (see Yu.S. Il'yashenko [18], J. Ecalle [13]).

As one of the consequences a singular point S of (1.1) can be either a center, or a focus or a tangential limit point, i.e., the limit point of trajectories with a limit tangent at S. □

This paper, essentially expository, is a review of various aspects (analytical, geometrical, dynamical) of polynomial systems with a center, under the assumptions declared above.

A large part is devoted to the *identification* by means of the coefficients of $P(x, y)$, $Q(x, y)$, of systems with a center.

This includes the consideration of *hamiltonian* systems and of *reversible* ones.

The rest of the paper deals with the *central region*, the *period function* and with *isochronous centers*.

The integration problem, i.e., the determination of first integrals for systems with a center will be considered here only occasionally.
Nor systems with more than one center will be given a special consideration.

□

As we shall see the behavior of (1.1) strongly depends on n. So it is convenient to distinguish among the four cases

$$n = 2; \quad n = 4, 6, \ldots; \quad n = 3; \quad n = 5, 7, \ldots$$

In what follows particular attention will be paid to *quadratic* systems ($n = 2$) and to *cubic* ones ($n = 3$). □

2. Degeneracy. Quasi homogeneity. O-symmetry.

As a rule, calculations are simplified at no expense of generality by assuming $S = O = (0, 0)$.
Then (1.1) can be written as

$$(S) \qquad \begin{aligned} \dot{x} &= \alpha x + \beta y + p(x, y), \\ \dot{y} &= \gamma x + \delta y + q(x, y) \end{aligned}$$

where

$$(2.1) \qquad p(x, y) = \sum_{2}^{n}{}_{j}\, p_j(x, y), \quad q(x, y) = \sum_{2}^{n}{}_{j}\, q_j(x, y)$$

and

$$(2.2) \qquad p_j(x, y) = \sum_{0}^{j}{}_{\nu}\, p_{j-\nu, j} x^{j-\nu} y^{\nu}, \quad q_j(x, y) = \sum_{0}^{j}{}_{\nu}\, q_{j-\nu, \nu} x^{j-\nu} y^{\nu}$$

are homogeneous polynomials of degree j. □

Definition 2.1. *We shall say that* (S) *is*

totally degenerate
if $\alpha = \beta = \gamma = \delta = 0$,

semidegenerate
if $\alpha\delta - \beta\gamma = 0$, $\alpha + \delta = 0$, $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 > 0$,

nondegenerate
if $\alpha\delta - \beta\gamma > 0$.

It can be shown (see, for instance G. Sansone - R. Conti [42]) that O can be a center in each one of these cases, whereas if $\alpha + \delta \neq 0$, or $\alpha\delta - \beta\gamma < 0$, O cannot be a center.
From this it follows that if O is a center then, after a linear change of coordinates x, y, (S) can be written as

$$(S)_{\lambda,\mu} \qquad \begin{aligned} \dot{x} &= \lambda y + p(x, y), \\ \dot{y} &= \mu x + q(x, y) \end{aligned}$$

where $\lambda = \mu = 0$ corresponds to total degeneracy, $\lambda \neq 0$, $\mu = 0$ to semidegeneracy and $\lambda\mu < 0$ to nondegeneracy. □

For $n = 2$ and $n = 3$ we shall use the notations

$$(Q)_{\lambda,\mu} \qquad \begin{aligned} \dot{x} &= \lambda y + ax^2 + bxy + cy^2 \\ \dot{y} &= \mu x + kx^2 + lxy + my^2, \end{aligned}$$

$$(C)_{\lambda,\mu} \qquad \begin{aligned} \dot{x} &= \lambda y + ax^2 + bxy + cy^2 + Ax^3 + Bx^2y + Cxy^2 + Dy^3 \\ \dot{y} &= \mu x + kx^2 + lxy + my^2 + Kx^3 + Lx^2y + Mxy^2 + Ny^3, \end{aligned}$$

$$(C)^0_{\lambda,\mu} \qquad \begin{aligned} \dot{x} &= \lambda y + Ax^3 + Bx^2y + Cxy^2 + Dy^3 \\ \dot{y} &= \mu x + Kx^3 + Lx^2y + Mxy^2 + Ny^3. \end{aligned} \qquad \square$$

Definition 2.2. *If the polynomials $p(x, y), q(x, y)$ are homogeneous of the same degree n we say that (S) is* quasi homogeneous.

If (S) is quasi homogeneous and totally degenerate we say that (S) is homogeneous. □

If n is even the trajectories of (S) cannot be symmetric with respect to O. In fact this happens if and only if the transformation $(x, y) \mapsto (-x, -y)$ leaves (S) unchanged, i.e., if and only if

$$p_j(x, y) = q_j(x, y) = 0, \quad (x, y) \in \mathbb{R}^2, \quad j \text{ even } \leq n \tag{2.3}$$

hold.

If n is odd then (2.3) make sense and we have

Definition 2.3. *When n is odd and* (2.3) *hold we say that (S) is O*-symmetric.

For $n = 3$ O-symmetry is equivalent to quasi-homogeneity.
For $n = 5, 7, \ldots$ if (S) is quasi homogeneous then (S) is also O-symmetric, but not viceversa. □

3. Polar coordinates.

Introducing polar coordinates $\rho, \theta, x = \rho\cos\theta,\ y = \rho\sin\theta,\ (S)$ becomes

$$(\Sigma)\qquad \begin{aligned}\dot\rho &= [\alpha\cos^2\theta + (\beta+\gamma)\cos\theta\sin\theta + \delta\sin^2\theta]\rho + \sum\nolimits_{j\,2}^{n} \rho^j r_j(\theta)\\ \dot\theta &= [\gamma\cos^2\theta + (\delta-\alpha)\cos\theta\sin\theta - \beta\sin^2\theta] + \sum\nolimits_{j\,2}^{n} \rho^{j-1} s_j(\theta)\end{aligned}$$

where $r_j(\theta)$, $s_j(\theta)$ are homogeneous polynomials in $\cos\theta$, $\sin\theta$ of degree $j+1$, namely,

$$(3.1)\qquad \begin{aligned} r_j(\theta) &= p_j(\cos\theta,\sin\theta)\cos\theta + q_j(\cos\theta,\sin\theta)\sin\theta\\ s_j(\theta) &= -p_j(\cos\theta,\sin\theta)\sin\theta + q_j(\cos\theta,\sin\theta)\cos\theta.\end{aligned} \qquad \square$$

Using Euler's identity for homogeneous functions we have the identities

$$(3.2)\qquad (j+1)r_j(\theta) + \frac{ds_j(\theta)}{d\theta} = d_j(\theta),$$

where

$$(3.3)\qquad \begin{aligned} d_j(\theta) &= \frac{\partial p_j}{\partial x}(\cos\theta,\sin\theta) + \frac{\partial q_j}{\partial y}(\cos\theta,\sin\theta) =\\ &= \sum\nolimits_{\nu\,0}^{j-1} (j-\nu) p_{j-\nu,j} \cos^{j-\nu-1}\theta\sin^{\nu}\theta +\\ &+ \sum\nolimits_{\nu\,1}^{j} \nu q_{j-\nu,j} \cos^{j-\nu}\theta\sin^{\nu-1}(\theta).\end{aligned}$$

$\square$

Notice that

$$(3.4)\qquad r_j(\theta+\pi) = (-1)^{j+1} r_j(\theta),\ s_j(\theta+\pi) = (-1)^{j+1} s_j(\theta). \qquad \square$$

In polar coordinates $(S)_{\lambda,\mu}$ becomes

$$(\Sigma)_{\lambda,\mu}\qquad \begin{aligned}\dot\rho &= (\lambda+\mu)\rho\cos\theta\sin\theta + \sum\nolimits_{j\,2}^{n} \rho^j r_j(\theta)\\ \dot\theta &= (\mu\cos^2\theta - \lambda\sin^2\theta) + \sum\nolimits_{j\,2}^{n} \rho^{j-1} s_j(\theta) \qquad \square\end{aligned}$$

