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ON TOTAL FUNCTIONAL STABILITY WITH TWO MEASURES

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A concept of total functional stability in terms of two different measures is given.
The theoretical results are obtained by mean of an extension of Liapunov's direct method.

Introduction.

In this paper we consider the problem of total stability, or "stability under persistent perturbations", for general differential systems defined on open subsets of $R^+ \times R^n$.
The total stability problem of the zero solution of a differential equation was first examined, in a general formulation, in a paper by Dubosin [1] and in subsequent works by Malkin [8],[9] and Gorsin [3].
Movchan's theory of stability in terms of two metrics [12] appeared on mathematical scene in 1960.
From this we obtain, for example, Liapunov's stability [15], Rumiantsev's partial stability [16] and [18], the stability of sets by Peiffer and Rouche [14].
In the present work we deal with a new method (founded on auxiliary functions measure depending) for characterization of total stability in terms of two different measures, it is developed by introducing the total functional stability and the total eventual functional stability, two notions of new formulation.

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As we shall see the ideas which constitute a starting point for theory construction lie in [4], [5], [7], [17], [19], [20] and [21], [22], [23], [24].

1. Preliminaries.

First, it is useful to remenber two classical definitions.

Definition 1. A function $a : R^+ \to R^+$ continuous, strictly increasing, so that $a(0) = 0$ is said to be a class K function ([4]).

Definition 2. A function $h : R^+ \times R^n \to R^+$ continuous s.t. $\inf h(t, y) = 0$ for each $t \in R^+$ is said to be a class H function ([6]).

Assumption A. Given $h_0, h \in H$ with $sup\, h = l > 0$ suppose that $\exists \lambda > 0$, $\exists m \in K$ s.t. $h_0(t, y) < \lambda$ implies $h(t, y) \leq m[h_o(t, y)] < m(\lambda)$ ([7]).

Notations. Let $p = \min[m(\lambda), l]$, $\eta \in]0, p]$ and $\eta' \in]0, \eta[$ we denote $Q_\eta = \{(t, y) \in R^+ \times R^n : 0 < h(t, y) \leq \eta\}$, $Q = Q_p$ and $Q_\eta \times \eta = Q_\eta \times]0, \eta]$, $F_r A =$ the boundary of set A, A'=closure of A, $R'^+ =]0, \infty[$.
Consider the differential equation with initial condition

$$\dot{y} = Y(t, y), Y \in C^{(0)}(Q' \to R^n), y(t_0) = y_0 \tag{1.1}$$

and their perturbative

$$\dot{y} = Y(t, y) + P(t, y) = Z(t, y), P \in C^0(Q' \to R^n). \tag{1.2}$$

Because $Y, P \in C^0$ then $\forall (t_0, y_0)$ the relative Cauchy problem admit continuous solutions $y'(t, t_0, y_0)$ defined on $J'^+(t_0, y_0)$ for (1.1) and $y(t, t_0, y_0)$ defined on $J^+(t_0, y_0)$ for (1.2).
Let us list the following classes of functions for convenience:

$I(Q_\eta) = \{g = g(t, y) \in C^0(Q_\eta \to R'^+)\}$;
$I'(Q_\eta) = \{e = e(t.y) \in C^0(Q_\eta \to R^+), \sup e > 0\}$;
$I''(Q_\eta) = \{Y(t, y) \in C^0(Q_\eta \to R^n)\}$;
$L(Q_\eta \times \eta) = \{V(t, y, h) : Q_\eta \times \eta \to R,$ continuous $\}$;
$L'(Q_\eta \times \eta) = \{V(t, y, h) \in L(Q_\eta \times \eta) : \lim_{h\to 0^+} V = 0\}$;
$L''(Q_\eta \times \eta) = \{V(t, y, h) \in L(Q_\eta \times \eta) : V \in C^1\}$.

Let $\mathcal{F}$ be a family of positive continuous functions that depend upon the parameter ϵ, $\mathcal{F} = \{f_\epsilon = f_\epsilon(t, y) > 0, (t, y) \in Q_\epsilon, \epsilon > 0\}$.

Definition 3. The system (1.1) is said to be *functionally* (h_0, h)*-totally stable* if $(\forall \epsilon > 0)(\forall t_0 \in R^+)(\exists \delta(\varepsilon, t_0) > 0)(\forall y_0 : (t_0, y_0) \in Q_\epsilon, h_0(t_0, y_0) < \delta)(\forall P : ||P|| \leq f_\epsilon \text{ on } Q_\epsilon)\ (h(t, y(t, t_0, y_0)) < \epsilon\ \forall t \geq t_0)$.

