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ON THE NUMERICAL CALCULATION OF HADAMARD FINITE-PART INTEGRALS

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This paper is dedicated to the memory of Claudio Barone, friend and collaborator

In this paper we consider a simple method for calculating integrals possessing strong singularities, to be interpreted in the Hadamard finite-part sense. We partition the original interval of integration and then integrate over the subintervals by using suitably modified low-order Gaussian-type quadratures. Convergence is shown under suitable assumptions and numerical evidence supports the theoretical findings.

1. Introduction.

Strongly singular integrals appear in many branches of applied mathematics, of which fluid dynamics and fracture mechanis are among the most important ones. They arise from mixed boundary value problems, upon use of the boundary integral method for their solution. If the unknown function on the boundary is a tangential derivative, usually one obtains a singular integral equation, containing a Cauchy principal value integral. If the problem is instead

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reformulated for an unknown function which is a normal derivative, strongly singular integrals appear, see e.g. [6], [8], [10], [11], [12].

Direct numerical methods for Hadamard finite-part integrals are under current investigation in the recent literature. Special formulae for their evaluation have been introduced, see e.g. [13], [14]. In this note we would like to propose a very simple low-order scheme for their calculation, which stems from recent investigations of the author in related fields, [1], [2], [16].

2. Definitions.

We consider here the problem of evaluating the Hadamard finite-part integral

$$I_\alpha^{FP} = \int_0^1 \frac{f(t)}{|t-c|^\alpha}\, dt, \quad 0 < c < 1, \quad \alpha \in \mathbb{R}^+. \tag{1}$$

The finite-part integral can be defined by subtracting out from the integrand a suitable term so as to make the result finite: usually this term is given by the Taylor series of the integrand. Assuming m to be the integer part of α, and assuming f to be m times continuously differentiable, we are then led to the definition, [14]

$$\int_0^1 t^{-\alpha} f(t)\, dt = \sum_{k=0}^{m-1} \frac{(-1)^k f^{(k)}(1)}{(1-\alpha)_{k+1}} + \frac{(-1)^m}{(1-\alpha)_m} \int_0^1 t^{m-\alpha} f^{(m)}(t)\, dt, \tag{2}$$

with $(a)_k = \Gamma(a+k)/\Gamma(a)$, where now the last integral exists in the ordinary sense.

If α is an integer, the above expression needs to be modified by a logarithmic term, see [14], but the finite-part integral can also be defined as a suitable derivative under the sign of a corresponding Cauchy principal value integral, so that, e.g. [13]

$$\int_0^1 \frac{f(t)}{|t-c|^2}\, dt = \frac{d}{dc} ⨍_0^1 \frac{f(t)}{t-c}\, dt. \tag{3}$$

None of the above formulae are used in the algorithm we propose here, although it is interesting to note that this second approach is used in [13] to derive the quadrature rule for the finite-part integral by differentiating the corresponding rule for the principal value integral.

3. The method.

Let us subdivide the basic interval [0, 1] by means of the breakpoints $t_k = kh$, $k = 0, \dots, n$, $h = n^{-1}$. Then

$$I_\alpha^{FP} = \sum_{k=1}^{n} I_k, \quad I_k \equiv \int_{(k-1)h}^{kh} \frac{f(t)}{|t-c|^\alpha}\, dt. \tag{4}$$

Evidently, in general only one of the I_k's gives again a Hadamard finite-part integral. Let us assume that $c \in ((m-1)h, mh)$, so that the strongly singular integral is I_m. We split I_m into two finite-part integrals, by means of the change of variable $s = \phi^{-1}(t) \equiv \frac{1}{h}[t-(m-1)h]$. Then I_m can be put into the following form, letting $s^* = \phi^{-1}(c)$,

$$\begin{gathered} I_m = \int_0^{s^*} \frac{g(s)}{(s^*-s)^\alpha}\, ds + \int_{s^*}^{1} \frac{g(s)}{(s-s^*)^\alpha}\, ds \equiv I_{m_1} + I_{m_2}, \\ g(s) = f(\phi(s))\phi'(s)\left|\frac{s-s^*}{\phi(s)-\phi(s^*)}\right|^\alpha. \end{gathered} \tag{5}$$

The proposed quadrature over each $[(k-1)h, kh]$, $k = 1, \dots, n$, is a low-order Gaussian type rule. We replace each integral I_k by a $q-$nodes quadrature as follows

$$I_k \cong \sum_{i=1}^{q} w_i f(x_i), \quad k \neq m, \quad I_m \cong \sum_{i=1}^{q} w_i g(x_i). \tag{6}$$

We devise the algorithm to converge for the number of subintervals tending to infinity, while keeping the order of the quadrature in each subinterval the same and low enough so as to easily determine its weights and nodes.

