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AN EXISTENCE RESULT FOR A NONLINEAR PROBLEM IN A LIMIT CASE

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In this paper we study the problem (1) in the limit case $p = N$. We prove that such a problem admits at least a solution u if some suitable norms of f and g are small enough. Furthermore we show that u is such that the function $w = \frac{e^{\mu|u|}-1}{\mu} sign(u)$ belongs to $W_0^{1,N}(\Omega)$, where μ is some costant.

1. Introduction.

Let Ω be an open bounded set of $\mathbb{R}^N$ and let us consider the following problem:

$$\text{(1)} \qquad \begin{cases} -\text{div}\,(a(x, u, Du)) = H(x, u, Du) + f - \text{div}\,(g) \text{ in } \mathscr{D}'(\Omega) \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

We suppose that $-\text{div}(a(x, u, Du))$ is a Leray-Lions operator acting on $W_0^{1,p}(\Omega)$, for some $p > 1$, and $H(x, s, \zeta)$ is a Carathéodory function such that for a.e. $x \in \Omega$, any $s \in \mathbb{R}$ and any $\zeta \in \mathbb{R}^N$ it holds

$$\text{(2)} \qquad -c_0 a(x, s, \zeta) \cdot \zeta \le H(x, s, \zeta) sign(s) \le \gamma a(x, s, \zeta) \cdot \zeta$$

for some $\gamma > 0$, $c_0 \ge 0$. By the usual assumptions on $a(x, s, \zeta)$ (see (9) for the case $p = N$), it is clear that $H(x, u, Du)$ is a non linear term which grows at

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most like $|Du|^p$ with respect to Du. In this paper we study problem (1) in the case $p = N$ and we prove that it admits at least a solution if the source terms (f and g) satisfy a suitable smallness assumption.

The main idea of the proof is to consider a standard approximate problem (see [4]–[9]), labeled by ε, which always admits a solution u_ε. Then we use a suitable function of u_ε as a test function in the approximate problem obtaining an apriori estimate for $\|u_\varepsilon\|_{W_0^{1,N}(\Omega)}$. Finally we prove that, after passing to a subsequence, u_ε strongly converges in $W_0^{1,N}(\Omega)$ to a function u which is a solution of (1); moreover this solution is such that the function $w = \frac{N-1}{\gamma}(e^{\frac{\gamma}{N-1}|u|} - 1)sign(u)$ belongs to $W_0^{1,N}(\Omega)$. We also show, by an example, that problem (1) in general does not admit a unique solution. Finally we compare our hypotheses on f to some others sufficient to guarantee the existence of a solution for (1) in $W_0^{1,N}(\Omega) \cap L^\infty(\Omega)$ (see [11]), and we verify that the former are less restrictive than the latter.

In the literature problems having the structure of (1) have been widely studied. There exist some papers where solutions either in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ or just in $W_0^{1,p}(\Omega)$ are found. For the first type of results let us recall, e.g., [6], [7], [8], and [17] where some further structure hypotheses, like sign conditions, on the zero order term $H(x, s, \zeta)$ are made. Also [2] and [11] are concerned with bounded solutions. In these papers only a growth condition on H is assumed and an a priori estimate is obtained by imposing a suitable smallness assumption on the source terms. On the other hand in [4] and [5] one proves the existence of a solution in $W_0^{1,p}(\Omega)$ (possibly unbounded) for problem similar to (1), where H satisfies a sign condition; while in [9] problem (1) is studied when $p < N$. About this last paper we remark that for $p < N$ it is sufficient to assume that the source terms f and g are in $L^{N/p}$ and $L^{N/(p-1)}$ respectively. As we will show, in the limit case $p = N$ the natural spaces for f and g are not Lebesgue spaces but the Zygmund ones.

2. Preliminary results.

In this section we recall some definitions and classical properties about rearrangements and then we introduce the Zygmund-spaces.

Let Ω be an open bounded set of $\mathbb{R}^N$ and let $\phi(x) : \Omega \to \mathbb{R}$ be a measurable function, we define by $\mu_\phi(t) \equiv \left|\{x \in \Omega : |\phi(x)| > t\}\right|$ $(t \geq 0)$ the distribution function of ϕ and by $\phi^*(s) \equiv \sup\{t > 0 : \mu_\phi(t) > s\}$ $(s \in (0, |\Omega|))$ the decreasing rearrangement of ϕ.