Definition 4. If $\delta = \delta(\epsilon)$ we have the "*uniformity*".

Remark. If $f_\epsilon = const$ we obtain the (h_0, h)*-total stability* ([7], [24]).

Let $g = g(t, y) \in I(Q)$ be a continuous function corresponding upon the function Y so that for every $||P|| \leq g$ we have the following definition

Definition 5. The system (1.1) is said to be *uniformly* (h_0, h)*-totally eventually stable* if $(\forall \epsilon > 0)(\exists T > 0)(\exists \delta > 0)(\forall t_0 \geq T)(\forall y_0 : (t_0, y_0) \in Q_\epsilon, h_0(t_0, y_0) < \delta)$ then $(h(t, y(t, t_0, y_0)) < \epsilon \quad \forall t \geq t_0)$.

Definition 6. The system (1.1) is said to be (h_0, h)*-totally functionally stable* if $(\forall \epsilon > 0)(\exists \delta > 0)(\forall (t_0, y_0) \in Q_\epsilon : h_0(t_0, y_0) < \delta)(\exists \sigma > 0)\ (\exists f_\sigma \in I(Q))(\forall P : ||P|| \leq f_\sigma)(h(t, y(t, t_0, y_0)) < \epsilon\ \forall t \geq t_0)$.

Let $\varphi(t, y) \in I'(Q_\eta)$ and $E_t(\varphi = 0) = \{y \in Q : \varphi(t, y) = 0, \}$.

Definition 7. A function $V(t, y, h(t, y)) : Q \to R$ is said to be g*-definitively* $\neq 0$ on the set $E_t(\varphi = 0)$ if $(\exists g \in I(Q))(\forall \eta \in]0, p])(\exists \beta_\eta = \beta_\eta(t, y), \gamma_\eta = \gamma_\eta(t, y) \in I(Q_\eta))(\forall (t, y) \in Q_\eta : |\varphi(t, y)| < \gamma_\eta)\ (|V(t, y, h(t, y))| \geq \beta_\eta g)$.

Definition 8. We say that $V\dot{V} \geq 0$ on Q if $\forall \eta \in]0, p]$ we have $A^3_\eta = \{(t, y) \in Q_\eta : \varphi < \gamma_\eta,\ V\dot{V} < 0\} = \phi$.

2. Theoretical results and applications.

Theorem 1. *Suppose that*

1) three functions $V \in L'(Q \times p)$ *and* $a, b \in K$ *corresponding to* Y *exist;*

2) for every $\epsilon > 0$ *there exist three functions* $V_\epsilon \in L''(Q_\epsilon \times \epsilon)$, $f_\epsilon(t, y) \in F$ *and* $M_\epsilon \in \{I(Q_\epsilon) : \inf M_\epsilon > 0\}$*;*

3) on the set $Q_\epsilon \times \epsilon$ *the following conditions hold*

$a_1)$ $V_\epsilon(t, y, h) \leq V(t, y, h)$*;*

$b_1)$ $V_\epsilon(t, y, \epsilon) \geq a(\epsilon)$*;*

$c_1)$ $||\dfrac{\partial V_\epsilon}{\partial y}||;\ ||\dfrac{\partial v_\epsilon}{\partial h}\dfrac{\partial h}{\partial y}|| < M_\epsilon(t, y)$*;*

$d_1)$ $\dot{V}_\epsilon(t, y, h) = \dot{V}_\epsilon^{(1.1)} \leq -f_\epsilon(t, y)$.

Then the system (1.1) is functionally (h_0, h)*-totally stable; if* $V(t, y, h) \leq b(h)$ *we also obtain the "uniformity".*

Proof. Given $\epsilon > 0$ and $t_0 \in R^+$ by the Malkin generalized formula [24], from c_1) and d_1) we deduce

$$\dot{V}_\epsilon^{(1.2)} = \dot{V}_\epsilon^{(1.1)} + \frac{\partial V_\epsilon}{\partial y}P + \frac{\partial V_\epsilon}{\partial h}\frac{\partial h}{\partial y}P \leq -f_\epsilon + 2M_\epsilon ||P||. \tag{2.1}$$