4. Homogeneous systems.

We start with homogeneous systems (H. Forster [14], C.B. Collins [7]).
Let (S) be homogeneous, i.e., let $n \geq 2$ and

$$\dot{x} = \sum_{0}^{n}{}_{\nu} p_{n-\nu,\nu} x^{n-\nu} y^{\nu}, \; \dot{y} = \sum_{0}^{n}{}_{\nu} q_{n-\nu,\nu} x^{n-\nu} y^{\nu}. \tag{4.1}$$

If Γ is a trajectory of (4.1) then also $r\Gamma$, $r > 0$, is a trajectory. Therefore, either (4.1) has no cycle or all its trajectories are cycles and O is called a *global center*. $\square$

In polar coordinates (4.1) becomes

$$\dot{\rho} = \rho^{n} r_{n}(\theta), \; \dot{\theta} = \rho^{n-1} s_{n}(\theta) \tag{4.2}$$

where

$$\begin{aligned} r_n(\theta) &= \sum_{0}^{n}{}_{\nu} p_{n-\nu,n} \cos^{n-\nu+1}\theta \sin^{\nu}\theta + \\ &\qquad + \sum_{0}^{n}{}_{\nu} q_{n-\nu,n} \cos^{n-\nu}\theta \sin^{\nu+1}\theta \\ s_n(\theta) &= -\sum_{0}^{n}{}_{\nu} p_{n-\nu,n} \cos^{n-\nu}\theta \sin^{n-\nu+1}\theta + \\ &\qquad + \sum_{0}^{n}{}_{\nu} q_{n-\nu,n} \cos^{n-\nu+1}\theta \sin^{\nu}\theta. \end{aligned} \tag{4.3}$$

If the equation $s_n(\theta) = 0$ has a solution θ_0 then the ray $\theta = \theta_0$, $\rho > 0$, is a trajectory so that O is a tangential limit point.
Since $s_n(\theta)$ is a polynomial in $\cos\theta$, $\sin\theta$ of odd degree it follows

Theorem 4.1. *Homogeneous polynomial systems of* even *degree have an invariant line through O.* $\square$

If n is *odd* the assumption

$$s_n(\theta) \neq 0, \quad \theta \in \mathbb{R}$$

makes sense. If it holds we have $\dot{\theta} \neq 0$ and the trajectories can be represented by the solutions $\rho : \theta \mapsto \rho(\theta)$ of the linear equation

$$\frac{d\rho}{d\theta} = \frac{r_n(\theta)}{s_n(\theta)}\rho.$$

On the other hand from (3.2), (3.3) for $j = n$ we have

$$\rho(\theta) = \rho(\theta_0)\Big[\frac{s_n(\theta_0)}{s_n(\theta)}\Big]^{\frac{1}{n+1}} \exp \frac{1}{n+1}\int_{\theta_0}^{\theta} \frac{d_n(\varphi)}{s_n(\varphi)}d\varphi,\ \ \theta_0, \theta \in \mathbb{R}.$$

Since

$$\frac{d_n(\varphi)}{s_n(\varphi)} = \frac{d_n(\varphi+\pi)}{s_n(\varphi+\pi)}, \varphi \in \mathbb{R},$$

we have

Theorem 4.2. *Homogeneous polynomial systems of* odd *degree have a tangential limit point at* 0 *with tangent line* $\theta = \theta_0$ *if*

$$s_n(\theta_0) \quad \textit{for some} \quad \theta_0 \in \mathbb{R},$$

a focus if

$$s_n(\theta) \neq 0, \theta \in \mathbb{R}; \quad \int_{-\pi/2}^{\pi/2} \frac{d_n(\theta)}{s_n(\theta)}\, d\theta \neq 0,$$

a global center if

$$s_n(\theta) \neq 0, \quad \theta \in \mathbb{R}; \quad \int_{-\pi/2}^{\pi/2} \frac{d_n(\theta)}{s_n(\theta)}\, d\theta = 0. \qquad \square$$

5. Totally degenerate systems.

All the remaining systems to be classified are non necessarily homogeneous. $\square$

Examples show that if (S) is totally degenerate, but not homogeneous and $n = 4, 6, \ldots$ then O can be a tangential limit point, a focus or a (non global) center.
For instance O is a center for

$$\dot{x} = y^3,\ \dot{y} = -x^3 - x^n,\ n = 4, 6, \ldots \qquad \square \tag{5.1}$$

Also, there exist totally degenerate systems of degree $n = 5, 7, \dots$ which are O-symmetric but not homogeneous, or non O-symmetric, which have a center at O.

For instance O is a (global) center for

$$\dot{x} = y^3, \ \dot{y} = -x^3 - x^n, \ n = 5, 7, \dots \tag{5.2}$$

as well as for

$$\dot{x} = y^3, \ \dot{y} = -x^3 - x^4 - (n+1)x^n, \ n = 5, 7, \dots \qquad \square \tag{5.3}$$

We shall now examine in detail the case $n = 3$ of totally degenerate systems, i.e., systems $(C)_{0,0}$.

For $x = 0$ we have $\dot{x} = cy^2 + Dy^3$, so if $c \neq 0$ O cannot be a center nor a focus, so it is a tangential limit point. The same happens if $k \neq 0$.

If $c = k = 0$, in polar coordinates $(C)_{0,0}$ becomes

$$\dot{\rho} = \rho^2 r_2(\theta) + \rho^3 r_3(\theta), \ \dot{\theta} = \rho s_2(\theta) + \rho^2 s_3(\theta) \tag{5.4}$$

where

$$\begin{aligned} r_2(\theta) &= [a\cos\theta + b\sin\theta]\cos^2\theta + [l\cos\theta + m\sin\theta]\sin^2\theta \\ s_2(\theta) &= [(l-a)\cos\theta + (m-b)\sin\theta]\cos\theta\sin\theta \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} r_3(\theta) = {} & A\cos^4\theta + (B+K)\cos^3\theta\sin\theta + (C+L)\cos^2\theta\sin^2\theta + \\ & + (D+M)\cos\theta\sin^3\theta + N\sin^4\theta \\ s_3(\theta) = {} & K\cos^4\theta + (L-A)\cos^3\theta\sin\theta + (M-B)\cos^2\theta\sin^2\theta + \\ & + (N-C)\cos\theta\sin^3\theta - D\sin^4\theta. \end{aligned} \tag{5.6}$$

Let $(l-a)(m-b) \neq 0$ and $(l-a)\cos\theta_0 + (m-b)\sin\theta_0 = 0$. Then $\theta_0, 0, \pi/2$ are simple roots of $s_2(\theta) = 0$ and it follows (see, for instance, P. Hartman [40], pp. 220–221) the existence of trajectories having O as limit point with tangents $\theta = \theta_0, 0, \pi/2$.

If $l - a \neq 0$, $m - b = 0$ then $s_2(\theta) = (l-a)\cos^2\theta\sin\theta$ and $\theta = 0$ is a simple root of $s_2(\theta) = 0$, so $\theta = 0$ is a limit tangent.

Symmetrically, if $l - a = 0$, $m - b \neq 0$, $\theta = \pi/2$ is a limit tangent.

Let $l - a = m - b = 0$, so that $r_2(\theta) = a\cos\theta + b\sin\theta$, $s_2(\theta) \equiv 0$ and (5.4) becomes

$$\dot\rho = \rho^2(a\cos\theta + b\sin\theta) + \rho^3 r_3(\theta),\ \dot\theta = \rho^2 s_3(\theta).$$

If there exist θ_0 such that $s_3(\theta_0) = 0$ then $\theta = \theta_0, \theta_0 + \pi$, is an invariant line. Finally, let $r_2(\theta) = 0$, $\theta \in \mathbb{R}$, and

$$s_3(\theta) < 0, \quad \theta \in \mathbb{R} \tag{5.7}$$

hold. Then $\dot\theta = \rho^2 s_3(\theta) < 0$, $\rho > 0$, $\theta \in \mathbb{R}$, so O is the unique singular point of $(C)_{0,0}$.