We now address the question of how to determine the quadrature weights and nodes. The basic tool we use to evaluate each I_k is given by the following moment equations

$$\sum_{i=1}^{q} w_i x_i^j = M_j, \text{ for } k \neq m; \quad \sum_{i=1}^{q} w_i x_i^j = M_j^H, \quad \text{for } k = m, \tag{7}$$

where the right hand sides denote indeed the moments, to be defined below. The latter can be explicitly obtained. Recalling from [14] that

$$\int_0^1 \frac{t^j}{t^\alpha}\, dt = \frac{1}{j-\alpha+1}, \quad \alpha \in \mathbb{R}^+,$$

we indeed obtain, e.g. for the moments of the first integral in (5),

$$M_j^H \equiv \int_0^{s^*} \frac{t^j}{t^\alpha}\,dt = \frac{(s^*)^{j-\alpha+1}}{j-\alpha+1}, \quad \alpha \in \mathbb{R}^+. \tag{8}$$

In other words, in (6) we impose the quadrature to be extact for the low-order moments, with respect of the weight function $t^{-\alpha}$. For some values of j it is possible that the above integral exists either in the improper or even in the ordinary sense. More in general, in view of (4), we will also need to evaluate the following integrals, which always exist in the ordinary sense,

$$M_j \equiv \int_a^b \frac{t^j}{t^\alpha}\,dt = \frac{b^{j-\alpha+1} - a^{j-\alpha+1}}{j-\alpha+1}, \qquad a > 0, \quad b-a=h, \quad j = 0,1,2,\ldots. \tag{9}$$

For low-order schemes, say $q \le 4$, we can derive explicitly the quadrature nodes and weights by setting up a nonlinear system. The latter is then reduced to a nonlinear algebraic equation, which is finally solved by means of standard closed-form formulae. For the rest of this section, let us then fix the interval $[a,b]$ under consideration. Also, in every formula that follows, let M_j be understood as M_j^H, whenever the quadrature is sought for the Hadamard finite-part integrals (5), i.e. for $j = m$, while let it be simply M_j for the rules related to every other subinterval, $j \ne m$.

Let us define the following quantities

$$s_0 = 1, \quad s_l = (-1)^l \sum_{1 \le i_1 < i_2 < \cdots < i_l \le q} x_{i_1} \cdots x_{i_l}, \quad l = 1,\ldots,q, \tag{10}$$

as well as

$$s_0^{(j)} = 1, \quad s_l^{(j)} = (-1)^l \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_l \le q \\ i_1, i_2, \ldots, i_l \ne j}} x_{i_1} \cdots x_{i_l}, \quad j, l = 1,\ldots,q.$$

Let us observe then that

$$\prod_{i=1}^q (x - x_i) = \sum_{j=0}^q s_j x^{q-j}. \tag{11}$$

The system is constructed by imposing the quadrature formula to be exact for polynomials up to degree $2q-1$, to obtain

$$
\begin{aligned}
w_1 + w_2 + \cdots + w_q &= M_0,\\
w_1x_1 + w_2x_2 + \cdots + w_qx_q &= M_1,\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
w_1x_1^{q-1} + w_2x_2^{q-1} + \cdots + w_qx_q^{q-1} &= M_{q-1},\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
w_1x_1^{2q-1} + w_2x_2^{2q-1} + \cdots + w_qx_q^{2q-1} &= M_{2q-1}.
\end{aligned}
\tag{12}
$$

