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A COVARIANT AND EXTENDED APPROACH TO SOME PHYSICAL PROBLEMS WITH CONSTRAINED FIELD VARIABLES

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Many physical problems are described by means of systems S of partial differential equations, whose field variables $\underline{u}$ are restricted by relations of the type $\Phi_I(\underline{u}) = 0$. Some examples, to this regard, are the ultrarelativistic gases studied in the framework of Extended Thermodynamics, the relativistic magnetofluiddynamics and the Maxwell Equations in the relativistic form.
Here a general method is proposed to deal with problems of this kind; in particular, a new system S' is proposed in the independent variables $\underline{u}$, ψ_R which are not restricted. Moreover, the solutions of S' with $\Phi_I(\underline{u}) = 0$, $\psi_R = 0$, are the same of the original system S. The new system S' is expressed in the covariant form and is hyperbolic, under the assumption that the original system S satisfies these properties; $\Phi_I(\underline{u}) = 0$, $\psi_R = 0$ are satisfied as consequences of S' and of the intial conditions. The new variables ψ_R are only auxiliary quantities.

1. Introduction.

Let us consider the physical problems which are described by means of a quasi-linear system of partial differential equations of the type

$$\sum_{j=1}^{N} A^{\alpha}_{ij}(\underline{u})\partial_\alpha u_j = f_i(\underline{u}) \tag{1}$$

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for $i = 1, 2, \ldots, n$ and in the N variables $u_1, \ldots, u_N$.
When the variables $\underline{u}$ are independent and $n = N$, this system is un-constrained, and its hyperbolicity is easily defined as follows.

Definition 1. *The system* (1) *is hyperbolic* ([3], [4]) *in the time-direction* t^α *(such* $t^\alpha t_\alpha = -1$*) if and only if*

1) $\det(A^\alpha_{ij} t_\alpha) \neq 0$*;*
2) *for any four-vector* n^α *such that* $n^\alpha t_\alpha = 0$, $n^\alpha n_\alpha = 1$, *the eigenvalue problem* $\sum_{j=1}^{N} A^\alpha_{ij}(n_\alpha - \lambda t_\alpha)\delta u_j = 0$ *has real eigenvalues* λ *and* N *linearly independent l.i eigenvectors* δu_j.

We notice that the first of these conditions is equivalent to the following one

1′) *the system* $t_\alpha \sum_{j=1}^{N} A^\alpha_{ij}(\underline{u})\delta u_j = 0$, *in the independent unknowns* δu_j, *has only the solution* $\delta u_j = 0$.

However, in some physical problems, the N variables $\underline{u}$ are not independent, but constrained by M relations of the type

$$\Phi_I(\underline{u}) = 0 \quad \text{for} \quad I = 1, \ldots, M, \tag{2}$$

where the functions Φ_I are differentiable with respect to $\underline{u}$, expressed in covariant form, and functionally independent, i.e. the rectangular matrix

$$\frac{\partial \Phi_I}{\partial u_j} \quad I = 1, \ldots, M; \; j = 1, \ldots, N$$

has rank M. In this way there remain $N - M$ independent variables.
If $n = N - M$, the above definition can be easily extended and becomes

Definition 2. *The system* (1) *under the constraints* (2) *is hyperbolic in the time-direction* t_α *if and only if*

1) *the system* $t_\alpha \sum_{j=1}^{N} A^\alpha_{ij}(\underline{u})\delta u_j = 0$, $\sum_{j=1}^{N} \frac{\partial \Phi_I}{\partial u_j}\delta u_j = 0$, *in the independent unknowns* δu_j, *has only the solution* $\delta u_j = 0$*;*
2) *the problem*

$$(n_\alpha - \lambda t_\alpha) \sum_{j=1}^{N} A^\alpha_{ij}(\underline{u})\delta u_j = 0, \qquad \sum_{j=1}^{N} \frac{\partial \Phi_I}{\partial u_j}\delta u_j = 0,$$

in the independent unknowns δu_j, *has real eigenvalues* λ *and* N *1.i eigenvectors* δu_j.

In fact, if we choose $N-M$ parameters q_h such that $u_j = u_j(q_h)$ is the general solution of the constraints (2), the Definition 2 is exactly the Definition 1 written in the $N-M$ un-constrained variables q_h. This approach has been widely used in literature; see, for example, ref. [7] where the independent variables are n (partile density), e (energy density), π (dinamic pressure), U^α (4-velocity), q^α (heat flux), $t^{\langle\alpha\beta\rangle}$ (stress deviator), constrained by

$$U^\alpha U_\alpha = -1,\ q^\alpha U_\alpha = 0,\ t^{\langle\alpha\beta\rangle} U_\alpha = 0,\ t^{\langle\alpha\beta\rangle} g_{\alpha\beta} = 0,$$

with $g_{\alpha\beta}$ the metric tensor.

Finally, if $n > N-M$, we have also $n-(N-M)$ differential constraints. In this case we may conceive the idea of taking only $N-M$ equations from the system (1) and hoping that this reduced system is hyperbolic, according to Definition 2; more generally, we may take $N-M$ linear combinations of the equations of system (1), or, equivalently, we may multiply it on the left by a $(N-M)Xn$ matrix Z_{ki}. In order not to lose manifest covariance, we may allow the matrix Z_{ki} to have more than $N-M$ rows, but to have rank $N-M$; in this way the supplementary equations are only linear combinations of the others. Therefore, the following definition of hyperbolicity, for this constrained system, is proposed:

Definition 3. *The system* (1) *under the constraints* (2) *is hyperbolic in the time-direction t_α if and only if an mXn matrix Z_{ki} exists, such that,*

1) *Z_{ki} has rank $N-M$;*
2) *the system $t_\alpha \sum_{i=1}^{n}\sum_{j=1}^{N} Z_{ki} A^\alpha_{ij}(\underline{u})\delta u_j = 0$, $\sum_{j=1}^{N} \frac{\partial \Phi_I}{\partial u_j}\delta u_j = 0$, in the independent unknowns δu_j, has only the solutions $\delta u_j = 0$;*
3) *the problem*

$$(n_\alpha - \lambda t_\alpha)\sum_{i=1}^{n}\sum_{j=1}^{N} Z_{ki} A^\alpha_{ij}(\underline{u})\delta u_j = 0,\ \sum_{j=1}^{N} \frac{\partial \Phi_I}{\partial u_j}\delta u_j = 0,$$

in the independent unknowns δu_j, has real eigenvalues λ and N l.i eigenvectors δu_j.

If the system (1) is expressed in covariant form, also the matrix Z_{ki} must be covariant, to preserve this property.

Obviously, we may substitute the system (1) with

$$\sum_{i=1}^{n}\sum_{j=1}^{N} Z_{ki} A^\alpha_{ij}(\underline{u})\partial_\alpha u_j = \sum_{i=1}^{n} Z_{ki} f_i(\underline{u}), \tag{3}$$

and apply to it the Definition 2.

I remark that this arguments are inspired by the elegant Strumia's papers on this subject ([10], [11], [12]), altough the definition in this article is less restrictive than Strumia's one, as it will be seen in section 2. Also in this section, a method will be shown to find the matrix Z_{ki}.

The above definition of hyperbolicity for a constrained system appears a little complicated. To eliminate this drawback I propose a method based on the ideas of extended thermodynamics ([6], [7]) to introduce other independent variables ψ_R and to find a new system of equations that for $\psi_R = 0$ reduces to (1); moreover this new system has the same number of equations and of independent variables, it is hyperbolic if and only if the system (1) is hyperbolic, it gives $\psi_R = 0$ if we impose $\psi_R = 0$ only on a given time-like initial hypersurface, it is expressed in covariant form. These results will be obtained in Section 3 and are expressed by the system (17).