Exhaustive treatments of the theory of rearrangements can be found, e.g., in [12] and [16]; here we only recall the following Hardy-Littlewood inequality:

$$\int_\Omega |h(x)k(x)|\,dx \le \int_0^{|\Omega|} h^*(s)k^*(s)\,ds \tag{3}$$

for any h,k real measurable function defined in Ω.

The Zigmund-space $L^p(\log L)^q$, $p > 0$, $q \in \mathbb{R}$ (see [3]), consists of all measurable functions $\phi : \Omega \to \mathbb{R}$ such that the following quantity is finite

$$\|\phi\|_{L^p(\log L)^q} \equiv \left(\int_0^{|\Omega|} \left[\left(\log \frac{|\Omega|}{s} \right)^q \phi^*(s) \right]^p ds \right)^{1/p}. \tag{4}$$

We observe that the following inclusion relations hold:

$$L^{p_2}(\log L)^q \subset L^{p_1}(\log L)^q \quad \text{if} \quad 0 < p_1 < p_2, \tag{5}$$

$$L^p(\log L)^{q_2} \subset L^p(\log L)^{q_1} \quad \text{if} \quad q_1 < q_2. \tag{6}$$

In particular, $L^p(\log L)^q \subset L^p\ \forall q > 0, \forall p > 0$.

We finally recall a result contained in [1] which will be used in the next section:

Theorem 2.1. *If u is a real function in $W_0^{1,N}(\Omega)$, then*

$$u^*(s) \le \frac{\|Du\|_N}{N C_N^{1/N}} \left[\log \frac{|\Omega|}{s} \right]^{\frac{N-1}{N}} \quad \forall s \in (0, |\Omega|), \tag{7}$$

where C_N is the measure of the unit sphere of $\mathbb{R}^N$ ($C_N = \frac{\pi^{N/2}}{\Gamma(1+N/2)}$).

3. Existence theorem.

Let us consider the following problem:

$$\begin{cases} -\mathrm{div}(a(x,u,Du)) = H(x,u,Du) + f - \mathrm{div}(g) \text{ in } \mathcal{D}'(\Omega) \\ u \in W_0^{1,N}(\Omega) \end{cases} \tag{8}$$

where:

(i) Ω is an open bounded set of $\mathbb{R}^N$;

(ii) $a(x,s,\zeta) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function which satisfies, for a.e. $x \in \Omega$, any $s \in \mathbb{R}$ and any $\zeta, \zeta' \in \mathbb{R}^N$ with $\zeta \neq \zeta'$,

$$\left\{\begin{array}{l}(a(x,s,\zeta) - a(x,s,\zeta'))(\zeta - \zeta') > 0 \\ a(x,s,\zeta) \cdot \zeta \geq \alpha |\zeta|^N \\ |a(x,s,\zeta)| \leq \beta [b(x) + |s|^{N-1} + |\zeta|^{N-1}]\end{array}\right. \tag{9}$$

for some $\alpha > 0$, $\beta > 0$, $b \in L^{\frac{N}{N-1}}(\log L)^{\frac{(N-1)^2}{N}}$;

(iii) $H(x,s,\zeta) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that for some $\gamma > 0$, $c_0 \geq 0$

$$-c_0 a(x,s,\zeta) \cdot \zeta \leq H(x,s,\zeta) sign(s) \leq \gamma a(x,s,\zeta) \cdot \zeta \tag{10}$$

for a.e. $x \in \Omega$, any $s \in \mathbb{R}$, any $\zeta \in \mathbb{R}^N$;

(iv) $g \in L^{\frac{N}{N-1}}(\log L)^{\frac{(N-1)^2}{N}}$ and $f \in L^1(\log L)^{N-1} \equiv L(\log L)^{N-1}$.