To reduce it, let us consider the first q equations *in the unknowns* $w_1, \dots, w_q$. The matrix of this subsystem is the $q \times q$ Vandermonde matrix $(x_j^i)_{i=0,\dots,q-1;\,j=1,\dots,q}$. We reduce it to triangular form, by taking suitable linear combinations of *adjacent* rows. The system thus obtained has the matrix

$$
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1\\
 & x_2 - x_1 & x_3 - x_1 & \cdots & \cdots\\
 & & (x_3-x_1)(x_3-x_2) & \cdots & \cdots\\
 & & & \cdots & \cdots\\
 & & & & (x_q-x_1)\dots(x_q-x_{q-1})
\end{pmatrix}
$$

and the right hand side

$$
\begin{pmatrix}
M_0\\
M_1 - x_1M_0\\
M_2 - (x_1+x_2)\,M_1 + x_1x_2M_0\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
M_{q-1} - (x_1+\cdots+x_{q-1})M_{q-2} + \cdots + x_1\cdots x_{q-1}M_0
\end{pmatrix}
$$

The last equation gives then w_q. With an exchange of the indices, we then find the following expression for the weights:

$$
w_j = \prod_{\substack{i=1\\ i\neq j}}^{q} (x_j - x_i)^{-1}\left[\sum_{l=0}^{q-1} s_{q-l-1}^{(j)} M_l\right], \quad j = 1, \dots, q. \tag{13}
$$

For the nodes, we proceed on the bottom portion of (12) with the $q \times q$ Vandermonde matrix $\left(x_j^{q+i}\right)_{i=0,\dots,q-1;\,j=1,\dots,q}$. We take the $q-1+j$-th equation,

$j = 0, \ldots, q-1$, multiply it in turn q times by $x_1, \ldots, x_q$ and sum all these so obtained q equations, to introduce the quantities s_k:

$$\begin{aligned} M_{q+j} &= -M_{q+j-1}s_1 - x_1^{q-2+j} w_1[x_1x_2 + x_1x_3 + \cdots + x_1x_q] \\ &\quad - x_2^{q-2+j} w_2[x_1x_2 + x_2x_3 + \cdots + x_2x_q] - \cdots \\ &\quad - x_q^{q-2+j} w_q[x_1x_q + x_2x_q + \cdots + x_{q-1}x_q] \\ &= -M_{q+j-1}s_1 - M_{q+j-2}s_2 - x_1^{q-3+j} w_1 \prod_{2\le i_1<i_2<i_3\le q} x_{i_1}x_{i_2}x_{i_3} \\ &\quad -x_2^{q-3+j} w_2 \prod_{\substack{1\le i_1<i_2<i_3\le q \\ i_1,i_2,i_3\neq 2}} x_{i_1}x_{i_2}x_{i_3} - \cdots - x_q^{q-3+j} w_q \prod_{1\le i_1<i_2<i_3\le q-1} x_{i_1}x_{i_2}x_{i_3}. \end{aligned}$$

Proceeding inductively, completing each time the terms on the right hand side, so as to introduce the s_k's up to $k = q$, by adding and subtracting the missing terms, we obtain

$$M_{q+j} = -M_{q+j-1}s_1 - M_{q+j-2}s_2 - \cdots - M_j s_q, \quad j = 0, \ldots, q-1. \tag{14}$$

This is a linear system for the unknowns s_k. We can write it extensively by reordering the unknowns as follows

$$\begin{pmatrix} M_0 & M_1 & M_2 & \cdots & M_{q-2} & M_{q-1} \\ M_1 & M_2 & M_3 & \cdots & M_{q-1} & M_q \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{q-1} & M_q & M_{q+1} & \cdots & M_{2q-3} & M_{2q-2} \end{pmatrix} \begin{pmatrix} s_q \\ s_{q-1} \\ \cdots \\ s_1 \end{pmatrix} = - \begin{pmatrix} M_q \\ M_{q+1} \\ \cdots \\ M_{2q-1} \end{pmatrix} \tag{15}$$

Let us then define for $i = 0, \ldots, q-1$ the determinants

$$\Delta_i = \begin{vmatrix} M_0 & M_1 & M_2 & \cdots & M_{i-1} & -M_q & M_{i+1} & \cdots & M_{q-2} & M_{q-1} \\ M_1 & M_2 & M_3 & \cdots & M_i & -M_{q+1} & M_{i+2} & \cdots & M_{q-1} & M_q \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{q-1} & M_q & M_{q+1} & \cdots & M_{i+q-2} & -M_{2q-1} & M_{i+q} & \cdots & M_{2q-3} & M_{2q-2} \end{vmatrix} \tag{16}$$

as well as

$$\hat{\Delta}_i = \begin{vmatrix} M_0 & M_1 & M_2 & \cdots & M_{i-1} & M_{i+1} & \cdots & M_{q-2} & M_{q-1} & M_q \\ M_1 & M_2 & M_3 & \cdots & M_i & M_{i+2} & \cdots & M_{q-1} & M_q & M_{q+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{q-1} & M_q & M_{q+1} & \cdots & M_{i+q-2} & M_{i+q} & \cdots & M_{2q-3} & M_{2q-2} & M_{2q-1}. \end{vmatrix} \tag{17}$$