If $||P|| \leq \frac{f_\epsilon}{2M_\epsilon}$ on Q_ϵ we obtain $\dot{V}_\epsilon^{(2,1)} \leq 0$.
Because $V \in L'(Q \times p)\, \exists \delta' \in]0, \epsilon[$ so that $h(t_0, y) < \delta' \rightarrow V(t_0, y, h(t_0, y)) < a(\epsilon)$.
Let $(t_0, y_0) \in Q_\epsilon$ s.t. $h_0(t_0, y_0) < \delta = \min[\lambda, m^{-1}(\delta')]$ consequently $h(t_o, y_0) \leq m(h_0(t_0, y_o)) < \delta'$, hence, by a_1), we deduce

$$V_\epsilon(t_0, y_0, h(t_0, y_0)) \leq V(t_0, y_0, h(t_0, y_0)) < a(\epsilon). \tag{2.2}$$

Let $y(t) = y(t, t_0, y_0))$ a solution of (1.2), suppose that $\exists t' \in J^+(t_0, y_0)$ so that $h(t', y(t')) = \epsilon$ with $h(t, y(t)) < \epsilon \forall t \in [t_0, t'[$ then, by b_1) we have

$$V_\epsilon(t', y(t'), h(t', y(t'))) \geq a(\epsilon). \tag{2.3}$$

Therefore $\exists t'' \in]t_0, t'[$ s.t. $\dot{V}_\epsilon(t'', y(t''), h(t'', y(t'')) > 0$ it is a contradiction.
If $V(t, y, h) \leq b(h)$, fixed $\epsilon > 0$ we obtain $b(h) < a(\epsilon) \rightarrow h < b^{-1}(a(\epsilon))$.
Let $h_0(t_0, y_0) < \delta = \min[\lambda, m^{-1}(b^{-1}(a(\epsilon)))] \rightarrow h(t_0, y_0) \leq m[h_0(t_0, y_0] < b^{-1}(a(\epsilon))$ then

$$V(t_0, y(t_0, h(t_0, y(t_0))) < b[b^{-1}(a(\epsilon))] = a(\epsilon). \tag{2.4}$$

When $h(t_0, y_0) = 0$ the proof is trivial.

Theorem 2. *Assume that*

1) seven functions $g \in I(Q), \varphi \in I'(Q), V \in L''(Q \times p), W \in L'(Q \times p) \cap L''(Q \times p), a, b \in K, h \in H \cap C^1$ *exist so that*
a_2) $W(t, y, h) \geq a(h)$;
b_2) $\varphi(t, y) = 0 \Longrightarrow \dot{W} = \dot{W}(t, y, h(t, y)) \leq 0$;
c_2) $\forall \eta > 0, \forall \sigma \in I(Q_\eta)\, \exists \nu \in I(Q_\eta)(\inf \nu > 0), :\varphi \geq \sigma \Rightarrow \dot{W}(t, y, h(t, y)) \leq -(\nu g)(t, y)$;
d_2) $\dot{V}$ *is g-definitively* $\neq 0$ *and* $V\dot{V} \geq 0$ *on the set* $E_t(\varphi = 0)$;
e_2) $\dot{h}(t, y) \leq 0$;

2) two constants $L, M > 0$ *exist s.t. we have on the set* $Q \times p$.
f_2) $|V|, ||VY|| < Lg$;
g_2) $|\frac{\partial \varphi}{\partial t}|, ||\frac{\partial \varphi}{\partial y}||, ||\frac{\partial h}{\partial y}|| < M$;

$h_2)$ $||\frac{\partial V}{\partial y}||,\ |\frac{\partial V}{\partial h}| < Mg;$

$i_2)$ $||\frac{\partial W}{\partial y}||,\ |\frac{\partial W}{\partial h}| < Mg;$

Then, the system (1.1) *is functionally* (h_0, h)*-totally stable.*

Proof. From hypothesis $d_2)$, $\forall \eta \in\ ]0, p]$ two continuos functions $\beta_\eta, \gamma_\eta \in I(Q_\eta)$ exist and two sets

$$A^1{}_\eta(t, y) = \{(t, y) \in Q_\eta : \varphi < \gamma_\eta \rightarrow \dot{V} \leq -\beta_\eta g. \quad V \leq 0\},$$

$$A^2{}_\eta(t, y) = \{(t, y) \in Q_\eta : \varphi < \gamma_\eta \rightarrow \dot{V} \geq \beta_\eta g, \quad V \geq 0\}$$

with $A^1{}_\eta \cap A^2{}_\eta = \phi$.
Let $i = 1, 2$, denote by $n_i (\geq 3)$ the first integer s.t. one at least of the sets $B^i{}_\eta(t, y) = \{(t, y) \in A^i{}_\eta : \frac{n_i-3}{n_i}\gamma_\eta \leq \varphi(t, y) \leq \frac{n_i-2}{n_i}\gamma_\eta\}$ is non empty.
We set