Another idea is that of searching a new system with a less number of auxiliary variables ψ_R than in (17), and without the constraints (2); this idea is realized in Section 4 and the new system is expressed by (24).

All these results depend on the time-like congruence t_α that has been initially chosen. This problem is investigated in Section 5, under the assumption that the system (1) is hyperbolic in the direction t_α and the characteristic velocities do not exceed the speed of light.

Physical examples of application of this methodology are also considered in this paper; they are the equations of Extended Thermodynamics for ultra-relativistic gases (see in Section 2), those of relativistic fluiddynamics (see in Section 4 from eq. (30) to eq. (32)), those of relativistic magnetofluiddynamics (see in Section 3 from eq. (21) and Section 4, eq. (29)), those of covariant Maxwell electro-dynamics (see in Section 4 from eq. (33)).

It is also shown, in Section 2 from eq. (13), that the Einstein's equations in empty space are not hyperbolic.

2. A method to find the matrix Z_{ki}.

If system (1) is hyperbolic, it can be written in the form (3) which makes more easy to find the associated eigenvalues and eigenvectors. The problem now arises on how to find the matrix Z_{ki}. To this end, the following 4 steps can be accomplished.

Step 1) Let us consider the system $\delta\Phi_I = 0$, of M linearly independent equations in the N unknowns δu_j, i.e.,

$$\sum_{j=1}^{N} \frac{\partial \Phi_I}{\partial u_j} \delta u_j = 0, \tag{4}$$

which is a consequence of the constraints (2); it gives δu_j as a linear combination of $N-M$ free unknowns. Another possibility, useful in order not to lose the covariance, is to obtain δu_j as a linear combination of p free unknowns, with $p \geq N-M$, but by means of functions which are not functionally independent. More precisely, we may find

$$\delta u_j = U_{jj'}(\underline{u}) V_{j'}, \ \forall V_{j'}, \tag{5}$$

with $U_{jj'}$ a matrix of rank $N - M$.
For the applications it is useful to notice that $U_{jj'}$ is the matrix whose j-th row is the derivative of δu_j in equation (5), with respect to $V_{j'}$. This matrix is such that

$$\sum_{j=1}^{N} \frac{\partial \Phi_I}{\partial u_j} U_{jj'} = 0. \tag{6}$$

The practical meaning of this step is that the variables contribute to the equations, only by means of their projections onto the subspace tangential to the variety (2).
Step 2) Let us consider the matrix $t_\alpha \sum_{j=1}^{N} A^\alpha_{ij} U_{jj'}$; it has rank $N - M$, as proved in appendix 1. For the applications, we notice that its i-th row is the derivative with respect to $V_{j'}$ of the expression $t_\alpha \sum_{j=1}^{N} A^\alpha_{ij} \delta u_j$, after having substituted δu_j from equation (5).
After that, let us consider the system with this matrix of the coefficients and contracted on the left by the unknowns X_i,

$$\sum_{i=1}^{n} X_i t_\alpha \sum_{j=1}^{N} A^\alpha_{ij} U_{jj'} = 0. \tag{7}$$

I gives X_i as a linear combination of $n-(N-M)$ free unknowns. Alternatively, we may find X_i as a linear combination of q free unknowns, with $q \geq$

$n-(N-M)$, but by means of functionally dependent functions; this possibility is used in order not to lose the covariance. More precisely, we may find

$$X_i = \sum_{i'=1}^{q} \overline{X}_{i'} X_{i'i} \quad \forall \overline{X}_{i'}, \tag{8}$$

where the qXn matrix $X_{i'i}$ has rank $n-(N-M)$.
(Note that $X_{i'i}$ is the matrix whose i-th coulumn is the derivative of X_i in equation (8) with respect to $\overline{X}_{i'}$).
This matrix is such that the following relation holds

$$\sum_{i=1}^{n} X_{i'i} t_\alpha \sum_{j=1}^{N} A_{ij}^{\alpha} U_{jj'} = 0. \tag{9}$$

The practical meaning of this matrix is that it allows to separate the differential constraints from the other equations.
Step 3) Let us consider the system

$$\sum_{k=1}^{n} X_{i'k} Y_k = 0; \tag{10}$$

it gives

$$Y_k = \sum_{i=1}^{n} Y_{ki} \overline{Y}_i, \quad \forall \overline{Y}_i \quad \text{(free unknowns)}, \tag{11}$$

where Y_{ki} is a $n \times n$ matrix having rank $N-M$. Obviously, it is such that

$$\sum_{k=1}^{n} X_{i'k} Y_{ki} = 0. \tag{12}$$

Here too, the k-th row of Y_{ki} is the derivative of Y_k in eq. (11) with respect to $\overline{Y}_i$. The practical meaning of multiplyng the original system on the left by Y_{ki} is that of projecting it onto the subspace ortogonal to $X_{i'i}$; in this way, the constant (9) on the evolutive part of the equations becomes now a constraint also on its spatial part.
Step 4) The matrix Z_{ki} is sum of Y_{ki} and of a suitable solution of the system (9), i.e., the parameters μ_{ki}, exist such that

$$Z_{ki} = Y_{ki} + \sum_{i'=1}^{n} \mu_{ki}, X_{i'i}.$$

Obviously, the parameters $\mu_{ki'}$ must be such that the above mentioned properties of the matrix Z_{ki}, are preserved. The second term, in the expression of Z_{ki}, takes account of the fact that the differential constraints may still play a role, before to be neglected.

Having completed the description of these 4 step, let us prove the last of the them. The $2n \times n$ matrix $\begin{pmatrix} Y_{i'i} \\ X_{i'i} \end{pmatrix}$ has rank n (see Appendix 2); therefore the parameters $\lambda_{ki'}, \mu_{ki'}$ exist such that $Z_{ki} = \sum_{i'=1}^{n}(\lambda_{ki'}Y_{i'i} + \mu_{ki'}X_{i'i})$; from this it follows

$$t_\alpha \sum_{i=1}^{n}\sum_{j=1}^{N} Z_{ki}A^\alpha_{ij}\delta u_j = t_\alpha \sum_{i'=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{N}\lambda_{ki'}Y_{i'i}A^\alpha_{ij}\delta u_j,$$

where (9) has been used and also the fact that δu_j is a linear combination of $U_{jj'}$. If we impose now the condition 2) of Definition 3, we obtain that $t_\alpha \sum_{i'=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{N}\lambda_{ki'}Y_{i'i}A^\alpha_{ij}\frac{\partial u_j}{\partial q_h}\delta q_h = 0$ has only the solution $\delta q_h = 0$; this fact proves that the matrix $t_\alpha \sum_{i'=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{N}\lambda_{ki'}Y_{i'i}A^\alpha_{ij}\frac{\partial u_j}{\partial q_h}$ has rank $N - M$ and, consequently, $\sum_{i'=1}^{n}\lambda_{ki'}Y_{i'i}$ has rank $N - M$ (see Appendix 3).

In this way we see that $\sum_{i'=1}^{n}\lambda_{ki'}Y_{i'i}$ is also a solution of system (12) and has rank $N-M$, i.e., $\sum_{i'=1}^{n}\lambda_{ki'}Y_{i'i}$ satisfies the same properties of Y_{ki}; by substituting $\sum_{i'=1}^{n}\lambda_{ki'}Y_{i'i}$ with Y_{ki} we obtain $Z_{ki} = Y_{ki} + \sum_{i'=1}^{n}\mu_{ki'}X_{i'i}$.

Therefore Z_{ki} is determined except for $\mu_{ki'}$; from equation (8), we see that Z_{ki} is the sum of Y_{ki} and of a particular solution of the system $\sum_{i=1}^{n}X_i t_\alpha \sum_{j=1}^{N}A^\alpha_{ij}U_{jj'} = 0$.