Notice that from (16) and (17)

$$\Delta_i = (-1)^{q-i} \hat{\Delta}_i. \tag{18}$$

Also, let

$$
(19) \qquad \Delta_q = \hat{\Delta}_q = \begin{vmatrix} M_0 & M_1 & M_2 & \cdots & M_{q-3} & M_{q-2} & M_{q-1} \\ M_1 & M_2 & M_3 & \cdots & M_{q-2} & M_{q-1} & M_q \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{q-1} & M_q & M_{q+1} & \cdots & M_{2q-4} & M_{2q-3} & M_{2q-2} \end{vmatrix}
$$

By (11), it is easily verified that the roots of the following nonlinear algebraic equation are the sought quadrature nodes

$$
(20) \qquad \sum_{i=0}^{q} s_i x^{q-i} = 0.
$$

On solving (15) by Cramer's rule,

$$
s_{q-i} = \frac{\Delta_i}{\Delta_q}, \quad i = 0, \ldots, q-1.
$$

Substituting into (20) and using (18), we have

$$
0 = x^q + \sum_{i=0}^{q-1} s_{q-i} x^i = x^q + \sum_{i=0}^{q-1} \frac{\Delta_i}{\Delta_q} x^i = x^q + \sum_{i=0}^{q-1} (-1)^{q-i} \frac{\hat{\Delta}_i}{\hat{\Delta}_q} x^i.
$$

Upon multiplication by $\hat{\Delta}_q$, the latter is equivalent to

$$
\sum_{i=0}^{q} (-1)^i \hat{\Delta}_i x^i = 0,
$$

which in turn can be finally written in determinant form as

$$
(21) \qquad \begin{vmatrix} 1 & x & x^2 & \cdots & x^{q-2} & x^{q-1} & x^q \\ M_0 & M_1 & M_2 & \cdots & M_{q-2} & M_{q-1} & M_q \\ M_1 & M_2 & M_3 & \cdots & M_{q-1} & M_q & M_{q+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{q-1} & M_q & M_{q+1} & \cdots & M_{2q-3} & M_{2q-2} & M_{2q-1} \end{vmatrix} = 0.
$$

Its roots are the required quadrature nodes. In practice, the nodes are found by using the explicit formulae for the zeros of the $q-$th degree equation. This of course is possible only if $q \leq 4$.

Remark. The system (21) could be useful in case one wants to use a higher order formula. This could be accomplished in principle by using the zeros of the $q-$th degree equation as initial guesses for a rootfinding method for the equation of degree $q+1$, or as endpoints for the subinterval in which the next higher order quadrature node lies. The possibility of applying such technique could be explicitly checked by using Stieltjes theorem, see Theorem 15, p. 232 of [5].

Remark. The drawback of the procedure described in this section consists in the weights not being one-signed, contrary to what happens in standard Gaussian quadrature. Indeed, e.g. for $q = 1$ we have $w_1 = M_0$, so that $w_1 > 0$ only if $\alpha < 1$, which is the case of ordinary or improper quadrature. This fact might entail some kind of instability, mildly reflected in the figures of the examples. However a sort of ill conditioning is reported to occur often in the literature, for the numerical calculation of this type of integrals.

4. Convergence analysis.

We consider separately the two cases corresponding to the two intervals with endpoint c, and to all the other subintervals, respectively. We start with the analysis of the latter.

Since in all intervals not of the form (5) the integrals can be interpreted in the ordinary sense, the error analysis for the above formulae follows from the standard theory. Denoting by $p_m(t)$ the $m-th$ orthogonal polynomial over $[a,b]$ with respect to the weight t^γ, we have the following classical result, see [7], p. 334.