The trajectories are the graphs of the solutions $\theta \mapsto \rho(\theta)$ of the equation

$$\frac{d\rho}{d\theta} = \frac{r_3(\theta)}{s_3(\theta)}\rho + \frac{a\cos\theta + b\sin\theta}{s_3(\theta)}$$

so that

$$\rho(\theta)\exp I(\theta) - r = J_{a,b}(\theta), \quad r = \rho(0) \ge 0 \tag{5.8}$$

where

$$I(\theta) = -\int_0^\theta \frac{r_3(\varphi)}{s_3(\varphi)}\,d\varphi \tag{5.9}$$

$$J_{a,b}(\theta) = \int_0^\theta \frac{a\cos\varphi + b\sin\varphi}{s_3(\varphi)}\exp I(\varphi)\,d\varphi. \tag{5.10}$$

The graph of $\theta \mapsto \rho(\theta)$ represents a cycle if and only if $\rho(2\pi) = r$ and $\rho(\theta) > 0$, $\theta \in \mathbb{R}$, i.e., if and only if r satisfies

$$r[\exp I(2\pi) - 1] = J_{a,b}(2\pi) \tag{5.11}$$

and

$$r + \mu_{a,b} > 0 \tag{5.12}$$

where

$$\mu_{a,b} = \min\{J_{a,b}(\theta), 0 \le \theta \le 2\pi\} \tag{5.13}$$

Since $J_{a,b}(0) = 0$ we have

$$\mu_{a,b} \leq 0. \tag{5.14}$$

If

$$I(2\pi)J_{a,b}(2\pi) \neq 0 \tag{5.15}$$

there is only one (limit) cycle, namely the trajectory passing through $(J_{a,b}(2\pi)[\exp I(2\pi) - 1]^{-1}, 0)$ and O is a focus.
If

$$I(2\pi)J_{a,b}(2\pi) = 0, \quad I^2(2\pi) + J_{a,b}^2(2\pi) > 0 \tag{5.16}$$

there are no cycles and O is a focus.
If

$$I(2\pi) = J_{a,b}(2\pi) = 0, \quad \mu_{a,b} = 0 \tag{5.17}$$

O is a global center.
Finally, if

$$I(2\pi) = J_{a,b}(2\pi) = 0, \quad \mu_{a,b} < 0 \tag{5.18}$$

the trajectory through $(r, 0)$, $r > -\mu_{a,b}$, is a cycle and O is a limit point. □

Summing up we have

Theorem 5.1. *Let*

$$(C)_{0,0} \qquad \begin{aligned} \dot{x} &= ax^2 + bxy + cy^2 + Ax^3 + Bx^2y + Cxy^2 + Dy^3 \\ \dot{y} &= kx^2 + lxy + my^2 + Kx^3 + Lx^2y + Mxy^2 + Ny^3. \end{aligned}$$

Then O is a tangential limit point if either $c^2 + k^2 > 0$, or $c = k = 0$, $l - a \neq 0$, or $c = k = 0$, $m - b \neq 0$, or $c = k = 0$, $l - a = m - b = 0$ and the equation

$$\begin{aligned} K\cos^4\theta + (L - A)\cos^3\theta\sin\theta + (M - B)\cos^2\theta\sin^2\theta + \\ +(N - C)\cos\theta\sin^3\theta - D\sin^4\theta = 0 \end{aligned} \tag{5.19}$$

has one real root at least.

If $c = k = 0$, $l - a = m - b = 0$ and (5.19) *has no real root, then O is a focus if* (5.17) *holds and a tangential limit point if* (5.18) *hold.* □

Remark 5.1. If $(C)_{0,0}$ is homogeneous then either O is a focus or a global center, in accordance with Theorem 4.2 for $n = 3$.

If $a^2 + b^2 > 0$, O can be a tangential limit point as shown by

$$\dot{x} = xy + x^2 y + y^3, \quad \dot{y} = y^2 - x^3 - xy^2$$

corresponding to $a = 0$, $b = 1$, $\mu_{0,1} = -2$. $\square$

Remark 5.2. The condition $I(2\pi) = 0$ is satisfied, in particular if

$$3A + L = 0, \quad B + M = 0, \quad C + 3N = 0 \tag{5.20}$$

hold. $\square$

6. Semidegenerate systems.

Theorem 6.1. *Let $n = 2, 4, \ldots$ and let $(S)_{\lambda,0}$ be quasi homogeneous. Then O is a tangential limit point.*

Proof. In polar coordinates $(S)_{\lambda,0}$ quasi homogeneous is written as

$$\dot{\rho} = \lambda\rho \cos\theta \sin\theta + \rho^n r_n(\theta)$$

$$\dot{\theta} = -\lambda \sin^2\theta + \rho^{n-1} s_n(\theta).$$

Since n is even there are $\theta_0 \in \mathbb{R}$ such that $s_n(\theta_0) = O$.
If $s_n(\theta_0) = 0$ and $\sin\theta_0 = 0$ the line $\theta = \theta_0 = 0$ is invariant.
If $s_n(\theta_0) = 0$ but $\sin\theta_0 \neq 0$, since for n even we have $r_n(\theta + \pi) = -r_n(\theta)$ then for $\theta = \theta_0 + \pi$ $\dot{\rho}$ equals $\lambda\rho\cos\theta_0 \sin\theta_0 - \rho^n r_n(\theta_0)$ and $\dot{\theta}$ equals $-\sin^2\theta_0$. This means that the trajectories close enough to O cut across with the same orientation the two half lines of $\theta = \theta_0$ originating from O.
Then O is not a center nor a focus. $\square$

It remains to consider $(S)_{\lambda,0}$ in the two cases a) $n = 4, 6, \ldots$, $(S)_{\lambda,0}$ not quasi homogeneous, b) n odd. $\square$

Next example shows that O can be a center in case a). Systems

$$\dot{x} = y, \ \dot{y} = -x^3 - x^n; \ n = 4, 6, \ldots \tag{6.1}$$

are not quasi homogeneous and O is a center. $\square$

The rest of this Section shows that O can be a center also in case b) n odd. $\square$

Let us consider first the cubic case, i.e., $(C)_{\lambda,0}$.

Then we have (A.F. Andreev [2], V.A. Lunkevich-K.S. Sibirskii [23])

Theorem 6.2. *Let $(C)_{\lambda,0}$ be O-symmetric. Then O is a center if and only if*

$$\lambda K < 0 \tag{6.2}$$

and

$$\begin{cases} 3A + L = 0 \\ 2A(B+M) + K(C+3N) = 0 \\ 2N(B+M)^2 - M(B+M)(C+3N) - A(C+3N)^2 = 0 \end{cases} \tag{6.3}$$

hold.

Further O is a tangential limit point if and only if

$$\lambda \neq 0,\ \lambda K \geq 0. \qquad \square \tag{6.4}$$

Notice that (5.20) are a particular case of (6.3). $\square$

Independent of O-symmetry we have

Theorem 6.3. *If O is a center or a focus of $(C)_{\lambda,0}$ then*

$$k = 0,\ \lambda K < 0 \tag{6.5}$$

must hold.

In fact, for $y = 0$ $\dot{y}$ equals $(k + Kx)x^2$ so if $k = K = 0$ the line $y = 0$ is invariant and if $k \neq 0$ $\dot{y}$ does not change sign for $x \lessgtr 0$ close to $x = 0$ and the trajectories close to O will cut across the line $y = 0$ one way, so O cannot be a center nor a focus. $\square$

The converse of Theorem 6.3 is not valid as it is shown, for instance, by (Yu Shu-Xiang, Zhang Ji-Zhou [39])

$$\dot{x} = y + ax^2,\ \dot{y} = -x^3. \tag{6.6}$$

In fact if $a^2 \geq 2$ the parabolas $y + \frac{a \pm \sqrt{a^2-2}}{2}x^2 = 0$ are invariant.

It is easy to show that O is a (global) center for $a^2 < 2$. $\square$

For semidegenerate systems $(S)_{\lambda,0}$ of degree $n = 5, 7, \ldots, O$ can be a center, possibly a global one.