Let us introduce now some notes.

Note 1: We notice that the Definition 2 is less restrictive that Strumia's one [10]; in fact he considers only the case $n = N$ and imposes the further condition that the matrix Z_{ki} is the projector onto the subspace generated by the 1.i. columns of $U_{jj'}$.

Note 2: If system (1) is hyperbolic and δu_j is a solution of the system $t_\alpha \sum_{j=1}^{N}A^\alpha_{ij}(\underline{u})\delta u_j = 0$, $\sum_{j=1}^{N}\frac{\partial \Phi_I}{\partial u_j}\delta u_j = 0$, then δu_j satisfies also the system

considered in condition 2) of Definition 2; therefore we have $\delta u_j = 0$.
This note can be used to show that Einstein's equations in empty space are not hyperbolic. These equations are ([10], [2], [5]):

$$\partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\nu\lambda} + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\rho} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} = 0 \tag{13}$$

with $\Gamma^\alpha_{\mu\nu} = g^{\alpha\rho}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu})/2$ (Christoffel symbol).
They constitute a second order system of ten equations for the unknown symmetric tensor $g_{\rho\mu}$.
By defining $\omega_{\alpha\mu\nu} = \partial_\alpha g_{\mu\nu}$, it can be reduced to the first order system

$$\begin{cases} \partial_\alpha[g^{\alpha\rho}(\omega_{\nu\rho\mu} + \omega_{\mu\rho\nu} - \omega_{\rho\mu\nu}) - \delta^\alpha_\mu g^{\lambda\rho}\omega_{\nu\rho\lambda}] = W_{\mu\nu}(g_{\alpha\delta}, \omega_{\beta\gamma\delta}) \\ \partial_\alpha(\delta^\alpha_\beta g_{\mu\nu}) = \omega_{\beta\mu\nu} \\ (\delta^\alpha_\sigma\delta^\beta_\tau - \delta^\alpha_\tau\delta^\beta_\sigma)\partial_\alpha\omega_{\beta\mu\nu} = 0 \end{cases} \tag{14}$$

in the 50 unknowns $g_{\mu\nu}, \omega_{\beta\mu\nu}$.
We have $I = 0$ and $U_{jj'} = \delta_{jj'}$, because there is no constraint on the independent variables. However, the condition on note 3, which is necessary for the hyperbolicity of the system (14) is not satisfied. In fact, in this case, the system in note 2 is

$$\begin{cases} t_\alpha\delta[g^{\alpha\rho}(\omega_{\nu\rho\mu} + \omega_{\mu\rho\nu} - \omega_{\rho\mu\nu}) - \delta^\alpha_\mu g^{\lambda\rho}\omega_{\nu\rho\lambda}] = 0 \\ t_\alpha(\delta^\alpha_\beta g_{\mu\nu}) = 0 \\ (\delta^\alpha_\sigma\delta^\beta_\tau - \delta^\alpha_\tau\delta^\beta_\sigma)t_\alpha\delta\omega_{\beta\mu\nu} = 0. \end{cases} \tag{15}$$

Its general solution is

$$\delta g_{\mu\nu} = 0,\ \delta\omega_{\tau\mu\nu} = t_\tau(t_\mu V_\nu + t_\nu V_\mu + t_\mu t_\nu t^\lambda V_\lambda),$$

with V_μ an arbitrary four-vector. Therefore this system has solutions different from $\delta u_j = 0$, i.e., the Einstein's equations in empty space are not hyperbolic.
I conclude this section by illustrating the above 4 steps in the case of Extended Thermodynamic of an ultrarelativistic gas. This problem is described by the equations

$$\begin{cases} \partial_\alpha\ V^\alpha = 0 \\ \partial_\alpha\ T^{\alpha\beta} = 0 \\ \partial_\alpha\ A^{\alpha\beta\gamma} = I^{\beta\gamma}, \end{cases}$$

where the independent variables are V^μ (particle number-particle flux vector), $T^{\lambda\nu}$ (stress-energy-momentum tensor), constrained by $\Phi = 0$, with $\Phi = T^\lambda_\lambda$.

Moreover, in the case of a non degenerate gas, we have

$$A^{\alpha\beta\gamma} = (4/3)n^{-1}e^2(2U^\alpha U^\beta U^\gamma + g^{(\alpha\beta}U^{\gamma)}) + \\ +2(e/n)(g^{(\alpha\beta}q^{\gamma)} + 6U^{(\alpha}U^\beta q^{\gamma)}) + 6(e/n)t^{(\langle\alpha\beta\rangle}U^{\gamma)},$$

$$I^{\beta\gamma} = B_3 t^{\langle\beta\gamma\rangle} + 2B_4 q^{(\beta}U^{\gamma)},$$

$$n = (-V_\beta V^\beta)^{1/2}, U^\alpha = n^{-1}V^\alpha,$$

$$h^{\alpha\beta} = g^{\alpha\beta} + U^\alpha U^\beta, t^{\langle\alpha\beta\rangle} = (h^\alpha_\mu h^\beta_\nu - 1/3h^{\alpha\beta}h_{\mu\nu})T^{\mu\nu},$$

$$q^\alpha = -h^\alpha_\mu U_\nu T^{\mu\nu},\ e = U_\mu U_\nu T^{\mu\nu}.$$

An arbitrary single variable function A, which appears in the first version of these equations, has been dropped because it does not appear in other versions based on the kinetic theory.

For the sake of simplicity, let us take $t^\alpha = U^\alpha$.

Step 1) The system (4) is $g^{\alpha\beta}\delta T_{\alpha\beta} = 0$, whose solution is $\delta T^{\lambda\nu} = (g^{(\lambda}_\varphi g^{\nu)}_\rho - 1/4g^{\lambda\nu}g_{\varphi\rho})W^{\varphi\rho}$; therefore, the requested matrix $U_{jj'}$ is

$$U_{jj'} = \begin{pmatrix} g^\mu_\delta & 0 \\ 0 & g^{(\lambda}{}_\varphi g^{\nu)}_\rho - 1/4g^{\lambda\nu}g_{\varphi\rho} \end{pmatrix},\ \text{whose rank is 9.}$$

(The index j is described by $\mu\lambda\nu$, while j' is described by $\delta\varphi\rho$).

Step 2) For the sake of simplicity, let us aim to hyperbolicity holding in a neighbourhood of thermodinamical equilibrium; in this case we may calculate the coefficients of $\partial_\alpha V^\mu$, $\partial_\alpha T^{\lambda\nu}$ at equilibrium. The resulting matrix $U_\alpha \sum_{j=1}^{N} A^\alpha_{ij}U_{jj'}$ is

$$\begin{pmatrix} U_\delta & 0 \\ 0 & U_{(\varphi}g^\beta_{\rho)} - 1/4U^\beta g_{\varphi\rho} \\ A^{\beta\gamma}_\delta & A^{B\gamma}_{\varphi\rho} \end{pmatrix},$$

with $A^{\beta\gamma}_\delta = -(4/9)e^2n^{-2}[(h^{\beta\gamma} + 3U^\beta U^\gamma)U_\delta - 10h^{(\beta}{}_\delta U^{\gamma)}]$,

$$A^{\beta\gamma}_{\varphi\rho} = (2/3)(e/n)[-4(g^{\beta\gamma} + 4/3U^\beta U^\gamma)(U_\varphi U_\rho + 1/4g_{\varphi\rho}) + \\ + 10U^{(\beta}h^{\gamma)}_{(\varphi}U_{\rho)} + h_{\varphi\rho}h^{\beta\gamma} - 3h^\beta_{(\varphi}h^\gamma_{\rho)}].$$