$$C^i{}_\eta = \{(t, y) \in A^i{}_\eta : \frac{n_i - 3}{n_i}\gamma_\eta \leq \varphi(t, y) \leq \frac{n_i - 1}{n_i}\gamma_\eta\} \quad (\supset B^i{}_\eta),$$

$$T_\tau = \{(t, y) \in R^+ \times R^n : t^2 + ||y||^2 < \tau\}, \tau > 0,$$

$$\rho = \inf_{i=1,2}\{D(FrQ_\eta, FrA^i{}_\eta), D(FrA^i{}_\eta, FrC^i{}_\eta), D(FrC^i{}_\eta, FrB^i{}_\eta)\}$$

where $D(A, B)$=euclidean distance. Denote

$$A^i{}_{\eta\tau} = A^i{}_\eta \cap T_\tau, B^i{}_{\eta\tau} = B^i{}_\eta \cap T_\tau, C^i{}_{\eta\tau} = C^i{}_\eta \cap T_\tau, Q_{\eta\tau} = Q_\eta \cap T_\tau.$$

Consider the functions $\psi^i_{\eta\tau}(t, y) = 1\ \forall (t, y) : D[(t, y), B^i_{\eta\tau}] \leq \rho$; $\psi^i_{\eta\tau}(t, y) = 0\ \forall (t, y) : D[(t, y), B^i_{\eta\tau}] > \rho$ and the averaging functions [11]

$$\alpha^i{}_{\eta\tau}(z) = \int_{R\times R^n} \psi^i{}_{\eta\tau}(u) S_r(z - u)\, du \tag{2.5}$$

where $u, z \in R \times R^n,\ r = \frac{1}{2}\rho$.
We have the following properties [by (2.5)]

$j)$ $\alpha^i{}_{\eta\tau}(t, y) \in C^\infty$;

$2j)$ $0 \leq \alpha^i{}_{\eta\tau}(t, y) \leq 1$;

$3j)$ $|\frac{\partial \alpha^i{}_{\eta\tau}(t, y)}{\partial t}|,\ ||\frac{\partial \alpha^i{}_{\eta\tau}(t, y)}{\partial y}|| \leq M_\eta$ $(> 0$, constant);

$4j)$ $\alpha^i{}_{\eta\tau}(t, y) = 1$ when $(t, y) \in B^i{}_{\eta\tau}$, $\alpha^i{}_{\eta\tau}(t, y) = 0\ \forall (t, y) \notin A^i{}_{\eta\tau}$.

Since $\eta \leq p$ then $Q_\eta \subseteq Q$,we put, on Q_η

(2.6) $H_{\eta\tau}(t, y, h(t, y)) =$

$$= \Sigma_{(i=1,2)}(-1)^{i+1}\alpha^i{}_{\eta\tau}(t, y)V(t, y, h(t, y))(\eta - h(t, y)).$$

We have

$r)$ $H_{\eta\tau}(t, y, h) \leq 0$;
$2r)$ $H_{\eta\tau}(t, y, \eta) = 0$;
$3r)$ $H_{\eta\tau} = 0 \; \forall (t, y) \notin A^i{}_{\eta\tau}$;
$4r)$ $\dot{H}_{\eta\tau} \leq \Sigma_{(i=1,2)}(-1)^{i+1}(\eta - h)\dot{\alpha}^i_{\eta\tau}(t, y)V(t, y, h) \leq 2M_\eta \eta L g$.

We now define on $Q_{\eta\tau}$ the function ($\mu > 0$, $constant$)

(2.7) $$W^\mu{}_{\eta\tau}(t, y, h(t, y)) = W(t, y, h(t, y)) + \mu H_{\eta\tau}(t, y, h(t, y)).$$

By (2.7) we deduce

$s)$ $W^\mu{}_{\eta\tau}(t, y, h) \leq W(t, y, h)$;
$2s)$ $W^\mu{}_{\eta\tau}(t, y, \eta) = W(t, y, \eta) + \mu H_{\eta\tau}(t, y, \eta) \geq a(\eta)$;
$3s)$ $\dot{W}^\mu_{\eta\tau} = \dot{W} + \mu \dot{H}_{\eta\tau}$.