For instance, O is a center for semidegenerate systems which are quasi homogeneous, like

$$\dot{x} = y, \ \dot{y} = -x^n; \ n = 5, 7, \ldots \tag{6.7}$$

O-symmetric, not quasi homogeneous, like

$$\dot{x} = y, \ \dot{y} = -x^{n-2} - x^n; \ n = 5, 7 \ldots \tag{6.8}$$

not O-symmetric, like

$$\dot{x} = y + y^{n-2} + y^n, \ \dot{y} = -x^3; \ n = 5, 7 \ldots . \quad \square \tag{6.9}$$

7. Nondegenerate systems. The center/focus problem.

To deal with nondegenerate systems with a center, $(S)_{\lambda,\mu}$, $\lambda\mu < 0$, it is not restrictive to assume further $\lambda = 1, \mu = -1$, so that (S) and (Σ) reduce respectively to

$$\dot{x} = y + p(x, y), \ \dot{y} = -x + q(x, y), \tag*{$(S)_{1,-1}$}$$

$$\dot{\rho} = \sum_2^n {}_j \rho^j r_j(\theta), \dot{\theta} = -1 + \sum_2^n {}_j \rho^{j-1} s_j(\theta). \tag*{$(\Sigma)_{1,-1}$}$$

The nondegenerate case is, by far, the most intensively investigated since the classical work of Poincaré and Liapunov.

One of the reasons lies in the fact that O cannot be a tangential limit point, i.e.,

Theorem 7.1. *O is a center or a focus of $(S)_{1,-1}$.*

Proof. From $(\Sigma)_{1,-1}$ we see that the trajectories close enough to O wind around O itself so, either they are spirals tending to O and O is a focus, or they are cycles surrounding O and O is a center. $\square$

Due to this, in the nondegenerate case the identification problem is commonly referred to as the *center/focus problem*. $\square$

Another reason for privileging nondegenerate systems is that from Poincaré-Liapunov's work (see, for instance, V.V. Nemytskii-V.V. Stepanov [41],

G. Sansone-R. Conti [42]) it is known that such systems with a center are characterized by a *finite* number of algebraically independent conditions of the form $D_i = 0$, where D_i are polynomials of the coefficient of the system.

This is a very remarkable result, but its importance is more theoretical than practical. In fact, quoting from V.V. Nemytskii-V. V. Stepanov [41], p.123: "In order to make an effective use of these conclusions we must answer the following question: Given that right hand members of our equation are polynomials of degree n, to determine $N(n)$ such that all the equalities $D_i = 0$ for $i > N(n)$ are consequences of such equalities for $i \leq N(n)$. The problem of characterization of $N(n)$ is still unsolved".

And still *(1998)* it is.

It remains, therefore, to single out systems for which the question above is solved. $\square$

8. Nondegenerate systems of even degree.

For nondegenerate *quadratic* systems $(Q)_{1,-1}$, i.e.,

$$\begin{cases} \dot{x} = & y + ax^2 + bxy + cy^2 \\ \dot{y} = & -x + kx^2 + lxy + my^2, \end{cases} \tag{8.1}$$

the center/focus problem has been solved in various ways in terms of algebraic equalities satisfied by the coefficients (Li Chengzhi [20], D. Schlomiuk - J. Guckenheimer - R. Rand [34]). We have, for instance

Theorem 8.1. *Let $n = 2$. Then O is a center of* (8.1) *if and only if one of the following sets of conditions is satisfied:*

$$\begin{cases} (a+c)(b+2m) - (2a+l)(k+m) = 0 \\ k(a+c)^3 + (l-a)(a+c)^2(k+m)+ \\ \qquad +(m-b)(a+c)(k+m)^2 - c(k+m)^3 = 0 \end{cases} \tag{8.2}$$

$$2a + l = 0, \quad b + 2m = 0 \tag{8.3}$$

$$\begin{cases} 5(a+c) - (2a+l) = 0 \\ 5(k+m) - (b+2m) = 0 \\ c^2 + c(a+c) + k^2 + k(k+m) = 0. \end{cases} \tag{8.4}$$

$\square$

Remark 8.1. Notice the particular cases of (8.2):

$$a + c = 0, \quad k + m = 0 \tag{8.5}$$

$$a = c = l = 0 \tag{8.6}$$

$$b = k = m = 0. \qquad \square \tag{8.7}$$

Let now $n = 4, 6, \ldots$. Simple examples show that O can be a center both for nondegenerate quasi homogeneous systems like

$$\dot{x} = y, \dot{y} = -x - x^n; \quad n = 4, 6, \ldots, \tag{8.8}$$

and for nondegenerate non quasi homogeneous systems, like

$$\dot{x} = y, \dot{y} = -x - \frac{1}{2}x^2 - \frac{1}{2}x^n; \quad n = 4, 6, \ldots \qquad \square \tag{8.9}$$

9. Nondegenerate systems of odd degree.

It remains to examine nondegenerate systems of *odd* degree. $\square$

For cubic nondegenerate O-symmetric systems,

$$\begin{cases} \dot{x} = & y + Ax^3 + Bx^2y + Cxy^2 + Dy^3 \\ \dot{y} = & -x + Kx^3 + Lx^2y + Mxy^2 + Ny^3 \end{cases} \tag{9.1}$$

I.G. Malkin [24] and K.S. Sibirskii [36], using different methods, gave the solution of the center/focus problem as follows.

Theorem 9.1. *Let* $n = 3$. O *is a center of* (9.1) *if and only if one of the following sets of conditions is satisfied:*

$$\begin{cases} 3A + L + C + 3N = 0 \\ (3A + L)(B + D + K + M) - 2(A - N)(B + M) = 0 \\ 2(A + N)[(3A + L)^2 - (B + M)^2] + \\ \qquad +(3A + L)(B + M)(B - D + K - M) = 0, \end{cases} \tag{9.2}$$

$$\begin{cases} 3A + L + C + 3N = 0 \\ 2A - L + C - 2N = 0 \\ B + 3D - 3K - M = 0 \\ B + 5D + 5K + M = 0 \\ (A + 3N)(3A + N) - 16DK = 0. \qquad \square \end{cases} \tag{9.3}$$

Notice that (5.20) are a particular case of (9.2). □

Cubic nondegenerate systems which are not quasi homogeneous (i.e., non O-symmetric) but have a center at O, do exist as it is shown, for instance, by (N.A. Lukashevich [22])

$$\dot{x} = y + 2xy + 2y^3, \; \dot{y} = -x - y^2. \tag{9.4}$$ □

A remarkable class of such systems, known as Kukles' systems, is represented by

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -x + kx^2 + lxy + my^2 + Kx^3 + Lx^2y + Mxy^2 + Ny^3 \end{cases} \tag{9.5}$$

with $k^2 + l^2 + m^2 > 0$, $K^2 + L^2 + M^2 + N^2 > 0$.

Systems (9.5) with a center at O have been the object of intensive research (I.S. Kukles [19], A.P. Sadovskii [32], P. Mardešić - C. Rousseau - B. Toni [25], C.J. Christopher - J. Devlin [6]). □

Simple examples like the following ones show the existence of a center at O for nondegenerate systems of degrees $n = 5, 7, \dots$ and quasi homogeneous, like

$$\dot{x} = y, \; \dot{y} = -x - x^n; \; n = 5, 7, \dots, \tag{9.6}$$

or O-symmetric, non quasi homogeneous, like

$$\dot{x} = y, \; \dot{y} = -x - x^3 - x^n; \; n = 5, 7 \dots \tag{9.7}$$

or non O-symmetric, like

$$\dot{x} = y + y^2 + y^n, \; \dot{y} = -x; \; n = 5, 7 \dots. \tag{9.8}$$ □

10. Remarks about the identification problem.

The identification problem of polynomial systems with a center, arose more than one century ago and it is still alive and a challenging one.

What precedes shows that systems for which it was solved are a minority. In fact the problem is solved for quadratic systems, but it is

- still unsolved for systems of degree $n = 4, 6, \dots$ which are not quasi homogeneous or which are quasi homogeneous but nondegenerate,

- still unsolved for cubic systems which are not O-symmetric and are semidegenerate or nondegenerate.

For $n = 5, 7, \ldots$ the problem is solved only for homogeneous systems.

This situation accounts for the search for conditions which are only necessary or only sufficient ones.