Theorem 3.1. *If (i)–(iv) hold and moreover*

$$\frac{\|f\|_{L(\log L)^{N-1}}}{N^N C_N} + \frac{1}{N^{N-2} C_N^{\frac{N-1}{N}}} \cdot \|g\|_{L^{\frac{N}{N-1}}(\log L)^{\frac{(N-1)^2}{N}}} < \alpha \Big(\frac{N-1}{\gamma}\Big)^{N-1}, \tag{11}$$

then there exists at least a solution u of (8). Furthermore this solution is such that the function

$$w = \frac{e^{\mu|u|} - 1}{\mu} sign(u) \tag{12}$$

belongs to $W_0^{1,N}(\Omega)$, where $\mu = \frac{\gamma}{N-1}$.

Proof. Step 1: An approximate problem.

Consider two sequences $\{f_\varepsilon\}$ and $\{g_\varepsilon\}$ such that:

$$\left\{\begin{array}{l} f_\varepsilon \to f \text{ in } L^1(\Omega) \text{ and a.e. in } \Omega;\ g_\varepsilon \to g \text{ in } (L^{\frac{N}{N-1}}(\Omega))^N \\ |f_\varepsilon| \leq |f| \text{ a.e. in } \Omega; |g_\varepsilon| \leq |g| \text{ a.e. in } \Omega \\ \{f_\varepsilon\} \text{ and } \{g_\varepsilon\} \in L^\infty(\Omega). \end{array}\right. \tag{13}$$

These sequences can be obtained, e.g., by truncation of f and g_i respectively ($g = (g_1, \cdots, g_N)$).

Define, for $\varepsilon > 0$,

$$H_\varepsilon(x,s,\zeta) = \frac{H(x,s,\zeta)}{1 + \varepsilon |H(x,s,\zeta)|}, \tag{14}$$

from which it follows that H_ε satisfies (10) as well as $|H_\varepsilon(x, s, \zeta)| \leq |H(x, s, \zeta)|$. Since, by the very definition, $|H_\varepsilon(x, s, \zeta)| \leq \frac{1}{\varepsilon}$, a classical result on nonlinear problems (see [14] and [15]) allows us to say that the approximate problem:

$$(15) \qquad \begin{cases} -\mathrm{div}(a(x, u_\varepsilon, Du_\varepsilon)) = H_\varepsilon(x, u_\varepsilon, Du_\varepsilon) + f_\varepsilon - \mathrm{div}(g_\varepsilon) \text{ in } \mathcal{D}'(\Omega) \\ u_\varepsilon \in W_0^{1,N}(\Omega) \cap L^\infty(\Omega) \end{cases}$$

admits a solution u_ε.

Step 2: A priori estimate for $\|u_\varepsilon\|_{W_0^{1,N}}$.

Let us define the functions

$$(16) \qquad w_\varepsilon = \frac{e^{\mu|u_\varepsilon|} - 1}{\mu} sign(u_\varepsilon)$$

$$(17) \qquad v_\varepsilon = e^{\gamma|u_\varepsilon|} w_\varepsilon.$$

We observe that $v_\varepsilon \in W_0^{1,N}(\Omega) \cap L^\infty(\Omega)$ and, since $\mu + \gamma = N\mu$ and $sign(w_\varepsilon) = sign(u_\varepsilon)$, the following equalities hold

$$(18) \qquad e^{\gamma|u_\varepsilon|} = (1 + \mu|w_\varepsilon|)^{N-1},$$

$$(19) \qquad Dw_\varepsilon = e^{\mu|u_\varepsilon|} Du_\varepsilon = (1 + \mu|w_\varepsilon|) Du_\varepsilon.$$

If we use v_ε as a test function in the weak formulation of (15) we obtain

$$(20) \quad \int_\Omega a(x, u_\varepsilon, Du_\varepsilon) Dw_\varepsilon e^{\gamma|u_\varepsilon|}\, dx =$$
$$= \int_\Omega w_\varepsilon f_\varepsilon e^{\gamma|u_\varepsilon|}\, dx + \int_\Omega (Dw_\varepsilon + \gamma|w_\varepsilon| Du_\varepsilon) g_\varepsilon e^{\gamma|u_\varepsilon|}\, dx +$$
$$+ \int_\Omega (H_\varepsilon(x, u_\varepsilon, Du_\varepsilon) sign(u_\varepsilon) - \gamma a(x, u_\varepsilon, Du_\varepsilon) Du_\varepsilon) |w_\varepsilon| e^{\gamma|u_\varepsilon|}\, dx.$$