Let $n_i \geq 4$ $(i = 1, 2)$ then $\varphi \geq \frac{1}{4}\gamma_\eta$ and by $c_2)$ $\exists \nu \in I(Q_\eta)$ s.t. $\dot{W} \leq -\nu g$; this implies

(2.8) $$\dot{W}^\mu_{\eta\tau} \leq -\nu g + 2\mu M_\eta L g \eta;$$

if $\mu \leq \inf \frac{\nu}{4M_\eta L\eta}$ we have $\dot{W}^\mu_{\eta\tau} \leq -\frac{\nu g}{2}$ on $Q_{\eta\tau}$.
Let $n_1 = 3, n_2 = 4$ we deduce

(2.9) $$B^1{}_{\eta\tau} = \{(t, y) \in A^1{}_{\eta\tau} : 0 \leq \varphi(t, y) \leq \frac{\gamma_\eta}{3}\}, \quad \varphi > \frac{1}{4}\gamma_\eta \; \forall (t, y) \notin B^1{}_{\eta\tau}.$$

On the set $B^1{}_{\eta\tau}$ we have $\alpha^1{}_{\eta\tau} = 1$ hence $H_{\eta\tau} = (-1)^2(\eta - h)V = (\eta - h)V$ and

(2.10) $$\dot{H}_{\eta\tau} = -\dot{h}V + (\eta - h)\dot{V} \leq (\eta - h)\dot{V} \leq -\eta' \beta_\eta g$$

where $\eta' = (\eta - h)$ then $\dot{W}^\mu_{\eta\tau} \leq -\mu\eta'\beta_\eta g$.
Let $(t, y) \notin B^1{}_{\eta\tau}$ then $\exists \nu' \in I(Q_\eta)$ so that $\dot{W} \leq -\nu' g$, therefore $\dot{W}^\mu_{\eta\tau} \leq -\nu' g + 2\mu M_\eta L g \eta$ is true for $\mu \leq \inf \frac{\nu'}{4M_\eta L\eta} = \mu'$ we have $\dot{W}^\mu_{\eta\tau} \leq -\frac{\nu' g}{2}$.
Choose μ so that $-\frac{1}{2}\nu' g \leq -\mu\eta'\beta_\eta g$, it is verified when $\mu \leq inf \frac{1}{2}\frac{\nu'}{\eta'\beta_\eta} = \mu$”; then, if $\mu \leq min(\mu', \mu$”), we obtain $\dot{W}^\mu_{\eta\tau} \leq -\mu\eta'\beta_\eta g$. Similarly by elaborating the cases $n_1 \geq 4$, $n_2 \geq 3$; $n_1 = n_2 = 3$ we verify hypothesis $d_1)$. Since τ is arbitrary we obtain the proof.

Application of Theorem 2 to analytical mechanics.

The motion equations of a nonstationary mechanical system, with ([13]) variable masses $m_j(q,\dot q,t)$, under the action potential, gyroscopic and dissipative forces have the form ([13], [25])

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right)-\frac{\partial T}{\partial q_i}=f_i+g_{is}\dot q_s+\frac{\partial U}{\partial q_i}-\frac{\partial R}{\partial \dot q_i}\quad (i,s=1,\ldots,n). \tag{2.11}$$

Here $T=\frac{1}{2}a_{is}\dot q_i\dot q_s+a_i\dot q_i+T_0=T_2+T_1+T_0$ is the kinetic energy ([2]), $g_{is}=g_{is}(q,t)=-g_{si}(q,t)$ the gyroscopic coefficient, $R=R(\dot q)$ the dissipative function, $U=U(q)$ the forcing function, $f_i=f_i(q,\dot q)$ the generalized forces of masses variation. In this section the symbol Σ is deleted, $||\cdot||$ =euclidean norm.

Theorem 3. *Suppose that the following hypotheses hold on the set* $F=\{(t,q,\dot q): t\in R^+, ||q||, ||\dot q||<S\}$, $S>0$,

$a_3)$ a_i, a_{ij}, T_0, U *are continuous and bounded with their derivatives;*
$b_3)$ f_i, g_{is} *bounded and continuous with* $f_i(q,0)+\dfrac{\partial U}{\partial q_i}\equiv 0$;
$c_3)$ $\dfrac{\partial a_{is}}{\partial t}\dot q_i\dot q_j+\dfrac{\partial a_i}{\partial t}\dot q_i+\dfrac{\partial T_0}{\partial t}-f_i\dot q_i\equiv 0$, $\dfrac{\partial T_0}{\partial q_i}q_i=T_0$, $\dfrac{\partial a_{rs}}{\partial q_i}q_i=a_{rs}$, $\dfrac{\partial a_r}{\partial q_i}q_i=a_r$, $\dfrac{\partial R}{\partial \dot q_i}\dot q_i=mR$ $(m=const>0)$;
$d_3)$ $T_0+U\le 0$, $h(t,q,\dot q)=T_2-T_0-U$, $U(0)=0$, $g=m$, $h_0=a(h)$ *with* $a\in K$;
$e_3)$ $\varphi=R\ge b(||\dot q||)$, $b\in K$;
$f_3)$ $\forall\rho\in]0,S[\ \exists\rho'(t,q,\dot q)\in I(F)$ *s.t.* $||q||\ge\rho\Longrightarrow T_0(q)\ge m\rho'$.