Hence the recourse to computer algebra (which will not be considered here), the attention paid to hamiltonian systems, and the emphasis on reversible systems. These two classes have in common the property that O cannot be a focus. □

11. Hamiltonian systems.

Restricting to polynomial systems the wellknown definition of hamiltonian systems we have

Definition 11.1. *A polynomial system $(S)_{\lambda,\mu}$ of degree n is said to be* hamiltonian *if it can be written as*

$$\dot{x} = H_y(x, y), \ \dot{y} = -H_x(x, y)$$

where $H(x, y)$ is a polynomial of degree $n + 1$, $H(0, 0) = 0$.

To recognize whether $(S)_{\lambda,\mu}$ is hamiltonian it is sufficient (as well as necessary) to verify whether

$$p_x(x, y) + q_y(x, y) = 0, \quad (x, y) \in \mathbb{R}^2 \tag{11.1}$$

holds.

Since $p_x(x, y)+q_y(x, y)$ is the divergence of the vector $(p(x, y), q(x, y))$ (11.1) is also called the *divergence condition.* □

If (11.1) holds, then

$$[-\mu x - q(x, y)]dx + [\lambda y + p(x, y)]dy =$$
$$= \frac{1}{2}d(-\mu x^2 + \lambda y^2) - q(x, y)dx + p(x, y)dy$$

is the differential of $[-\mu x^2 + \lambda y^2 + R(x, y)]/2$, where $R(x, y)$ is a polynomial of degree $n + 1$ with no term of degree ≤ 2.

Then the trajectories of $(S)_{\lambda,\mu}$ are represented by the family of algebraic curves of degree $n + 1$

$$-\mu x^2 + \lambda y^2 + R(x, y) = r, \quad r \in \mathbb{R},$$

i.e., by the level curves of the algebraic surface $z = -\mu x^2 + \lambda y^2 + R(x, y)$.

It follows that O cannot be a focus, so that we have

Theorem 11.1. *If the divergence condition holds, then O is a center or a tangential limit point of $(S)_{\lambda,\mu}$.* □

Notice that if the divergence condition holds and O is a tangential limit point then O is the limit point of finitely many trajectories. □

Since O is a center or a focus if $(S)_{\lambda,\mu}$ is nondegenerate, from Theorem 11.1 it follows the criterion

Theorem 11.2. *If $(S)_{\lambda,\mu}$ is hamiltonian and nondegenerate then O is a center.* □

Nondegeneracy is not a necessary condition as it is shown, for instance, by

$$\dot{x} = y, \ \dot{y} = -2x^3.$$

Actually, Theorem 11.2 can be improved, so as to cover systems like the preceding one, as follows

Theorem 11.3. *If $(S)_{\lambda,\mu}$ is hamiltonian and O is an isolated critical point of the algebraic curve $-\mu x^2 + \lambda y^2 + R(x, y) = 0$ then O is a center.* □

Notice that the theorems above are independent of n. □

Many systems considered in the previous Sections are hamiltonian.
$(Q)_{\lambda,\mu}$ is hamiltonian if and only if (8.3) hold.
$(C)_{\lambda,\mu}$ is hamiltonian if and only if (8.3) and (5.20) hold. □

When $(S)_{\lambda,\mu}$is quasi homogeneous the divergence condition of Theorem 11.2 can be weakened as follows (M.A.M. Alwash-N.G. Lloyd [1])

Theorem 11.4. *Let $(S)_{1,-1}$ be quasi homogeneous of degree n. Then O is a center if there exist $\alpha \in \mathbb{R}$ such that*

$$\begin{aligned}(x^2 + y^2)[p_x(x, y) + q_y(x, y)] &= \\ &= \alpha[xp(x, y) + yq(x, y)], \ (x, y) \in \mathbb{R}^2\end{aligned} \tag{11.2}$$

and either n is even, or n is odd and $\alpha \neq n + 1$, or n is odd, $\alpha = n + 1$, and

$$\int_0^{2\pi} d_n(\theta)\, d\theta = 0$$

where $d_n(\theta)$ is the polynomial of degree $n + 1$ defined by (3.3) for $j = n$.

For $\alpha = 0$ (11.2) reduces to the divergence condition (11.1). □

12. Reversible systems.

Beside hamiltonian (polynomial) systems there is another class of systems for which O cannot be a focus, namely that of reversible systems satisfying

Definition 12.1. *We say that* $(S)_{\lambda,\mu}$ *is* reversible *with respect to a straight line* l *through* O *if it is invariant with respect to reflection about* l *and a reversion of time* t.

If $(S)_{\lambda,\mu}$ is reversible O cannot be a focus, but it is not necessarily a center as it is shown, for instance, by system (6.6), $a^2 \geq 2$.
However O is a center if $(S)_{\lambda,\mu}$ is also nondegenerate. We thus have a very simple and useful criterion, going back to Poincaré, namely (cfr. V.V. Nemytskii-V.V. Stepanov [41], p.122)

Theorem 12.1. *Let* $(S)_{\lambda,\mu}$ *be non degenerate. Then* O *is a center if* $(S)_{\lambda,\mu}$ *is reversible.* □

Examples show that Theorem 12.1 is not invertible. For instance O is a center of the nondegenerate system (12.6) below, but the system is not reversible. □

We shall now express reversibility of $(S)_{\lambda,\mu}$ in terms of λ, μ, $p(x,y)$, $q(x,y)$. $(S)_{\lambda,\mu}$ is reversible about the line $l : y = 0$ if and only if the transformation $(x, y, t) \mapsto (x, -y, -t)$ leaves $(S)_{\lambda,\mu}$ unchanged. This means that

$$(12.1) \qquad \begin{cases} p(x,y) = & -p(x,-y) \\ q(x,y) = & q(x,-y) \end{cases} \qquad , (x,y) \in \mathbb{R}^2$$

are satisfied.

Let now $l : \alpha x - \beta y = 0,\ \alpha^2 + \beta^2 = 1$.

The rotation of axes

$$u = \beta x + \alpha y, \quad v = \alpha x - \beta y$$

sends l into the line $v = 0$ and it transforms $(S)_{\lambda,\mu}$ into

$$(12.2) \qquad \begin{cases} \dot{u} & = \alpha\beta(\lambda+\mu)u - (\beta^2\lambda - \alpha^2\mu)v + \\ & +\beta p(\beta u + \alpha v, \alpha u - \beta v) + \alpha q(\beta u + \alpha v, \alpha u - \beta v) \\ \dot{v} & = (\alpha^2\lambda - \beta^2\mu)u - \alpha\beta(\lambda+\mu)v + \\ & +\alpha p(\beta u + \alpha v, \alpha u - \beta v) - \beta q(\beta u + \alpha v, \alpha u - \beta v). \end{cases}$$

Therefore, changing t into $-t$ (12.2) coincides with $(S)_{\lambda,\mu}$ if and only if

$$\begin{cases} p(x,y) = & -\beta p(\beta x+\alpha y, \alpha x-\beta y) - \alpha q(\beta x+\alpha y, \alpha x-\beta y) \\ q(x,y) = & -\alpha p(\beta x+\alpha y, \alpha x-\beta y) + \beta q(\beta x+\alpha y, \alpha x-\beta y) \end{cases} \tag{12.3}$$

$(x, y) \in \mathbb{R}^2$, and

$$\alpha(\lambda+\mu) = 0 \tag{12.4}$$

hold. Therefore $(S)_{\lambda,\mu}$ is reversible about the line $l : \alpha x - \beta y = 0$ if and only if (12.3) and (12.4) hold.

Assuming $\lambda = -\mu = 1$ we have (T.R. Blows-N.G. Lloyd [3])

Theorem 12.2. *Let* (12.3) *hold for some* $\alpha, \beta, \alpha^2+\beta^2 = 1$. *Then* O *is a center for* $(S)_{1,-1}$. □

Remark 12.1. Equalities (12.3) allow to verify whether a *given* line $\alpha x - \beta y = 0$ is a reversibility line for $(S)_{1,-1}$. However, when used for the search for *possible* reversibility lines they may lead to calculations usually getting longer and longer as the degree of $(S)_{1,-1}$ is increased.

Some help may be obtained by observing that a reversibility line l is also an *orthogonality line*, i.e., at every point $P \in l$ the vector $(\dot{x}, \dot{y})$ is necessarily orthogonal to the ray OP.