Now hypotheses (9) and (10), together with identities (18) and (19), give

$$(21) \qquad \alpha \|Dw_\varepsilon\|_N^N \leq \int_\Omega |w_\varepsilon|(1 + \mu|w_\varepsilon|)^{N-1} |f_\varepsilon|\, dx +$$
$$+ N \int_\Omega |Dw_\varepsilon|(1 + \mu|w_\varepsilon|)^{N-1} |g_\varepsilon|\, dx \leq$$
$$\leq \int_\Omega |w_\varepsilon|(1 + \mu|w_\varepsilon|)^{N-1} |f| dx + N \int_\Omega |Dw_\varepsilon|(1 + \mu|w_\varepsilon|)^{N-1} |g|\, dx,$$

where the last inequality follows from (13). At this point we recall that for every $\theta > 0$ and $\rho > 1$ there exists a constant $\tilde{C}(\rho, \theta)$ such that

$$(1+x)^{\rho} \le (1+\theta)x^{\rho} + \tilde{C}(\rho,\theta) \quad \forall x \ge 0. \tag{22}$$

Using (22) and Hölder's inequality in (21), we get

$$\begin{aligned}
\alpha\|Dw_\varepsilon\|_N^N &\le \tilde{C}\int_\Omega |w_\varepsilon||f|\,dx + (1+\theta)\int_\Omega \mu^{N-1}|w_\varepsilon|^N|f|\,dx + \\
&+ N\tilde{C}\int_\Omega |Dw_\varepsilon||g|\,dx + N(1+\theta)\int_\Omega \mu^{N-1}|Dw_\varepsilon||w_\varepsilon|^{N-1}|g|\,dx \le \\
&\le \tilde{C}\|f\|_1^{\frac{N-1}{N}}\left(\int_\Omega |w_\varepsilon|^N|f|\,dx\right)^{\frac{1}{N}} + (1+\theta)\mu^{N-1}\int_\Omega |w_\varepsilon|^N|f|\,dx + \\
&+ N\tilde{C}\|g\|_{\frac{N}{N-1}}\|Dw_\varepsilon\|_N + N(1+\theta)\mu^{N-1}\|Dw_\varepsilon\|_N\left(\int_\Omega |w_\varepsilon|^N|g|^{\frac{N}{N-1}}\,dx\right)^{\frac{N-1}{N}}
\end{aligned} \tag{23}$$

where $\tilde{C} = \tilde{C}(N-1,\theta)$ is the constant appearing in (22). Finally, Hardy-Littlewood inequality (3) and estimate (7) give

$$\begin{aligned}
\alpha\|Dw_\varepsilon\|_N^N &\le \tilde{C}\|f\|_1^{\frac{N-1}{N}}\frac{\|Dw_\varepsilon\|_N}{NC_N^{\frac{1}{N}}}\left(\int_0^{|\Omega|}(\log\frac{|\Omega|}{s})^{N-1}f^*(s)ds\right)^{1/N} + \\
&+ (1+\theta)\mu^{N-1}\frac{\|Dw_\varepsilon\|_N^N}{N^NC_N}\int_0^{|\Omega|}(\log\frac{|\Omega|}{s})^{N-1}f^*(s)\,ds + N\tilde{C}\|Dw_\varepsilon\|_N\|g\|_{\frac{N}{N-1}} + \\
&+ (1+\theta)\mu^{N-1}\frac{\|Dw_\varepsilon\|_N^N}{N^{N-2}C_N^{\frac{N-1}{N}}}\left(\int_0^{|\Omega|}(\log\frac{|\Omega|}{s})^{N-1}(|g|^*(s))^{\frac{N}{N-1}}ds\right)^{\frac{N-1}{N}} = \\
&= \tilde{C}\|f\|_1^{\frac{N-1}{N}}\frac{\|Dw_\varepsilon\|_N}{NC_N^{\frac{1}{N}}}\|f\|_{L(\log L)^{N-1}}^{\frac{1}{N}} + (1+\theta)\mu^{N-1}\frac{\|Dw_\varepsilon\|_N^N}{N^NC_N}\|f\|_{L(\log L)^{N-1}} + \\
&+ N\tilde{C}\|Dw_\varepsilon\|_N\|g\|_{\frac{N}{N-1}} + (1+\theta)\mu^{N-1}\frac{\|Dw_\varepsilon\|_N^N}{N^{N-2}C_N^{\frac{N-1}{N}}}\|g\|_{L^{\frac{N}{N-1}}(\log L)^{\frac{(N-1)^2}{N}}}.
\end{aligned} \tag{24}$$