Then the system (2.11) *is functionally* (h_0,h)*-totally stable.*

Proof. The explicit form of (2.11) is

$$\dot a_{is}\dot q_s+a_{is}\ddot q_s+\dot a_i-\frac{1}{2}\frac{\partial a_{rs}}{\partial q_i}\dot q_r\dot q_s-\frac{\partial a_s}{\partial q_i}\dot q_s-\frac{\partial T_0}{\partial q_i}=f_i+g_{is}\dot q_s+\frac{\partial U}{\partial q_i}-\frac{\partial R}{\partial \dot q_i}. \tag{2.12}$$

From $d_3)$ taking $W=T_2-T_0-U=h$ we have $a_2)$; by (2.12), $b_3)$ and $c_3)$ follows $\dot W\le -mR\le 0$, since $\varphi=R$, when $R=0$ we have $\dot W\le 0$; and hence $b_2)$, $e_2)$.
Let $\sigma>0$ and $\varphi>\sigma$ then $\dot W=-mR<-m\sigma$, put $g=m$, $\nu=\sigma$, we have $c_2)$.

Given $V = \frac{\partial T}{\partial \dot{q}_i} q_i$ we obtain, by (2.11) and f_3)

(2.13) $$\dot{V} = 3T_2 + 2T_1 + T_0 + f_i q_i + g_{ir} q_i \dot{q}_r + \frac{\partial U}{\partial q_i} q_i - \frac{\partial R}{\partial \dot{q}_i} q_i.$$

By b_3), b_3) we deduce that in the set $\varphi = 0$ we obtain $\dot{V} = T_0$ and $V\dot{V} = 0$ if we suppose $||q|| > \rho \in]0, S[\rightarrow T_0 > m\rho'$ i.e. d_2).

Theorem 4. *Suppose that six functions $M \in I(Q)$, $V \in L''(Q \times p)$, V_*, $\psi : R^+ \rightarrow R^+$, $a, b \in K$ with the following properties exist*

a_4) $a(h) \leq V(t, y, h) \leq b(h)$;
b_4) $V_*(t, y, h) \geq 0$;
c_4) $\dot{V}(t, y, h) \leq -V_*(t, y, h) + \psi(t)$;
d_4) $\int_0^{+\infty} \psi(\tau)\, d\tau < +\infty$;
e_4) $||\frac{\partial V}{\partial y}||$, $||\frac{\partial V}{\partial h} \frac{\partial h}{\partial y}|| \leq M(t, y)$ $(\inf M > 0)$.

Then, put $g = \dfrac{V_*}{2M}$ *the system* (1.1) *is uniformly* (h_0, h)*-totally eventually stable.*

Proof. Put

(2.14) $$U(t, y, h(t, y)) = V(t, y, h(t, y)) + \int_t^{+\infty} \psi(\tau)\, d\tau$$

we deduce, by c_4) $\dot{U} = \dot{V}(t, y, h) - \psi(t) \leq -V_*$.
From the generalized Malkin's formula

(2.15) $$\dot{U}^{(2.2)} = \dot{U}^{(1.2)} + \frac{\partial V}{\partial y} P + \frac{\partial V}{\partial h} \frac{\partial h}{\partial y} P \leq -V_* + 2M||P||.$$

Therefore $\|P\| \leq \frac{V_*}{2M}$ implies $\dot{U}^{(1.2)} \leq 0$ on Q.
For any $\epsilon \in]0, p[$ there exist a $T > 0$ so that $\int_{t_0}^{+\infty} \psi(\tau)\, d\tau < \frac{a(\epsilon)}{2}$ for $t_0 > T$.
Let's assume $\delta' = b^{-1}[\frac{a(\epsilon)}{2}]$, if we choose $(t_0, y_0) \in Q_\epsilon$ with $t_0 \geq T$ and $h_0(t_0, y_0) < \min(\lambda, m^{-1}(\delta'))$ then $h(t_0, y_0) < m(m^{-1}(\delta')) = \delta'$ and $b(h(t_0, y_0)) < b(\delta') = \frac{a(\epsilon)}{2}$.
Consider a solution $y(t) = y(t, t_0, y_0)$ of (1.2) we have