This means that a reversibility line is part of the algebraic curve of degree $n+1$

$$\rho\dot{\rho} = xp(x,y) + yq(x,y) = 0.$$

Therefore if $\rho\dot{\rho} = 0$ does not contain a real line through O then there are no reversibility lines at all, whereas if $\rho\dot{\rho} = 0$ contains a real line l through O one has only to apply (12.3) to verify whether l is actually a reversibility line. □

If $xp(x,y) + yq(x,y) = 0$ identically, i.e., for $(x, y) \in \mathbb{R}^2$, then

$$y + p(x,y) = y[1 - q(x,y)/x], \quad -x + q(x,y) = -x[1 - q(x,y)/x]$$

for $x \neq 0$. Since by assumption, $y + p(x, y)$, $-x + q(x, y)$ must be relatively prime, $\rho\dot{\rho} = 0$ cannot be valid for $(x, y) \in \mathbb{R}^2$.

Therefore if we denote by l_O the number of orthogonality lines and by l_r the number of reversibility lines of $(S)_{1,-1}$ we have

$$0 \le l_r \le l_O \le n+1. \quad \square \tag{12.5}$$

Remark 12.2. Reversibility and O-symmetry are properties independent of each other.
For instance system (6.6) is reversible with respect to the line $x = 0$, but for $a \neq 0$ it is not O-symmetric. Recall that for $a^2 < 2$ O is a center.

Conversely the system (N.A. Saharnikov [33], C. Rousseau-D. Schlomiuk [28])

$$\dot{x} = y - x^3 + xy^2, \; \dot{y} = -x - 7x^2y + 3y^3 \tag{12.6}$$

is O-symmetric and according to (9.3) O is a center. We have

$$\rho\dot{\rho} = -[x^2 + (2\sqrt{3}+3)y^2][x^2 - (2\sqrt{3}-3)y^2]$$

so that there are two orthogonality lines, namely

$$l' : \quad x - \sqrt{2\sqrt{3}-3}\,y = 0;$$
$$l'' : \quad x + \sqrt{2\sqrt{3}-3}\,y = 0.$$

Since (12.6) is O-symmetric if l', l'' were also reversibility lines they ought to be orthogonal each other, which is not. Therefore $0 = l_r < l_O = 2$. □

Remark 12.3. If $(S)_{1,-1}$ is O-symmetric *and* it is reversible about a line l then it is reversible also about the line l' orthogonal to l.

Therefore, an O-symmetric system is not reversible at all, i.e., $l_r = 0$, or the lines of reversibility come in pairs of orthogonal lines, i.e., $l_r = 2, \ldots, n+1$. □

13. Geometrical classification of centers. Central region.

We shall now consider some geometrical aspects of centers.

Let O be a center of a polynomial system, let Γ_O denote the family of cycles γ surrounding O and no other singular point and let intγ denote the region of $\mathbb{R}^2$ interior to γ.

Then

$$\mathcal{N}_O = \bigcup_{\gamma \in \Gamma_O} \operatorname{int}\gamma \tag{13.1}$$

is a region of $\mathbb{R}^2$ whose boundary $\partial\mathcal{N}_O$ is the finite union of trajectories of $(S)_{\lambda,\mu}$.

According to Poincaré $\partial\mathcal{N}_O$ cannot be a cycle.

If $\partial\mathcal{N}_O = \phi$, i.e., $\mathcal{N}_O = \mathbb{R}^2$, O is a global center or a center of *type* A.

If $\partial\mathcal{N}_O \neq \phi$ then $\partial\mathcal{N}_O$ is the finite union of connected components and we have, a priori, the following possibilities:

O is of *type* B if $\partial\mathcal{N}_O \neq \phi$ does not contain singular points, i.e., it is the finite union of open unbounded trajectories;

O is of *type* C if $\partial\mathcal{N}_O$ is unbounded but it contains one singular point at least;

O is of *type* D if $\partial\mathcal{N}_O$ is bounded. □

The region

$$\mathcal{C}_O = \mathcal{N}_O \setminus \{O\}$$

will be called the *central region* of O. □

14. Centers of types A and B.

M. Galeotti and M. Villarini [17] extending Theorem 4.1 proved that every polynomial system of even degree has one unbounded trajectory at least. Therefore we have

Theorem 14.1. *If O is a global center of a polynomial system of degree n then n is odd.* □

As we have seen already there exist polynomial systems of an arbitrary odd degree for which O is a global center. Therefore it makes sense to pose

Problem 14.1. *To identify all the polynomial systems (of odd degree) having a global center.*

Using the extension of $(S)_{\lambda,\mu}$ to the Poincaré's sphere, M. Sabatini [29] gave a partial solution. □

If O is a center of type B then $\partial\mathcal{N}_O$ is the union of k open unbounded trajectories so that type B can be divided into subtypes B^k.

From a result of M. Galeotti [16] we have $k \leq n-1$ and examples (R. Conti [10], [12]) show the existence of centers of type B^{n-1} for each $n \geq 2$. Therefore, denoting by $k(n)$ the maximum of k with respect to n we have

$$k(n) = n - 1. \qquad \square \tag{14.1}$$

15. Period function. Isochronous centers. Linearization.

Let us now introduce some notion of a dynamical character.

Definition 15.1. *Let O be a center of $(S)_{\lambda,\mu}$and let $T(P)$ denote the period of the cycle passing through $P \in \mathcal{C}_O$. The function $P \mapsto T(P)$ is called the* period function *associated with the center O.*

Definition 15.2. *If the period function $P \mapsto T(P)$ is constant for $P \in \mathcal{C}_O$ we say that O is an* isochronous center. □

Studying regularity properties of the period function M. Villarini [37] proved, in particular, the necessary condition expressed by

Theorem 15.1. *If O is an isochronous center of $(S)_{\lambda,\mu}$ then $(S)_{\lambda,\mu}$ is nondegenerate.*

For another proof see C.J.Christopher-J. Devlin [6]. □

Theorem 15.1 accounts once more for the preference given to nondegenerate systems. □

A classical result due to Poincaré and Liapunov, of great theoretical importance, reduces, roughly speaking, the isochronism of O to the existence of an analytical transformation $(x, y) \mapsto (u, v)$ of a certain type which *linearizes* $(S)_{1,-1}$, that is, sends $(S)_{1,-1}$ into $\dot{u} = v$, $\dot{v} = -u$.
For a precise formulation see P. Mardešić - C. Rousseau - B. Toni [25]. □

Another necessary condition of isochronism was proved by B. Schuman [35], C.J. Christopher-J.Devlin [6], namely

Theorem 15.2. *If O is an isochronous center of $(S)_{1,-1}$ and $(S)_{1,-1}$ is quasi homogeneous then $(S)_{1,-1}$ is not hamiltonian.* □

If $n = 2$, since $(Q)_{1,-1}$ is quasi homogeneous, it follows that hamiltonian quadratic systems with an isochronous center do not exist (W.S. Loud [21]).
On the contrary, examples like the following show that hamiltonian systems of degree n with an isochronous center do exist for $n > 2$.

Let $m = 2, 3, \ldots$ and consider the systems of degree $2m - 1 = 3, 5, \ldots$

$$(15.1) \qquad \begin{cases} \dot{x} &= y + mxy^{m-1} + my^{2m-1} \\ \dot{y} &= -x - y^m \end{cases} \qquad m = 2, 3, \ldots$$

reducing to (9.4) for $m = 2$.

Systems (15.1) are hamiltonian and either not O-symmetric for $m = 2, 4, \ldots$ or O-symmetric but not quasi homogeneous for $m = 3, 5, \ldots$ O is a global isochronous center. In fact if $(x, y) : t \mapsto (x(t), y(t))$ is any solution of (15.1), by differentiating $\dot{y} = -x - y^m$ we have $\ddot{y} = -y$ so that $y(t + 2\pi) = y(t)$, hence from $x = -\dot{y} - y^m$, we have also $x(t + 2\pi) = x(t)$.