By hypothesis, the quantity

$$\eta \equiv \alpha - \left(\frac{\|f\|_{L(\log L)^{N-1})}}{N^NC_N} + \frac{\|g\|_{L^{\frac{N}{N-1}}(\log L)^{\frac{(N-1)^2}{N}}}}{N^{N-2}C_N^{\frac{N-1}{N}}}\right)\mu^{N-1} \tag{25}$$

is positive. Then from (25), choosing for example $\theta = \frac{1}{2}(\frac{\eta}{\alpha-\eta}) \equiv \bar{\theta}$, we obtain

$$\|Dw_\varepsilon\|_N \leq \tag{26}$$

$$\leq \left\{\frac{2}{\eta}\tilde{C}(N-1,\bar{\theta})\big(N^{-1}C_N^{-\frac{1}{N}}\|f\|_1^{\frac{N-1}{N}}\|f\|^{\frac{1}{N}}_{L(\log L)^{N-1}} + N\|g\|_{\frac{N}{N-1}}\big)\right\}^{\frac{1}{N-1}}.$$

Because of the definition of w_ε, the above inequality states also that u_ε is a bounded sequence in $W_0^{1,N}(\Omega)$ and so, up to a subsequence, we can say

$$\begin{cases} w_\varepsilon \to w \text{ a.e. in } \Omega,\ u_\varepsilon \to u \text{ a.e. in } \Omega \\ w_\varepsilon \rightharpoonup w \text{ weakly in } W_0^{1,N}(\Omega),\ u_\varepsilon \rightharpoonup u \text{ weakly in } W_0^{1,N}(\Omega), \end{cases} \tag{27}$$

where

$$w = \frac{e^{\mu|u|}-1}{\mu} sign(u). \tag{28}$$

Step 3: Strong convergence of Du_ε in $(L^N(\Omega))^N$.

Let us define, for $k \geq 0$, the functions:

$$G_k(s) = \begin{cases} s-k & \text{if } s \geq k \\ 0 & \text{if } -k < s < k \\ s+k & \text{if } s \leq -k \end{cases} \tag{29}$$

$$T_k(s) = \begin{cases} k & \text{if } s \geq k \\ s & \text{if } -k < s < k \\ -k & \text{if } s \leq -k. \end{cases} \tag{30}$$

Obviously we have:

$$D(u_\varepsilon - u) = D(G_k(u_\varepsilon)) - D(G_k(u)) + D(T_k(u_\varepsilon)) - D(T_k(u)). \tag{31}$$

In order to show that Du_ε strongly converges in $(L^N(\Omega))^N$ we firstly prove that

$$\limsup_{\varepsilon\to 0}\int_\Omega |DG_k(u_\varepsilon)|^N\,dx \to 0 \text{ as } k \to \infty. \tag{32}$$

To this aim we can use the same arguments contained in [9] to say that

$$\alpha\int_\Omega |DG_k(u_\varepsilon)|^N\,dx \leq \int_\Omega G_k(u_\varepsilon)(1+\mu|w_\varepsilon|)^{N-1}f_\varepsilon\,dx + \tag{33}$$

$$+\int_\Omega DG_k(u_\varepsilon)(1+\mu|w_\varepsilon|)^{N-1}g_\varepsilon\,dx + \gamma\int_\Omega Du_\varepsilon(1+\mu|w_\varepsilon|)^{N-1}|G_k(u_\varepsilon)|g_\varepsilon\,dx \leq$$

$$\leq \int_\Omega |G_k(u_\varepsilon)|(1+\mu|w_\varepsilon|)^{N-1}|f_\varepsilon|\,dx + N\int_\Omega |DG_k(w_\varepsilon)|(1+\mu|w_\varepsilon|)^{N-1}|g_\varepsilon|\,dx.$$