The system (7) is

$$XU_\delta + X_{\beta\gamma}(8/9)e^2n^{-2}[-2U^\beta U^\gamma U_\delta + 5h^\beta_\delta U^\gamma] = 0,$$

$$X_\beta[(U_{(\varphi}g^\beta_{\rho)} - 1/4U^\beta g_{\varphi\rho}] + X_{\beta\gamma}(2/9)en^{-1}[-16U^\beta U^\gamma(U_\varphi U_\rho + \\ + 1/4g_{\varphi\rho}) + 3(h_{\varphi\rho}U^\beta U^\gamma - 3h^\gamma_\varphi h^\beta_\rho)] = 0,$$

where the relation $h^{\beta}_{\delta} X_{\beta\gamma} U^{\gamma} = 0$ (which comes from the first of these equations, contracted with $h^{\delta}_{\delta'}$) has been used. Moreover, $X_{\beta\gamma}$ is a symmetric and traceless tensor.
The above system has the solution

$$X = (16/3)e^2 n^{-2}\overline{X},\ X_{\beta} = (32/3)en^{-1}\overline{X}U_{\beta},$$

$X_{\beta\gamma} = \overline{X}(g_{\beta\gamma} + 4U_{\beta}U_{\gamma})$. Consequently, the matrix $X_{ii'}$ is $(i' = 1)$

$$((16/3)e^2 n^{-2},\ (32/3)en^{-1}U_{\beta},\ (g_{\beta\gamma} + 4U_{\beta}U_{\gamma})).$$

Step 3) We have to consider now the system

$$(16/3)e^2 n^{-2}Y + (32/3)en^{-1}U_{\beta}Y^{\beta} + (g_{\beta\gamma} + 4U_{\beta}U_{\gamma})Y^{\beta\gamma} = 0,\ \ g_{\beta\gamma}Y^{\beta\gamma} = 0.$$

To find a covariant solution to this system, we notice that $Y_1, Y_2, \overline{Y}^{\lambda\nu}$ exist such that

$$Y^{\beta\gamma} = Y_1 g^{\beta\gamma} + Y_2 U^{\beta}U^{\gamma} + (1/3)[3g^{\beta}_{\lambda}g^{\gamma}_{\nu} - (g_{\lambda\nu} + U_{\lambda}U_{\nu})g^{\beta\gamma} + \\ - (4U_{\lambda}U_{\nu} + g_{\lambda\nu})U^{\beta}U^{\gamma}]\overline{Y}^{\lambda\nu};$$

(In fact Y_1, Y_2 can be found by contracting this expression with $g_{\beta\gamma} + U_{\beta}U_{\gamma}$, $(1/3)(g_{\beta\gamma} + 4U_{\beta}U_{\gamma})$ respectively; after that $\overline{Y}^{\lambda\nu} = Y^{\beta\gamma}$ is a possible choice for $\overline{Y}^{\lambda\nu}$). By using this expression the above equations become conditions which give Y, Y^{β}, Y_1, Y_2; the result is

$$Y = \overline{Y},\ Y^{\beta} = g^{\beta}_{\lambda}\overline{Y}^{\lambda},$$

$$Y^{\beta\gamma} = -(4/9)en^{-2}(e\overline{Y} + 2nU_{\lambda}\overline{Y}^{\lambda})(g^{\beta\gamma} + 4U^{\beta}U^{\gamma}) + \\ + (1/3)[3g^{\beta}_{\lambda}g^{\gamma}_{\nu} - (g_{\lambda\nu} + U_{\lambda}U_{\nu})g^{\beta\gamma} - (4U_{\lambda}U_{\nu} + g_{\lambda\nu})U^{\beta}U^{\gamma}]\overline{Y}^{\lambda\nu};$$

after that, we see that the matrix Y_{ki} is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & g^{\beta'}_{\beta} & 0 \\ eC^{\beta'\gamma'} & 2nC^{\beta'\gamma'}U_{\beta} & C^{\beta'\gamma'}_{\beta\gamma} \end{pmatrix}$$

where $C^{\beta'\gamma'} = -(4/9)en^{-2}(4U^{\beta'}U^{\gamma'} + g^{\beta'\gamma'})$,

$$C^{\beta'\gamma'}_{\beta\gamma} = g^{\beta'}_{\beta}g^{\gamma'}_{\gamma} - (1/3)g_{\beta\gamma}g^{\beta'\gamma'} - (1/3)U_{\beta}U_{\gamma}g^{\beta'\gamma'} + \\ - (1/3)U^{\beta'}U^{\gamma'}g_{\beta\gamma} - (4/3)U_{\beta}U_{\gamma}U^{\beta'}U^{\gamma'}.$$

Step 4) If we take $\mu_{ki'} = 0$, we obtain $Z_{ki} = Y_{ki}$ and the Definition 2 is satisfied, as it can be easily seen. For the sake of brevity, I avoid to report the proof of this statement. I simply note that the system multiplied on the left by Z_{ki} give 3 equations; if we add to the third one of them the first one, premultiplied by $eC^{\beta'\gamma'}$, and the second one, premultiplied by $2nC^{\beta'\gamma'}U_\beta$, we obtain

$$\begin{cases} \partial_\alpha \; V^\alpha = 0 \\ \partial_\alpha \; T^{\alpha\beta} = 0 \\ \partial_\alpha \; A^{\alpha\beta'\gamma'} - (1/3)(g^{\beta'\gamma'} + 4U^{\beta'}U^{\gamma'})U_\beta U_\gamma \partial_\alpha A^{\alpha\beta\gamma} = I^{\beta'\gamma'}, \end{cases}$$

where the identities $g_{\beta\gamma}A^{\alpha\beta\gamma} = 0$, $U_\beta U_\gamma I^{\beta\gamma} = 0$, $g_{\beta\gamma}I^{\beta\gamma} = 0$, have been used. This system is the conterpart of (3), for this particular case.

3. An extended approach to systems with constrained field.

In this section a new system is searched, in the variables $\underline{u}$, Ψ_R (auxiliary quantities), which for $\Psi_R = 0$ reduces to system (1); if (1) is hyperbolic, then also the new system is hyperbolic.
To this end, let T_k^R be $n - (N - M)$ l.i. solutions of the system

$$T_k Z_{ki} = 0, \tag{16}$$

expressed in covariant form and let ω^α be a four-vectorial function such that $t^\alpha\omega_\alpha \neq 0$ (for example we may choose $\omega^\alpha = t^\alpha$).
Let us consider the system

$$\sum_{i=1}^{n}\sum_{j=1}^{N} Z_{ki} A_{ij}^\alpha(\underline{u})\partial_\alpha u_j + \sum_{R=1}^{n-(N-M)} \omega^\alpha T_k^R \partial_\alpha \Psi_R = \sum_{i=1}^{n} Z_{ki} f_i(\underline{u}) \tag{17}$$

in the variables u_j, Ψ_R constrained by $\Phi(\underline{u}) = 0$, so that we have $N - M + [n - (N - M)] = n$ independent variables.
This system has the advantage to have an equal number of equations and of independent variables; when $\Psi_R = 0$ it reduces to system (10), which is equivalent to system (1) except for some conditions to be imposed in the initial manyfold (see the previous section).
Moreover we prove that

1) the system (17) is hyperbolic if this property is satisfied by the system (1),
2) if $\Psi_R = 0$ on an intial hypersurface Σ, then $\Psi_R = 0$ will propagate off Σ.