If O is an isochronous center $\partial \mathcal{N}_O$ cannot contain singular points. □

Therefore we have one more necessary condition for isochronism namely

Theorem 15.3. *If O is an isochronous center of $(S)_{1,-1}$ then O is of type B^k, $1 \leq k \leq n-1$, if n is even, and of type B^k, $1 \leq k \leq n-1$, or of type A if n is odd.* □

Example (15.1) shows the existence of systems of any odd degree having O as a global isochronous center. Therefore it makes sense to consider a particular case of Problem 14.1, namely

Problem 15.1. *Identify polynomial systems $(S)_{1,-1}$ (of odd degree) having O as a global isochronous center.* □

16. Isochronous centers: $n = 2$.

Identification of systems $(S)_{1,-1}$ having O as an isochronous center is part of the problem of identification of systems having O as a center. This sub-problem has been solved in full for quadratic and for O-symmetric cubic systems. □

Let $n = 2$. Then we have (W.S. Loud [21], P.Mardešić - C. Rousseau - B. Toni [25]):

Theorem 16.1. *The quadratic system $(Q)_{1,-1}$ has an isochronous center at O if and only if a linear change of coordinates x, y and a scaling of time t bring $(Q)_{1,-1}$ to one of the systems*

$$\dot{x} = y(1 + x), \quad \dot{y} = -x + y^2 \tag{16.1}$$

$$\dot{x} = y(1 + x), \quad \dot{y} = -x - \frac{1}{2}x^2 + \frac{1}{2}y^2 \tag{16.2}$$

$$\dot{x} = y(1 + x), \quad \dot{y} = -x + \frac{1}{4}y^2 \tag{16.3}$$

$$\dot{x} = y(1 + x), \quad \dot{y} = -x - \frac{1}{2}x^2 + 2y^2. \quad \square \tag{16.4}$$

By using the method of invariants I.I. Pleshkan - K.S. Sibirskii [27] obtained a different identification of a system $(Q)_{1,-1}$ with an isochronous center at O, based directly on the coefficients, namely

Theorem 16.2. *The quadratic system $(Q)_{1,-1}$ has an isochronous center at O if and only if one of the following sets of conditions is satisfied:*

$$a-l=0, \quad c=0, \quad b-m=0, \quad k=0; \tag{16.5}$$

$$a-c-l=0, \quad a+c=0, \quad b+k-m=0, \quad k+m=0; \tag{16.6}$$

$$\begin{cases} 4a+6c-l=0, \quad b-6k-4m=0 \\ \alpha(\alpha^2+\gamma^2)+\beta(\beta^2-3\delta^2)=0 \\ (\alpha^2+\gamma^2)\gamma+(3\beta^2-\delta^2)\delta=0; \end{cases} \tag{16.7}$$

$$\begin{cases} 4a+10c-3l=0, \quad 3b-10k-4m=0 \\ \alpha(\alpha^2+\gamma^2)-27\beta(\beta^2-3\delta^2)=0 \\ (\alpha^2+\gamma^2)\gamma-27(3\beta^2-\delta^2)\delta=0 \end{cases} \tag{16.8}$$

where

$$\alpha=b+k-m, \quad \beta=-b+3k+m, \quad \gamma=-a+c+l, \quad \delta=-a-3c+l. \qquad \square$$

17. Isochronous centers: $n = 3$.

Let us now consider the isochronism of the center O for a cubic nondegenerate O-symmetric system $(C)^0_{1,-1}$.

An analog of Theorem 16.1 is (R. Conti [11], P. Mardešić - C. Rousseau - B. Toni [25]):

Theorem 17.1. *O is an isochronous center of $(C)^0_{1,-1}$ if and only if a linear change of coordinates x, y and scaling of time t transform $(C)^0_{1,-1}$ into one of the systems*

$$\dot{x}=y(1+x^2), \quad \dot{y}=-x(1-y^2) \tag{17.1}$$

$$\dot{x}=y(1-3x^2+y^2), \quad \dot{y}=-x(1-x^2+3y^2) \tag{17.2}$$

$$\dot{x}=y(1+9x^2-2y^2), \quad \dot{y}=-x(1-3y^2) \tag{17.3}$$

$$\dot{x}=y(1-9x^2+2y^2), \quad \dot{y}=-x(1+3y^2). \tag{17.4}$$

□

An analog of Theorem 16.2 is (I.I. Pleshkan [26]):

Theorem 17.2. *O is an isochronous center of $(C)^0_{1,-1}$ if and only if one of the following sets of conditions is satisfied:*

$$\left\{\begin{array}{lll} A+C=0, & A-L=0, & A+N=0 \\ B-M=0, & D=0, & K=0 \end{array}\right. \tag{17.5}$$

$$\left\{\begin{array}{lll} 3A+C=0, & 3A-L=0, & A+N=0 \\ B+3D=0, & B+3K=0, & B-M=0 \end{array}\right. \tag{17.6}$$

$$\left\{\begin{array}{l} 3A+L+C+3N=0, \quad 9A-5L+5C-9N=0 \\ B+3D-3K-M=0, \quad B+6D+6K+M=0 \\ (3A+7N)(7A+3N)-100DK=0 \\ (A+N)[(3A+L)^2-(B+M)^2]- \\ \qquad\qquad -2(3A+L)(B+M)(D-K)=0. \end{array}\right. \tag{17.7}$$

□

Notice that each one of the conditions (17.5), (17.6), (17.7) is a particular case of (9.2). Therefore if (9.3) hold then the center O is *non* isochronous.

□

Non O-symmetric nondegenerate cubic systems $(C)_{1,-1}$ with center at O are not yet identified.

In spite of that there are subclasses of such systems with an *isochronous* center at O which have been identified. We refer to Kukles systems (9.5) (C.J. Christopher - J. Devlin [6]) and to systems with "degenerate infinity", i.e., systems whose Poincaré sphere has the equator filled with singular points (J. Chavarriga - M. Sabatini [5]). □

18. Isochronous centers. Cauchy-Riemann systems. Commutativity.

A sufficient condition for isochronism of a center of (S) is given by (N.A. Lukashevich [22], I.I. Pleshkan [26]):

Theorem 18.1. *Let O be a center of* (1.1). *Then O is isochronous if $P(x,y)$ and $Q(x,y)$ satisfy the Cauchy-Riemann equations*

$$P_x(x,y)-Q_y(x,y)=0, \quad P_y(x,y)+Q_x(x,y)=0. \tag{18.1}$$

□

Notice that such systems cannot be Hamiltonian. □

For $(Q)_{1,-1}$ (18.1) are equivalent to (16.6), for $(C)_{1,-1}$ to (17.6). □

Equations (18.1) can be interpreted as a property of commutativity (M. Villarini [38]). We say that

$$(S) \qquad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

and

$$(T) \qquad \dot{x} = R(x, y), \quad \dot{y} = S(x, y)$$

commute if

$$(18.2) \quad RP_x + SP_y - PR_x - QR_y = 0, \quad RQ_x + SQ_y - PS_x - QS_y = 0.$$

If $R = Q$, $S = -P$ we say that (S) and

$$(S^{\perp}) \qquad \dot{x} = Q(x, y), \quad \dot{y} = -P(x, y)$$

are *orthogonal* each other.

Then (18.2) become

$$Q(P_x - Q_y) - P(P_y + Q_x) = 0, \quad Q(P_y + Q_x) - P(P_x - Q_y) = 0$$

so that (18.1) means that the two systems (S), $(S^{\perp})$ commute.
Starting from this remark, M. Villarini [38] extended Theorem 8.1 as follows.

Two systems (S) and (T), of degrees n and m, are said to be *transversal* each other if

$$P(x, y)S(x, y) - Q(x, y)R(x, y) \neq 0$$

for all (x, y) which are not singular points for both (S) and (T).

For instance (S) and $(S^{\perp})$ are obviously transversal each other.

We then have

Theorem 18.2. *Let (S) commute with some transversal system. Then O is an isochronous center of (S).* □

The invertibility of this result has been studied in detail by M. Sabatini [30], [31] and it is still a source of research. □

19. Uniform isochronism.

Let $\lambda = 1, \mu = -1$ so that O is a center or a focus. In both cases the trajectories close to O wind around O so we can denote by $T(P)$ the time it takes to the trajectory γ_p to make a complete turn around O.

When O is a center $P \mapsto T(P)$ is the period function already defined.