(2.16) $$a(h(t, y(t))) \leq U(t, y(t), h(t, y(t))) \leq b(h(t_0, y_0)) + \int_0^{+\infty} \psi(\tau) d\tau.$$

From (2.16) follows

$$a(h(t, y(t))) < a(\epsilon) \quad \Longrightarrow h(t, y(t)) < \epsilon \quad \forall t \geq t_0.$$

Application to differential system.

Let us consider the differential system defined on $\mathcal{D} = \{(t, x, y) : t \geq 0, x \in R, |y| \leq 1\}$

$$(2.17)\qquad \begin{cases} \dot{x} = [1 - \lambda(t, x, y)]y - \dfrac{r(t)}{p(t)}xy \\ \dot{y} = \dfrac{\dot{r}(t) - f(t, x, y)}{r(t)}y - \dfrac{p(t)q(x)}{r(t)}[1 - \lambda(t, x, y)] + xq(x)] \end{cases}$$

where $q \in C^0(R^+ \to R)$, $p, r \in C^1(R^+ \to R)$ and $\inf(|p|, |r|) > 0$.

Theorem 5. *Suppose that*

$a_5)$ $0 < \frac{r}{p}$, $\Delta(t) = 3\frac{\dot{r}}{r} - \frac{\dot{p}}{p} \geq 0$, $\int_0^{+\infty} (rp^{-1}\Delta)(u)du < +\infty$;
$b_5)$ $xq > 0$ *for* $x \neq 0$, $pf \geq 0$ *on* $\mathcal{D}$;
$c_5)$ $\exists$ *a constant* $L > 0$ *s.t.* $2\int_0^x q(u)\,du \leq Lx^2$, $\forall x \in R$, $\frac{r}{p} \leq L$;
$d_5)$ $h(x, y) = \int_0^x q(u)\,du + \dfrac{r}{p}y^2$, $h_0(x, y) = \sup\{|x|, |y|\}$;
$e_5)$ $g = fy^2\Lambda^{-\frac{1}{2}}$, *where* $\Lambda = p^2(q^2 + \frac{r^2}{p^2}y^2)$.

Then, the system (2.17) is uniformly (h_0, h)*-totally eventually stable.*

Proof. Let $V = h$, $V_* = 2\frac{f}{p}y^2$, $\psi = \frac{r}{p}\triangle$ we have $a_4)$, $b_4)$, $d_4)$ immediately; $c_4)$ is valid because

$$(2.18)\qquad \dot{V} = p^{-1}(r\triangle - 2f)y^2 \leq p^{-1}(r\triangle - 2fy^2).$$

From $c_5)$, $d_5)$ follows $h \leq 2Lh_0{}^2$ and by $\|(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y})\|^2 \leq 4(q^2 + \frac{r^2}{p^2})$ we deduce $e_4)$.

Theorem 6. *Suppose that,* $\forall P(t, y) \in I'(Q)$, *two functions* $V_P{}^* \in L(Q \times p)$ *and* $V_P \in L''(Q \times p)$ *exist so that*

$a_6)$ $a(h(t, y)) \leq V_P{}^*(t, y, h(t, y)) \leq b(h(t, y))$ *with* $a, b \in K$;
$b_6)$ $\forall \sigma > 0\ \exists f_\sigma \in I(Q)$ *so that when* $\|P\| \leq f_\sigma$ *the following conditions hold:*
 $z)$ $|V_P(t, y, h) - V_P{}^*(t, y, h)| < \sigma$,
 $2z)$ $\dot{V}_P{}^{(1.2)}(t, y, h) \leq 0$.