In fact the system (17) is equivalent to

$$\sum_{i=1}^{n}\sum_{j=1}^{N} Z_{ki} A_{ij}^{\alpha} \frac{\partial u_j}{\partial q_h} \partial_\alpha q_h + \sum_{R=1}^{n-(N-M)} \omega^\alpha T_k^R \partial_\alpha \psi_R = \sum_{i=1}^{n} Z_{ki} f_i. \tag{18}$$

Condition 1) of Definition 1 is equivalent to impose that the system

$$t_\alpha \sum_{i=1}^{n}\sum_{j=1}^{N} Z_{ki} A_{ij}^{\alpha} \frac{\partial u_j}{\partial q_h} \delta q_h + \sum_{R=1}^{n-(N-M)} t_\alpha \omega^\alpha T_k^R \delta\psi_R = 0 \tag{19}$$

has only the solution $\delta q_h = \delta\psi_R = 0$.
If we multiply eq. (19) by T_k^S, we obtain $\sum_{R=1}^{n-(N-M)} t_\alpha \omega^\alpha T_k^S T_k^R \delta\psi_R = 0$, from which $\delta\psi_R = 0$ because T_k^R are l.i.; after that, the system (19) gives $\delta q_h = 0$ because the system (1) is hyperbolic and condition 2) of Definition 3 is satisfied. Let us now see that the system (18) satisfies also the condition 2) of Definition 1. To this end, let δq_k, for $k = 1, \dots, N - M$, be $N - M$ l.i. solutions of the condition 2) in Definition 2, corresponding to the eigenvalue λ_k; let $\delta\psi_R$ be the vector of components $\delta\psi_{RP} = \delta_{RP}$.
The vectors $(\delta q_k, 0_R)$ and $(0_k, \delta\psi_P)$ are n l.i. eigenvectors of the system (18), corresponding to the eigenvalues λ_k and $\lambda' = (n_\alpha \omega^\alpha)/(t_\alpha \omega^\alpha)$, respectively. In fact they satisfy the system

$$(n_\alpha - \lambda t_\alpha)\Big[\sum_{i=1}^{n}\sum_{j=1}^{N} Z_{ki} A_{ij}^{\alpha} \frac{\partial u_j}{\partial q_h} \delta q_h + \sum_{R=1}^{n-(N-M)} \omega^\alpha T_k^R \delta\psi_R\Big] = 0.$$

In this way we have proved that the system (17) is hyperbolic. Let us prove now the statement 2). To this end, we multiply the system (17) by T_k^S, obtaining that

$$\sum_{R=1}^{n-(N-M)} T_k^S T_k^R \omega^\alpha \partial_\alpha \psi_R = 0, \quad \text{i.e.,} \quad \omega^\alpha \partial_\alpha \psi_R = 0;$$

this relation can be written also as

$$t^\alpha \partial_\alpha \psi_R = (t^\mu \omega_\mu)^{-1} \omega_\gamma h^{\gamma\alpha} \partial_\alpha \psi_R, \tag{20}$$

with $h^{\gamma\alpha} = g^{\gamma\alpha} + t^\gamma t^\alpha$.

If $\psi_R = 0$ on the initial hypersurface Σ (from which $h^{\gamma\alpha}\partial_\alpha\psi_R = 0$ also holds in Σ), we see from eq. (20) that $\psi_R = 0$ holds also off Σ, as we desired to prove.

Let us now apply this methdology to a particular physical problem, the equations of relativistic magnetofluiddynamics. These equations are the conservation laws of mass and energy-momentum and the maxwell's equations, i.e.,

$$\begin{cases} \partial_\alpha(nu^\alpha) = 0 \\ \partial_\alpha[(e + p + b^2)u^\alpha u^\beta + (p + b^2/2)g^{\alpha\beta} - b^\alpha b^\beta] = 0 \\ \partial_\alpha(u^\alpha b^\beta - b^\alpha u^\beta) = 0 \end{cases}$$

in the variables e (total energy-density), V^α, b^α (related to the electromagnetic field), constrained by $V_\alpha b^\alpha = 0$. Moreover, we have $n = (-V^\gamma V_\gamma)^{-1/2}$ (rest-mass density), $u^\alpha = n^{-1}V^\alpha$ (four-velocity), $p = p(n, e)$.

Hyperbolicity of this system has already been proved, for example in refs. ([10], [11],[9], [1], [8]). By appliying the present methodology, with $t^\alpha = u^\alpha$, $\omega^\alpha = u^\alpha$, we obtain that $n = 9$, $N = 9$, $M = 1$,

$$U_{jj'} = \begin{pmatrix} 1 & 0_\gamma & 0_\gamma \\ 0_\gamma & g_{\gamma\delta} & 0_{\gamma\delta} \\ 0_\gamma & n^{-1}b_\delta u_\gamma & h_{\gamma\delta} \end{pmatrix}; \; X_{i'i} = \begin{pmatrix} 0 & 0_\beta & u_\beta \\ 0_\gamma & 0_{\gamma\beta} & 0_{\gamma\beta} \\ 0_\gamma & 0_{\gamma\beta} & 0_{\gamma\beta} \end{pmatrix};$$

$$Y_{kj} = \begin{pmatrix} 1 & 0^\beta & 0^\beta \\ 0^\lambda & g^{\lambda\beta} & 0^{\lambda\beta} \\ 0^\lambda & 0^{\lambda\beta} & h^{\lambda\beta} \end{pmatrix}; \; \mu_{ki'} = \begin{pmatrix} 0 & 0^\gamma & 0^\gamma \\ b^\lambda & 0^{\lambda\gamma} & 0^{\lambda\gamma} \\ 0^\lambda & 0^{\lambda\gamma} & 0^{\lambda\gamma} \end{pmatrix};$$

$$Z_{kj} = \begin{pmatrix} 1 & 0^\beta & 0^\beta \\ 0^\lambda & g^{\lambda\beta} & b^\lambda u^\beta \\ 0^\lambda & 0^{\lambda\beta} & h^{\lambda\beta} \end{pmatrix}; \; T_k^1 = \begin{pmatrix} 1 \\ 0^\lambda \\ u^\lambda \end{pmatrix}.$$

Consequently, the system (17), in this case, is

$$\text{(22)} \qquad \begin{cases} \partial_\alpha(nu^\alpha) = 0 \\ \partial_\alpha[(e + p + b^2)u^\alpha u^\lambda + (p + b^2/2)g^{\alpha\lambda} - \\ \quad -b^\alpha b^\lambda] + u^\gamma b^\lambda \partial_\alpha(u^\alpha b_\gamma - b^\alpha u_\gamma) = 0 \\ h^{\beta\gamma}\partial_\alpha(u^\alpha b_\beta - b^\alpha u_\beta) + u^\lambda u^\alpha \partial_\alpha \psi = 0. \end{cases}$$

Obviously, the system (22) in the unknowns $e, V^\alpha, b^\alpha, \psi$ constrained by $V_\alpha b^\alpha = 0$ is hyperbolic, is expressed in covariant form and has an equal number of equations and of independent variables.