Theorem 1. *For the above described quadrature formulae the following error estimates hold if the integrand function ψ is at least $2m$ times continuously differentiable*

$$E_m\{\psi\} \equiv \int_a^b t^\gamma \psi(t)\,dt - \sum_{j=1}^m w_j \psi(x_j) = \frac{\psi^{(2m)}(\xi)}{(2m)!} \int_a^b p_m^2(t) t^\gamma\,dt. \tag{22}$$

We then need to estimate the integral term, in terms of the interval length $b-a=h$. Theorem 4, p. 203 of [7] holds for any generic family of orthogonal polynomials. Thus the zeros x_k, $k = 1(1)q$, of $p_q(t)$ must lie in $[a,b]$. Since in (22) $p_q(t)$ is a monic polynomial, the following estimate follows

$$p_q^2(t) = \prod_{k=1}^q (t-x_k)^2 = O(h^{2q}), \quad t \in [a,b].$$

On using this result in (22), for the integrals (4), with $k \neq m$, we then have the following estimate.

Proposition 2. *Assuming enough differentiability of the integrand, i.e. if $f \in C^{2q}[0, 1]$, for the proposed modified Gaussian quadrature over $[a, b]$ for I_k, $k \neq m$, the error estimate for some $\zeta \in [a, b]$, is given by*

$$|E_q\{f\}| \leq \frac{|f^{(2q)}(\zeta)|}{(2q)!}\frac{h^{2q-\alpha+1}}{1-\alpha} = O(h^{2q+1-\alpha}). \tag{23}$$

We now turn to the analysis of the quadratures for (5). Let us fix our attention on the first of (5). Assuming that the integrand is enough differentiable, i.e. $f \in C^{2q}[0, 1]$, or $g \in C^{2q-1}[0, 1]$, we can express it as a Taylor series with remainder, in the interval $[0, s^*]$.

$$I_{m_1} \equiv \int_0^{s^*} t^{-\alpha} g(t)\, dt = \int_0^{s^*} \sum_{i=0}^{2q-2} \frac{1}{i!} g^{(i)}(0) t^{i-\alpha}\, dt + \\ + \int_0^{s^*} R_{2q-1}(\xi_t) t^{-\alpha}\, dt, \tag{24}$$

where

$$R_{2q-1}(\xi_t) = \frac{1}{(2q-1)!} g^{(2q-1)}(\xi_t) t^{2q-1}, \quad 0 < \xi_t < t. \tag{25}$$

Then, recalling (8), observe that the first integral on the right hand side of (24) becomes

$$\sum_{i=0}^{2q-2} \frac{1}{i!} g^{(i)}(0) M_i^H. \tag{26}$$

Using now (7), which is exact for $j = 0, 1, \ldots, 2q-1$, we obtain from (24)

$$I_{m_1} = \sum_{i=0}^{2q-2} \frac{1}{i!} g^{(i)}(0) \sum_{j=1}^{q} w_j x_j^i + \int_0^{s^*} R_{2q-1}(\xi_t) t^{-\alpha}\, dt. \tag{27}$$

By interchanging the summation order and using once again the Taylor series, the first term on the right of (27) is

$$\sum_{j=1}^{q} w_j \sum_{i=0}^{2q-2} \frac{1}{i!} g^{(i)}(0) x_j^i = \sum_{j=1}^{q} w_j [g(x_j) - R_{2q-1}(\xi_{x_j})]. \tag{28}$$

Hence upon substituting (28) into (27) and in view of (25),

$$I_{m_1} - \sum_{j=1}^{q} w_j g(x_j) = \frac{-1}{(2q-1)!} \sum_{j=1}^{q} w_j g^{(2q-1)}(\xi_{x_j}) x_j^{2q-1} + \\ + \int_0^{s^*} R_{2q-1}(\xi_t) t^{-\alpha} dt. \tag{29}$$