If O is a focus then (V.I. Čemodanov [4]) if $P \mapsto T(P)$ is constant the angular velocity $\dot{\theta}$ of the ray OP is constant. This is no longer true when O is a center, so we have (R. Conti [12])

Definition 19.1. *When O is a center of $(S)_{1,-1}$ and $\dot{\theta}$ is constant we shall say that O is a* uniformly isochronous *center.* □

Since
$$\dot{\theta} = -1 + S(\rho, \theta), \quad \lim_{\rho \to 0} S(\rho, \theta) = 0$$
$\dot{\theta}$ is constant if and only if $\dot{\theta} = -1$, which, in turn, is equivalent to the fact that $(S)_{1,-1}$ becomes

$$\dot{x} = y + xR(x, y), \ \dot{y} = -x + yR(x, y), \tag{19.1}$$

where $R(x, y)$ is a polynomial of degree $n - 1$, $R(0, 0) = 0$. □

It can be proved (R. Conti [12])

Theorem 19.1. *If O is a uniformly isochronous center of $(S)_{1,-1}$ then O is a center of type B^k, $1 \le k \le n - 1$.*

This represents a contribution to solving Problem 14.1. □

The following example shows that contrary to the center a focus which is “uniformly isochronous” can be a “global” one.

Let, for instance,
$$\dot{x} = y - x(x^2 + y^2), \ \dot{y} = -x - y(x^2 + y^2).$$

Then $\dot{\rho} = -\rho^3$, $\dot{\theta} = -1$, hence
$$\rho^2(\theta) = [r^{-2} - 2\theta]^{-1}, \ r^2 = \rho^2(0), \ \theta < r^{-2}/2,$$
so that $\mathbb{R}^2 \setminus \{0\}$ is entirely covered by spirals, i.e., O is a “global” focus. □

Let O be uniformly isochronous, center or focus. Let

$$R(x, y) = \sum_{1}^{n-1} {}_{\nu} R_{\nu}(x, y) \tag{19.2}$$

$$R_\nu(x, y) = \sum_{j+l=\nu} r_{j,l}\, x^j y^l, \quad 1 \le \nu \le n-1. \tag{19.3}$$

Then from (19.1) we have

$$\dot\rho = \sum_{1}^{n-1}{}_{\nu}\, \rho^{\nu+1} \sum_{j+l=\nu} r_{j,l} \cos^j\theta \sin^l\theta. \tag{19.4}$$

From this, if $R(x, y)$ is homogeneous, i.e., (19.1) is quasi homogeneous, we have (R. Conti [12])

Theorem 19.2. *If* (19.1) *is quasi homogeneous, i.e., $R(x, y) = R_{n-1}(x, y)$, then O is a uniformly isochronous center if either n is even, or n is odd and*

$$\sum_{0}^{n-1}{}_{\nu}\, r_{n-1-\nu,\nu} \int_0^{2\pi} \cos^{n-1-\nu}\theta \sin^\nu\theta\, d\theta = 0. \qquad \square \tag{19.5}$$

For $n = 3$ we have that O is a uniformly isochronous center or focus for $(C)_{1,-1}$, O-symmetric or not, if and only if (16.5) and (17.5) hold.

An identification of systems $(C)_{1,-1}$ with a uniformly isochronous center is provided by (C.B. Collins [8])

Theorem 19.3. *O is a uniformly isochronous center of $(C)_{1,-1}$ if and only if* (16.5), (17.5) *hold and, in addition, we have*

$$A + C = 0 \tag{19.6}$$

$$a^2A + abB + b^2C = 0. \qquad \square \tag{19.7}$$

For n odd we propose

Problem 19.1. *Identify systems* (19.1) *of odd degree which are O-symmetric (not necessarily quasi homogeneous) having O as a (uniformly isochronous) center.* $\square$

20. More about the period function.

Let O be a center of the polynomial system $(S)_{\lambda,\mu}$. When O is non isochronous it is of interest to study the period function $P \mapsto T(P)$, $P \in \mathcal{C}_O = \mathcal{N}_O \setminus \{O\}$. □

First of all it can be shown (see M. Villarini [37]) that there are two possibilities, namely, either

$$\lim_{P\to 0} T(P) = +\infty \tag{20.1}$$

or

$$\lim_{P\to 0} T(P) < +\infty. \tag{20.2}$$

In particular, (20.2) holds if $P \mapsto T(P)$ is *bounded* on $\mathcal{C}_O$.
If $P \mapsto T(P)$ is bounded then $\partial\mathcal{N}_O$ cannot contain singular points, i.e., O is global or of type B^k.
Also, if $P \mapsto T(P)$ is bounded, O is not necessarily isochronous, as it is shown, for instance, by the two following examples.

According to (8.5) O is a center of

$$\dot{x} = y + xy, \ \dot{y} = -x + \frac{1}{2}y^2$$

O is of type B^1, $\mathcal{N}_O$ is the half plane $x + 1 > 0$, $\partial\mathcal{N}_O$ is the line $x + 1 = 0$.
It takes a finite time for $(x(t), y(t))$ to traverse the line $\partial\mathcal{N}_O$ so $P \mapsto T(P)$ is bounded. Nevertheless, none of the conditions of Theorem 16.2 is satisfied so O is not isochronous.

Next, O is a global non isochronous center of

$$\dot{x} = y, \quad \dot{y} = -x - x^3.$$

However $P \mapsto T(P)$ is bounded. In fact the trajectories can be represented as the graphs of $\theta \to \rho(\theta)$ satisfying

$$\frac{d\rho}{d\theta} = \frac{\rho^3\cos^3\theta\sin\theta}{1 + \rho^2\cos^4\theta}.$$

It follows that the period of $\theta \mapsto \rho(\theta)$ is

$$\int_0^{2\pi} \frac{d\theta}{1 + \rho^2(\theta)\cos^4\theta} < 2\pi.$$

What precedes suggests

Problem 20.1. *Identify centers of $(S)_{\lambda,\mu}$ whose period function is bounded.*

□

21. Centers of types C and D.

We shall now briefly consider polynomial systems $(S)_{\lambda,\mu}$ with a center O and $\partial\mathcal{N}_O$ containing one singular point S at least. S is necessarily the limit point of a finite number of open half trajectories $\subset \partial\mathcal{N}_O$.

As examples show S can be the limit point also of half trajectories not contained in $\partial\mathcal{N}_O$, finite in number or not.
In particular S can be a saddle point. $\square$

Let O be a center of type C. The set $\partial\mathcal{N}_O$ consists of k unbounded connected components and type C can be distinguished into subtypes C^k, according to the value of k.

Denoting by $k(n)$ the maximum of k for each $n \geq 2$ we have (M. Galeotti [16])

$$k(n) \leq n - 1, \quad n \geq 2. \tag{21.1}$$

Thus we can pose

Problem 21.1. *Establish whether* (21.1) *can be improved by*

$$k(n) = n - 1, \quad n \geq 2. \tag{21.2}$$

Examples show that (21.2) is true for $n = 2, 3$. $\square$

Let $n = 2$. If O is a center of type C then the region $\mathcal{N}_O$ is the interior of a convex angle whose vertex is a tangential limit point S.
S is non elementary if $\mathcal{N}_O$ is an open half plane, it is a saddle point otherwise. This case includes the so called Volterra-Lotka systems. $\square$

When $n > 2$ a further distinction among systems with a center of type C^k occurs by considering the total number σ of singular points belonging to $\partial\mathcal{N}_O$ and the total number ω of open trajectories contained in $\partial\mathcal{N}_O$, so that we have systems of subtypes $C^k_{\sigma,\omega}$.
A detailed description of such systems (following the solution of Problem 1.1) may be of interest, at least for $n = 3$. $\square$

Also centers of $(S)_{\lambda,\mu}$ of type D, i.e., with a bounded $\mathcal{N}_O$ can be divided into subtypes $D_{\sigma,\omega}$.

For $n = 2$ we have $\sigma = \omega = 1, 2, 3$ (see M. Frommer [15], R. Conti [9]).
For $n > 2$ a full description of subtypes $D_{\sigma,\omega}$ does not exist, even for $n = 3$.

$\square$

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