4. A new form for the system (17).

Let us consider firstly the case $n \geq N$ (from which $n-(N-M) \geq M$ holds), which I call case a. The system (17) in the independent variables $q_1, \ldots, q_{N-M}, \psi_1, \ldots, \psi_M, \psi_{M+1}, \ldots, \psi_{n-(N-M)}$ is hyperbolic. There are M columns of the matrix $\frac{\partial \Phi_I}{\partial u_j}$, which are l.i.; we can suppose, without loss of generality, that they are the last M columns, i.e., that the matrix

$$\frac{\partial \Phi_I}{\partial u_j} \quad \text{for } j = N-M+1, \ldots, N \quad \text{has rank } M.$$

Therefore we can take $q_1 = u_1, \ldots, q_{N-M} = u_{N-M}$ and the system (17) in the independent variables $u_1, \ldots, u_{N-M}, \psi_1, \ldots, \psi_M, \psi_{M+1}, \ldots \psi_{n-(N-M)}$ is hyperbolic. Let us consider the invertible change of variables, expressing the above ones in terms of $u_1, \ldots, u_{N-M}, u_{N-M+1}, \ldots, u_N, \psi_{M+1}, \ldots, \psi_{n-(N-M)}$, by means of the transformation law

$$\Phi_I(\underline{u}) = \psi_I \quad \text{for } I = 1, 2, \ldots, M. \tag{23}$$

In other words, we are obtaining $u_{N-M+1}, \ldots, u_N$ as functions of the previous variables, from eq. (23).
Obviously, the system (17), expressed in terms of the new variables, is still hyperbolic. By using eq. (23) it becomes

$$\sum_{i=1}^{n} \sum_{j=1}^{N} Z_{ki} A^{\alpha}_{ij} \partial_\alpha u_j + \sum_{R=1}^{M} \omega^\alpha T_k^R \partial_\alpha \Phi_R(\underline{u}) + \\ + \sum_{R=M+1}^{n-(N-M)} \omega^\alpha T_k^R \partial_\alpha \psi_R = \sum_{i=1}^{n} Z_{ki} f_i . \tag{24}$$

This new form for the system (17) has the advantage to have an equal number of equations and of independent variables; these last ones are $\underline{u}$, $\psi_{M+1}, \ldots,$ $\psi_{n-(N-M)}$ and are not constrained.
However, if we impose that $\Phi_I(\underline{u}) = 0$, $\psi_{M+1} = 0, \ldots, \psi_{n-(N-M)} = 0$ hold on the initial manifold, they will be satisfied also off it.
Let us consider now the case $n < N$ (from which $n-(N-M) < M$ holds), which I call case b.
There are M columns of the matrix $\frac{\partial \Phi_I}{\partial u_j}$, which are l.i.; we can suppose, without loss of generality, that they are the last M columns, i.e., that the matrix

$M^* = \frac{\partial \Phi_I}{\partial u_j}$ for $j = N - M + 1, \dots, N$, has rank M. If we use the Laplace's rule to calculate $\det M^* \neq 0$, we see that we can exchange some columns of M^* such that M^* becomes $\begin{pmatrix} M^*_{11} & M^*_{12} \\ M^*_{21} & M^*22 \end{pmatrix}$ with M^*_{11} of order $n - (N - M)$, M^*_{22} of order $N - n$ and moreover $\det M^*_{11} \neq 0$, $\det M^*_{22} \neq 0$. In other words, by exchanging the names of some u_j, we obrtain that $\dfrac{\partial \Phi_I}{\partial u_j}$ takes the form

$$\begin{pmatrix} \dfrac{\partial \Phi_{I'}}{\partial u_{j'}} & \dfrac{\partial \Phi_{I'}}{\partial u_{j''}} & \dfrac{\partial \Phi_{I'}}{\partial u_{j'''}} \\ \dfrac{\partial \Phi_{I''}}{\partial u_{j'}} & \dfrac{\partial \Phi_{I''}}{\partial u_{j''}} & \dfrac{\partial \Phi_{I'''}}{\partial u_{j'''}} \end{pmatrix}$$

for $I' = 1, \dots, n - (N - M)$; $I'' = n - (N - M) + 1, \dots, M$;

$$j' = 1, \dots, N - M; \quad j'' = N - M + 1, \dots, n; \quad j''' = n + 1, \dots, N$$

and we have

$$\det\left(\frac{\partial \Phi_{I'}}{\partial u_{j''}}\right) \neq 0; \ \det\left(\frac{\partial \Phi_{I''}}{\partial u_{j'''}}\right) \neq 0; \tag{25}$$

$$\det \begin{pmatrix} \dfrac{\partial \Phi_{I'}}{\partial u_{j''}} & \dfrac{\partial \Phi_{I''}}{\partial u_{j'''}} \\ \dfrac{\partial \Phi_{I''}}{\partial u_{j''}} & \dfrac{\partial \Phi_{I''}}{\partial u_{j'''}} \end{pmatrix} \neq 0.$$

From $(25)_3$ we see that we can take $q_1 = u_1, \dots, q_{N-M} = u_{N-M}$ and the system (17) in the independent variables $u_1, \dots, u_{N-M}, \psi_1, \dots, \psi_{n-(N-M)}$ is hyperbolic. Let us consider the invertible change of variables, expressing the above ones in terms of $u_1, \dots, u_{N-M}, u_{N-M+1}, \dots, u_n$, by means of the transformation law

$$\Phi_{I'}(\underline{u}) = \psi_{I'} \text{ for } I' = 1, 2, \dots, n - (N - M). \tag{26}$$

In other words, we are obtaining $u_{N-M+1}, \dots, u_n$ as functions of the previous variables, from eq. (26).
Obviously, the system (17), expressed in terms of the new variables, is still hyperbolic. By using eq. (26) it becomes

$$\sum_{i=1}^{n} \sum_{j=1}^{N} Z_{ki} A^{\alpha}_{ij} \partial_\alpha u_j + \sum_{R=1}^{n-(N-M)} \omega^\alpha T^R_k \partial_\alpha \Phi_R(\underline{u}) = \sum_{i=1}^{n} Z_{ki} f_i. \tag{27}$$

This system in the independent variables $u_1, \ldots, u_n$, is hyperbolic. The variables $u_{n+1}, \ldots, u_N$ can be obtained from $\Phi_{I''}(\underline{u}) = 0$ as it can be seen from $(25)_2$. Consequently, the system (27) is hyperbolic in the variables $u_1, \ldots, u_n$ constrained only by

$$\Phi_{I''}(\underline{u}) = 0 \quad \text{for} \quad I'' = n - (N - M) + 1, \ldots, M. \tag{28}$$

If we impose that $\Phi_I, (\underline{u}) = 0$ (for $I' = 1, \ldots, n - (N - M)$) holds on the initial manifold, it will be satisfied also off it.
The new system (27) has the advantage to have all evolutive equations, even if the variables $\underline{u}$ are still constrained. If we want to eliminate also this drawback, we can proceed in the following way:
Let us consider the system (1) and $N - n$ other scalar-valued equations of the type $\sum_{j=1}^{N} 0_j^\alpha \partial_\alpha u_j = 0$. We obtain a new system which finally leads to the case a.
Let us see now some examples of physical applications.

1) *Relativistic magnetofluiddynamics.*

We have already obtained the system (22) to describe this physical problem; in this system ψ is an auxiliary quantity, and the variables are constrained by $u_\alpha b^\alpha = 0$. We see now, by applying the methodology in case a, that the system (22) can be substituted by

$$\begin{cases} \partial_\alpha(nu^\alpha) = 0 \\ \partial_\alpha[(e + p + b^2)u^\alpha u^\lambda + (p + b^2/2)g^{\alpha\lambda} - b^\alpha b^\lambda] + \\ \qquad + u^\gamma b^\lambda \partial_\alpha(u^\alpha b_\gamma - b^\alpha u_\gamma) = 0 \\ h^{\beta\lambda}\partial_\alpha(u^\alpha b_\beta - b^\alpha u_\beta) + u^\lambda u^\alpha \partial_\alpha(u_\gamma b^\gamma) = 0, \end{cases} \tag{29}$$

in the variables e, V^α, b^α which are now not constrained.