Notice that for finite-part integrals the usual estimates involving absolute values do not hold, since the finite-part of a nonnegative integrand may have a negative value, see the discussion on p. 13 of [4]. However, to circumvent this difficulty, let us impose the following restriction on the values of q and α, namely that they satisfy

$$2q - \alpha > 0. \tag{30}$$

With this constraint, the integral of the remainder term (25) appearing as the last term in (29), becomes an ordinary integral. For the first term appearing on the right of (29) we have the following estimate

$$\frac{-1}{(2q-1)!} \sum_{j=1}^{q} w_j g^{(2q-1)}(\xi_{x_j}) x_j^{2q-1} \leq \\ \leq \frac{-1}{(2q-1)!} \min_{1 \leq j \leq q} \{g^{(2q-1)}(\xi_{x_j})\} \sum_{j=1}^{q} w_j x_j^{2q-1},$$

so that the following upper bound for (29) holds

$$I_{m_1} - \sum_{j=1}^{q} w_j g(x_j) \leq - \min_{1 \leq j \leq q} \{g^{(2q-1)}(\xi_{x_j})\} \frac{M_{2q-1}^H}{(2q-1)!} + \int_0^{s^*} R_{2q-1}(\xi_t) t^{-\alpha}\, dt.$$

From this we then obtain

$$\left| I_{m_1} - \sum_{j=1}^{q} w_j g(x_j) \right| \leq \frac{2}{(2q-1)!} \|g^{(2q-1)}\|_\infty M_{2q-1}^H.$$

On recalling (8) and the fact that $0 < s^* \leq h$, the following result holds true.

Proposition 3. *Assuming the integrand function g to be $2q-1$ times continuously differentiable, or alternatively that $f \in C^{2q}[0,1]$, the error for the finite-part integral over $[0, s^*]$ is bounded above by*

$$\left| I_{m_1} - \sum_{j=1}^{q} w_j g(x_j) \right| \leq \frac{2}{(2q-1)!} \|g^{(2q-1)}\|_\infty h^{2q-\alpha}.$$

A similar result holds for I_{m_2} as well. Now, by considering the partition (4), combining Propositions 2 and 3, we have the following convergence result.

Theorem 4. *If $f \in C^{2q}[0,1]$, the proposed modified piecewise gaussian scheme with q nodes is convergent to the finite-part integral. Denoting by C a suitable constant, the convergence rate is given by*

$$\left| I_\alpha^{FP} - \sum_{k=1}^{n} \sum_{j=1}^{q} w_j f(x_j) \right| \leq C h^{2q-\alpha}.$$

Proof. From the above considerations, indeed,

$$\left| I_\alpha^{FP} - \sum_{k=1}^{n} \sum_{j=1}^{q} w_j f(x_j) \right| \leq \frac{2}{(2q-1)!} \|g^{(2q-1)}\|_\infty h^{2q-\alpha} + $$

$$+ \sum_{\substack{i=1 \\ i \neq m}}^{n} \frac{|g^{(2q)}(\zeta)|}{(2q)!} \frac{h^{2q-\alpha+1}}{1-\alpha},$$

from which the claim. □

Remark. Note that in (27) we cannot expand the function up to the term of order $2q-1$, because otherwise the error term of order $2q$ contains x_j^{2q}, and $\sum_{j=1}^{q} w_j x_j^{2q}$ need not be M_{2q}^H. In this situation moreover it is not ensured that $0 < x_j < s^*$. However using one less term in the expansion does not affect the overall convergence rate of the algorithm.

5. Examples.

Here we relate our numerical experience. We consider integrals of type (1), where the integrand function is always $f(t) = \exp(t)$. In the tables we give the numerical evidence for various choices of α and of the number of quadrature nodes q. One advantage of the proposed rule is that it applies equally well to the case of noninteger α.

Table 1	$c = .3$	$\alpha = 2.0$	$q = 3$
nodes	value	difference	order
2	$-.45565832726028E+01$		
4	$-.45565831419417E+01$	$.13066E-06$	
8	$-.45565831274366E+01$	$.14505E-07$	3.17
16	$-.45565831272926E+01$	$.14404E-09$	6.65
32	$-.45565831272798E+01$	$.12847E-10$	3.49
64	$-.45565831272796E+01$	$.19895E-12$	6.01
128	$-.45565831272795E+01$	$.42633E-13$	2.22

Table 2	$c = .3$	$\alpha = 3.0$	$q = 3$
nodes	value	difference	order
2	$-.72511974696530E+01$		
4	$-.72511806456007E+01$	$.16824E-04$	
8	$-.72511778454465E+01$	$.28002E-05$	2.59
16	$-.72511778039581E+01$	$.41488E-07$	6.08
32	$-.72511777966793E+01$	$.72788E-08$	2.51
64	$-.72511777967447E+01$	$.65484E-10$	6.80