Then, the system (1.1) *is* (h_0, h)*-totally functionally stable.*

Proof. Given $\epsilon > 0$, from $b(h(t, y)) < \frac{1}{2}a(\epsilon)$ we obtain, by a_6), $h(t, y) < b^{-1}[\frac{1}{2}a(\epsilon)] = \epsilon^* < \epsilon$.
Therefore, given $t_0 \in R^+$ so that $(t_0, y_0) \in Q_\epsilon$, $h_0(t_0, y_0) < \min(\lambda, m^{-1}(\epsilon^*)) = \delta$ it follows

$$h(t_0, y_0) \leq m[h_0(t_0, y_0)] < m[m^{-1}(\epsilon^*)] = \epsilon^*, \tag{2.19}$$

$$V_P{}^*(t_0, y_0, h(t_0, y_0)) = e_1 \leq b(h(t_0, y_0)) < \frac{1}{2}a(\epsilon). \tag{2.20}$$

Let $y(t) = y(t, t_0, y_0)$ a solution of (1.2), if $\exists t' \in J^+(t_0, y_0)$ so that $h(t, y(t)) < \epsilon$ $\forall t \in [t_0, t'[$ and $h(t', y(t')) = \epsilon$, we obtain, by a_6)

$$V_P{}^*(t', y(t'), h(t', y(t'))) = e_2 \geq a(\epsilon) > e_1. \tag{2.21}$$

Let's assume $\sigma(t_0, \epsilon) = \frac{e_2 - e_1}{2}$ (> 0), if $||P|| < f_\sigma$ we obtain, from z)

$$|V_P(t_0, y_0, h(t_0, y_0)) - V_P{}^*(t_0, y_0, h(t_0, y_0))| < \sigma \Longrightarrow$$

$$\Longrightarrow V_P(t_0, y_0, h(t_0, y_0)) < \frac{e_1 + e_2}{2} \tag{2.22}$$

and

$$|V_P(t', y(t'), h(t', y(t'))) - V_P{}^*(t', y(t'), h(t', y(t')))| < \sigma \Longrightarrow$$

$$\Longrightarrow V_P(t', y(t'), h(t', y(t'))) > \frac{e_1 + e_2}{2}. \tag{2.23}$$

Because, from 2z) $V(t, y(t), h(t, y(t)))$ does not increase, this is a contradiction.

Application of Theorem 6.

Given two differential equations on $Q = \{(t, u) : t \geq 0,\ |u| < 1\}$

$$\dot{u} = -f(t, u), \text{ with } f(t, 0) = 0,\ \forall t \geq 0, \tag{2.24}$$

$$\dot{u} = -f(t, u) + P(t, u), \tag{2.25}$$

where $f, P \in C^0(Q \to R)$.

Theorem 7. *Suppose that:*

i) $\int_0^{u^2} |P(t,\tau)|\,d\tau \leq |P(t,u)|$;

2i) $\int_0^{u^2} P^2(t,\tau)\,d\tau \leq P^2(t,u)$;

3i) $a(|u|) \leq \int_0^{u^2} f(t,\tau)\,d\tau \leq b(|u|)$, $a, b \in K$.

Then $u = 0$ *solution of* (2.24) *is* (h_0, h)*-totally functionally stable, with* h *euclidean metrics and* $h_0 = a(h)$*,* $a \in K$ *with* $\lim_{r\to+\infty} a(r) = +\infty$.

Proof. Assume

$$V^*(t,u,P) = \int_0^{u^2} [f(t,\tau) + \text{ arctg } P^2(t,\tau)]\,d\tau,$$

$$V(t,u,P) = \int_0^{u^2} |f(t,\tau) - P(t,\tau)]\,d\tau$$

from arctg $P^2(t,\tau) \leq \frac{\pi}{2}$ we obtain

$$\int_0^{u^2} \text{arctg } P^2(t,\tau)\,d\tau \leq \int_0^{u^2} \frac{\pi}{2}\,d\tau = \frac{\pi}{2}u^2.$$

Furthermore, by 3i) we deduce

$$a(|u|) \leq \int_0^{u^2} [f(t,\tau) + \text{ arctg } P^2(t,\tau)]\,d\tau = \tag{2.26}$$

$$= V^* \leq b(|u|) + \frac{\pi}{2}u^2 = c(u) \in K.$$

From definition of V, V^*, denote $P' = \sup\{|P| \text{ on } Q\}$, we obtain

$$|V^* - V| = \left|\int_0^{u^2} [P(t,\tau) + \text{ arctang } P^2(t,\tau)]\,d\tau\right| \leq \tag{2.27}$$

$$\leq \int_0^{u^2} (|P| + P^2)\,d\tau \leq P' + P'^2.$$

Let $\sigma > 0$, suppose $0 < |P| < \frac{1}{2}[(4\sigma+1)^{\frac{1}{2}} - 1]$ then $|V - V^*| < \sigma$.

Deduce also

$$\dot{V}^{(2.25)} = [f - P]\dot{u}^{(2.25)} = [f - P][-f + P] = -[f - P]^2 \leq 0. \tag{2.28}$$

By Theorem 6 we conclude the proof.

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