2) *Relativistic fluiddynamics.*

This physical problem is described by the equations

$$\begin{cases} \partial_\alpha(nu^\alpha) = 0 \\ \partial_\alpha[(e + p)u^\alpha u^\lambda + pg^{\alpha\lambda}] = 0 \end{cases} \tag{30}$$

which can be obtained by substituting $b^\alpha = 0$ in eqs. (21).
If we consider e, V^α as variables, this system is not constrained; moreover, it is hyperbolic according to Definition 1. However, to obtain an example of application of the present methodology, we may take e, n, u^α as variables constrained by

$$u^\alpha u_\alpha + 1 = 0. \tag{31}$$

In this way the number of equations is $n = 5$; moreover, $N = 6$, $M = 1$. Let $t^\alpha = u^\alpha$. The matrix X_{ki} is identically null; consequently, Y_{ki} and Z_{ki} are the identity matrix.
Therefore, we are in the case b; the system corresponding to (27) is exactly the same initial system (30). However, if we consider the system (30). However, if we cosider the system (30) and the equation $0^\alpha \partial_\alpha e = 0$, we have $n = 6$, $N = 6$, $M = 1$. The matrices X_{ki}, Y_{ki}, Z_{ki}, T_k^1 are respectively

$$X_{ki} = \begin{pmatrix} 0 & 0^\lambda & 0 \\ 0^\mu & 0^{\mu\lambda} & 0 \\ 0 & 0^\lambda & 1 \end{pmatrix}; Y_{ki} = \begin{pmatrix} 1 & 0^\lambda & 0 \\ 0^\mu & g^{\mu\lambda} & 0 \\ 0 & 0^\lambda & 0 \end{pmatrix} = Z_{ki}; T_k^1 = \begin{pmatrix} 0 \\ 0^\lambda \\ 1 \end{pmatrix}.$$

Therefore the system (17) is given by eqs. (30) and $u^\alpha \partial_\alpha \psi_1 = 0$, where $\omega^\alpha = u^\alpha$ has been taken. Finally, the system (24) is given by

$$\text{(32)} \qquad \begin{cases} \partial_\alpha (n u^\alpha) = 0 \\ \partial_\alpha [(e + p) u^\alpha u^\lambda + p g^{\alpha\lambda}] = 0 \\ u^\alpha \partial_\alpha (u_\gamma u^\gamma) = 0 \end{cases}$$

in the variables e, n, u^α which are not constrained.

3) *Covariant Maxwell electrodynamics.*

The field equations are (see e.g. [10])

$$\text{(33)} \qquad \partial_\alpha F^{\alpha\beta} = -j^\beta; \ \partial_\alpha F^{\alpha\beta^*} = 0$$

where

$$F^{\alpha\beta} = t^\alpha E^\beta - t^\beta E^\alpha + \eta^{\alpha\beta\gamma\delta} H_\gamma t_\delta,$$

$$F^{\alpha\beta^*} = t^\alpha H^\beta - t^\beta H^\alpha - \eta^{\alpha\beta\gamma\delta} E_\gamma t_\delta.$$

The variables are E^α (relative electric field) and H^α (relative magnetic field), constrained by

$$\text{(34)} \qquad \Phi_1 = t^\alpha E_\alpha = 0; \ \Phi_2 = t^\alpha H_\alpha = 0.$$

Moreover t^α is a field-independent time-direction, such that $t^\alpha t_\alpha = -1$, and $\eta^{\alpha\beta\gamma\delta}$ is the four-dimensional Levi-Civita's symbol. Therefore we have $n = 8$, $N = 8$, $M = 2$,

$$U_{jj'} = \begin{pmatrix} h^\mu_\gamma & 0 \\ 0 & h^\mu_\gamma \end{pmatrix}, X_{i'i} = \begin{pmatrix} t^\gamma & 0^\gamma \\ 0^{*\gamma} & 0^{*\gamma} \\ 0^\gamma & t^\gamma \end{pmatrix},$$

$$Y_{ki} = \begin{pmatrix} h^{\nu}_{\lambda} & 0 \\ 0 & h^{\nu}_{\lambda} \end{pmatrix} = Z_{ki}, T^1_k = \begin{pmatrix} -t^{\lambda} \\ 0^{\lambda} \end{pmatrix}, T^2_k = \begin{pmatrix} 0^{\lambda} \\ -t^{\lambda} \end{pmatrix},$$

where $0^{*\gamma}$ denotes a 6×4 matrix with null elements.
Consequently the system (17), in this case, is

$$\begin{cases} t^{\alpha} h^{\gamma\lambda} \partial_{\alpha} E_{\gamma} + \eta^{\alpha\lambda\gamma\delta} t_{\delta} \partial_{\alpha} H_{\gamma} - t^{\lambda} t^{\alpha} \partial_{\alpha} \psi_1 = -h^{\lambda\gamma} j_{\gamma} \\ -\eta^{\alpha\lambda\gamma\delta} t_{\delta} \partial_{\alpha} E_{\gamma} + t^{\alpha} h^{\gamma\lambda} \partial_{\alpha} H_{\gamma} - t^{\lambda} t^{\alpha} \partial_{\alpha} \psi_2 = 0. \end{cases}$$

The system (24) is

$$\text{(35)} \qquad \begin{cases} t^{\alpha} h^{\gamma\lambda} \partial_{\alpha} E_{\gamma} + \eta^{\alpha\lambda\gamma\delta} t_{\delta} \partial_{\alpha} H_{\gamma} - t^{\lambda} t^{\alpha} \partial_{\alpha} (t_{\gamma} E^{\gamma}) = -h^{\lambda\gamma} j_{\gamma} \\ -\eta^{\alpha\lambda\gamma\delta} t_{\delta} \partial_{\alpha} E_{\gamma} + t^{\alpha} h^{\gamma\lambda} \partial_{\alpha} H_{\gamma} - t^{\lambda} t^{\alpha} \partial_{\alpha} (t_{\gamma} H^{\gamma}) = 0. \end{cases}$$

in the variables E^{γ}, H^{γ} which are not constrained.
We notice that (35) can be written also as

$$\text{(36)} \qquad \partial_{\alpha}(F^{\alpha\lambda} + E^{\alpha} t^{\lambda}) = -h^{\lambda\gamma} j_{\gamma}; \quad \partial_{\alpha}(F^{\alpha\lambda^*} + H^{\alpha} t^{\lambda}) = 0,$$

where also the conservative form is preserved.

5. Hyperbolicity in every time-like direction.

The problem of characterizing the system, which are hyperbolic in every time-like direction, is still open. Strumia has obtained a very interesting result, in the case of symmetric hyperbolic systems (see appendix of ref. [11]). The general case remains still to be investigated. Here I rest content of the following result. Let us choose a particular t_{α} and find the matrix Z_{ki} depending on this t_{α}. Then the condition 2) of defintion 2 holds for every other time-direction t^*_{α} if the following two condition are satisfied

1) the system (10) is hyperbolic in the time-direction t_{α};
2) the characteristic velocities of system (10) (i.e., the solutions λ of condition 3) in Definition 2) do not exceed the speed of light.

In fact, let us assume, by absurd, that a four-vector t^*_{α} exists, such that $t^*_{\alpha} t^{*\alpha} = -1$ and the system

$$t^*_{\alpha} \sum_{i=1}^{n} \sum_{j=1}^{N} Z_{ki} A^{\alpha}_{ij} \delta u_j = 0, \quad \sum_{j=1}^{N} \frac{\partial \Phi_I}{\partial u_j} \delta u_j = 0,$$

in the independent unknowns δu_j, has a non null solution δu^*_j.