Table 3	$c = .3$	$\alpha = 4.0$	$q = 3$
nodes	value	difference	order
2	$-.14821395736609E+02$		
4	$-.14820314707160E+02$	$.10810E-02$	
8	$-.14819616622133E+02$	$.69809E-03$	.63
16	$-.14819542295120E+02$	$.74327E-04$	3.23
32	$-.14819518768671E+02$	$.23526E-04$	1.66
64	$-.14819517167844E+02$	$.16008E-05$	3.88
128	$-.14819516687770E+02$	$.48007E-06$	1.74
256	$-.14819516621064E+02$	$.66706E-07$	2.85

Table 4	$c = .3$	$\alpha = 2.3$	$q = 2$
nodes	value	difference	order
2	$-.39406944207095E+01$		
4	$-.39383772911318E+01$	$.23171E-02$	
8	$-.39376364230834E+01$	$.74087E-03$	1.65
16	$-.39375800011181E+01$	$.56422E-04$	3.71
32	$-.39375624892431E+01$	$.17512E-04$	1.69
64	$-.39375611513341E+01$	$.13379E-05$	3.71
128	$-.39375607357169E+01$	$.41562E-06$	1.69
256	$-.39375607039963E+01$	$.31721E-07$	3.71
512	$-.39375606941603E+01$	$.98360E-08$	1.69

Table 5	$c = .3$	$\alpha = 2.3$	$q = 3$
nodes	value	difference	order
2	$-.39375609431062E+01$		
4	$-.39375607242366E+01$	$.21887E-06$	
8	$-.39375606935678E+01$	$.30669E-07$	2.84
16	$-.39375606931948E+01$	$.37300E-09$	6.36
32	$-.39375606931504E+01$	$.44338E-10$	3.07
64	$-.39375606931498E+01$	$.62528E-12$	6.15

Table 6	$c = .3$	$\alpha = 2.3$	$q = 4$
nodes	value	difference	order
2	$-.39375606931797E+01$		
4	$-.39375606931510E+01$	$.28715E-10$	
8	$-.39375606931498E+01$	$.11608E-11$	4.63
16	$-.39375606931497E+01$	$.10303E-12$	3.49

From the numerical experiments carried out, it seems that the weights not being of one sign have some influence on the order of the convergence of the method, especially for values of α moderately high. However, convergence is attained in all the examples, at an average rate at times larger than the one theoretically predicted by Theorem 4.

6. Extensions.

As mentioned in the introduction, this investigation originates from some other questions related to the calculation of line integrals, if the parametrization of the integration path is not analytically known, [1], [2], [16]. The curve needs evidently to be replaced by a suitable piecewise polynomial interpolant. The interpolation error then plays an essential role in the algorithm, in the sense that it would not make sense to have a highly convergent quadrature if the interpolation scheme is poor, and vice-versa.

Let the integral in consideration be

$$I_L^H \equiv \int_L \frac{f(Q)}{|Q-P|^\alpha}\,dQ = \int_0^1 \frac{f(r(t))|r'(t)|}{|r(t)-r(t_0)|^\alpha}\,dt\,.$$

Let r_p be the union of the piecewise polynomial interpolants to r, each constructed over $p+1$ nodes of the interval $[0, 1]$. The integral I_L^H is then calculated by replacing the integration path L with its interpolant L_p, constructed as outlined above. In other words, we approximate I_L^H by means of

$$I_{L_p}^H \equiv \int_0^1 \frac{f(r_p(t))|r_p'(t)|}{|r_p(t)-r_p(t_0)|^\alpha}\,dt\,,$$

and the latter is then calculated by the algorithm presented in Section 3. The analysis can be done following the steps of [1] and [15]. For the convergence of this method, we have then the following result.

Theorem 5. *Suppose that the integration path L is parametrized by the function $r \in C^{p+2}[0, 1]$, with $|r'| > 0$. Assume that the composition map $f \circ r \in C^{2q}[0, 1]$, and is the restriction to L of a twice continuously differentiable function of several variables, defined in an open neighborhood of L. Then for the calculation of the hypersingular integral I_L^H by means of a piecewise gaussian quadrature rule of the type (4), the error is given by the expression*

$$E_n(I_L^H) = O(h^{\min[\tilde{p},2q-\alpha]}), \quad \tilde{p} = \begin{cases} p & p \text{ even} \\ p+1 & p \text{ odd} \end{cases}$$

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