Let n^α be defined by $t^*_\alpha = -t^\gamma t^*_\gamma t_\alpha + n_\alpha$.
We have $t_\alpha n^\alpha = 0$ and $n_\alpha n^\alpha \geq 0$, if $n_\alpha n^\alpha = 0$ it would follow that $n^\alpha = 0$, and condition 2) of Definition 2 would be violated; consequently we have $n_\alpha n^\alpha > 0$.
Let us define $\zeta_\alpha = n_\alpha (n_\gamma n^\gamma)^{-1/2}$.
From condition 2) we have that the eigenvalues λ of the problem

$$(\zeta_\alpha - \lambda t_\alpha) \sum_{i=1}^{n} \sum_{j=1}^{N} Z_{ki} A^\alpha_{ij} \delta(u_j = 0, \ \sum_{j=1}^{N} \frac{\partial \Phi_I}{\partial u_j} \delta u_j = 0,$$

are such that $|\lambda| \leq 1$.
Now this problem is equivalent to

$$[n_\alpha - \lambda t_\alpha (n_\gamma n^\gamma)^{1/2}] \sum_{i=1}^{n} \sum_{j=1}^{N} Z_{ki} A^\alpha_{ij} \delta u_j = 0, \sum_{j=1}^{N} \frac{\partial \Phi_I}{\partial u_j} \delta u_j = 0,$$

which has the solution $\lambda = t^\gamma t^*_\gamma (n_\delta n^\delta)^{-1/2}$, $\delta u_j = \delta u^*_j$.
(See what we assumed by absurd). Therefore we have
$|t^\gamma t^*_\gamma| (n_\delta n^\delta)^{-1/2} \leq 1$, from which $(t^\gamma t^*_\gamma)^2 \leq -1 + (t^\delta t^*_\gamma)^2$!!
This absurd result proves our statement.
I conclude this section noticing that the characteristic velocities of system (17) are those of system (10) and $\lambda' = (n^\alpha \omega_\alpha)/(t^\alpha \omega_\alpha)$. (See in section III). The first ones, of these, do not exceed the speed of light. Regarding λ', we have that $(\lambda')^2 \leq 1$ holds iff $(n^\alpha \omega_\alpha)^2 \leq (t^\alpha \omega_\alpha)^2$ for every n^α such that $n^\alpha n_\alpha = 1$, $n^\alpha t_\alpha = 0$. This relation can be written in the references frames where t_α, ω_α have the components $t_\alpha \equiv (1, 0, 0, 0), \omega_\alpha \equiv (\omega_0, \omega_1, 0, 0)$; it reads $(n_1 \omega^1)^2 \leq (\omega^0)^2$ for every n_i such that $n_i n^i = 1$. Now $(n_1 \omega^1)^2$ assumes its maximum value for $n_1 = 1, n_2 = n_3 = 0$; this maximum is $(\omega^1)^2$. Therefore we must have $(\omega^1)^2 \leq (\omega^0)^2$, i.e., $\omega^\alpha \omega_\alpha \leq 0$.
Consequently ω^α must be chosen as a time-like or a light-like 4-vector. If we choose $\omega^\alpha = t^\alpha$, this condition is surely satisfied.

Appendix 1.

I prove here that the matrix $t_\alpha \sum_{j=1}^{N} A^\alpha_{ij} U_{jj'}$, has rank $N - M$. Let δQ_j be a solution of the system

$$t_\alpha \sum_{j',j=1}^{N} A^\alpha_{ij} U_{jj'} \delta Q_{j'} = 0. \tag{A.1}$$

If δP_j is defined by $\delta P_j = \sum_{j'=1}^{N} U_{jj'}\delta Q_{j'}$, we have that $t_\alpha \sum_{j=1}^{N} A^{\alpha}_{ij}\delta P_j = 0$, $\sum_{j=1}^{N} \frac{\partial \Phi_I}{\partial u_j}\delta P_j = 0$, from which $t_\alpha \sum_{i=1}^{n}\sum_{j=1}^{N} Z_{ki} A^{\alpha}_{ij}\delta P_j = 0$, $\sum_{j=1}^{N} \frac{\partial \Phi_I}{\partial u_j}\delta P_j = 0$; from condition 2) of Definition 2, it follows that $\delta P_j = 0$, i. e.,

$$\sum_{j'=1}^{N} U_{jj'}\delta Q_{j'} = 0. \tag{A.2}$$

Vice versa if δQ_j, is a solution of system (A.2), then it satisfies also (A.1). Therefore the systems (A.1) and (A.2) have the same solutions; to this end it is necessary that the matrices of their coefficients have the same rank ρ (We have $N - \rho$ free unknowns). Consequently, $t_\alpha \sum_{j=1}^{N} A^{\alpha}_{ij} U_{jj'}$ and $U_{jj'}$ have the same rank, i.e., $N - M$.

Appendix 2.

I prove now that the $2n \times n$ matrix $\begin{pmatrix} Y_{i'i} \\ X_{i'i} \end{pmatrix}$ has rank n. Let C_{hi} be $N - M$ l.i. solutions of $\sum_{i=1}^{n} X_{i'i} C_{hi} = 0$; these solutions can be chosen also orthonormal. From system (9) it follows that

$$Y_{ki} = \sum_{h=1}^{N-M} \lambda_{kh} C_{hi}. \tag{A.3}$$

If λ_{k1} is a linear combination of $\lambda_{k2}, \ldots, \lambda_{k(N-M)}$, i.e., $\lambda_{k1} = \sum_{p=2}^{N-M} q_p \lambda_{kp}$, then (A.3) gives $Y_{ki} = \sum_{p=2}^{N-M} \lambda_{kp}(C_{pi} + q_p C_{1i})$, i.e., Y_{ki} are linear combinations of $N - M - 1$ vectors; this is not possible because Y_{ki} has rank $N - M$. Similarly no other λ_{kh} is a linear combination of the remaining ones. In other words, the matrix L with elements λ_{kh} has rank $N - M$.
Let us consider now the system

$$\sum_{i=1}^{n} Y_{i'i} x_i = 0, \ \sum_{i=1}^{n} X_{i'i} x_i = 0, \tag{A.4}$$

and prove that it has only the solution $x_i = 0$. In this way our aim will be achieved. From $(A.4)_2$ we have $x_i = \sum_{h=1}^{N-M} u_h C_{hi}$; by substituting this and (A.3) in $(A.4)_1$ we obtain

$$\sum_{h,r=1}^{N-M} \lambda_{i'h} C_{hi} \mu_r C_{ri} = 0, \quad \text{i.e.,} \quad LCC^T \underline{u} = 0.$$

This system can be written as

$$L\underline{u} = 0 \tag{A.5}$$

because $CC^T = I_{N-M}$.
Now L is a $nX(N-M)$ matrix with rank $N-M$; therefore we have $\underline{u} = 0$ from (A.5), i.e., $x_i = 0$.

Appendix 3.

Let us prove the following theorem
Let us consider a $n \times n$ matrix A and a $n \times p$ matrix B; moreover let AB have rank p. It follows that A has rank $\rho \geq p$.
In fact the system $A^T \underline{q} = 0$ has $n - \rho$ 1.i. solutions $\underline{q}_1, \ldots, \underline{q}_{n-\rho}$. Let Q be an invertible $n \times n$ matrix whose first $n-\rho$ rows are $\underline{q}_1^T, \ldots, \underline{q}_{n-\rho}^T$. It follows that QAB has rank p and its first $n-\rho$ rows have null elements; the remaining rows are ρ in number and, consequently, $p \leq \rho$. By applying this theorem with $A = \sum_{i'=1}^{n} \lambda_{ki'} Y_{i'i}$ and $B = t_\alpha \sum_{i=1}^{n} \sum_{j=1}^{N} A_{ij}^{\alpha} \frac{\partial u_j}{\partial q_h}$, we have that the rank ρ of $\sum_{i'=1}^{n} \lambda_{ki'} Y_{i'i}$ is such that $\rho \geq N-M$. Now the rank of $Y_{i'i}$ is $N-M$, from which $\rho \leq N-M$. Finally, we have $\rho = N-M$, a property which I used in section II.

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