Now we can apply Vitali's Theorem to all terms appearing on the right hand side of (33). Let us consider, for example, the last integral. If E is any measurable subset of Ω, using (13), we get

$$\int_E |DG_k(w_\varepsilon)|(1+\mu|w_\varepsilon|)^{N-1}|g_\varepsilon|\,dx \leq \tag{34}$$
$$\leq 2^{N-2}\Big(\int_E |Dw_\varepsilon||g|dx + \mu^{N-1}\int_E |Dw_\varepsilon||w_\varepsilon|^{N-1}|g|\,dx\Big).$$

Now the last inequality together with Hölder's inequality, Hardy-Littlewood inequality and Theorem 2.1 gives

$$\int_E |DG_k(w_\varepsilon)|(1+\mu|w_\varepsilon|)^{N-1}|g_\varepsilon|\,dx \leq \tag{35}$$
$$\leq 2^{N-2}\left(\|Dw_\varepsilon\|_N\Big(\int_0^{|E|}(|g|^*(s))^{\frac{N}{N-1}}\,ds\Big)^{\frac{N-1}{N}} + \right.$$
$$\left. + \mu^{N-1}\frac{\|Dw_\varepsilon\|_N^N}{N^{N-1}C_N^{\frac{N-1}{N}}}\Big(\int_0^{|E|}(\log\frac{|\Omega|}{s})^{N-1}(|g|^*(s))^{\frac{N}{N-1}}\Big)^{\frac{N-1}{N}}\,ds\right).$$

The equiintegrability of the sequence $\{|DG_k(w_\varepsilon)|(1+\mu|w_\varepsilon|)^{N-1}|g_\varepsilon|\}$ follows from (26), hypothesis (iv) and the inclusion relations in the Zygmund-spaces. We can treat analogously the other terms in (33) and so thanks to the Vitali's Theorem it holds

$$\alpha \limsup_{\varepsilon\to 0}\int_\Omega |DG_k(u_\varepsilon)|^N\,dx \leq \tag{36}$$
$$\leq \int_\Omega |G_k(u)|(1+\mu|w|)^{N-1}|f|\,dx + N\int_\Omega |DG_k(w)|(1+\mu|w|)^{N-1}|g|\,dx.$$

By the previous calculations it is also clear that the functions $|u|(1+\mu|w|)^{N-1}|f|$ and $|Dw|(1+\mu|w|)^{N-1}|g|$ belong to $L^1(\Omega)$. Then (32) follows from (36).

Finally, as in [9], one can prove

$$DT_k(u_\varepsilon) \to DT_k(u) \text{ strongly in } (L^N(\Omega))^N \tag{37}$$

for every fixed $k \geq 0$. Indeed (37) can be obtained by suitably adapting the arguments in [9], taking into account Theorem 2.1.

So step 3 is completely proved.

Now it is easy to show, passing to the limit in (15), that u is a solution of problem (8) (see [7]). □

Remark I. We observe that in general problem (8) does not admit a unique solution. To this aim it is sufficient to consider problem (8) when $N = 2$, $a(x, s, \zeta) = \zeta$, $f = 0$, $g = 0$, $H(x, s, \zeta) = |\zeta|^2$ and $\Omega = B_R = \{x \in \mathbb{R}^2 : |x| < R\}$:

$$\text{(38)} \qquad \begin{cases} -\Delta u = |Du|^2 \text{ in } \mathcal{D}'(B_R) \\ u \in H_0^1(B_R). \end{cases}$$

In this case Theorem 3.1 assures the existence of a solution u such that:

$$\text{(39)} \qquad w = e^{|u|} - 1 \in H_0^1(B_R).$$

A function that satisfies (38) as well as (39) is obviously $u = 0$. Furthermore (see also [13]) the following family of radial functions:

$$\text{(40)} \qquad \begin{aligned} \psi(x) = \psi(|x|) &= -\int_{|x|}^{R} \frac{1}{r \log r + Cr}\, dr = \\ &= \log \left| \frac{\log |x| + C}{\log R + C} \right|, \quad \forall C < -\log R, \end{aligned}$$

is a solution of problem (38), as a straightforward calculation shows. On the other hand $\psi(x)$ does not satisfy condition (39), indeed

$$\text{(41)} \qquad \int_{B_R} |D(e^{\psi(|x|)} - 1)|^2\, dx = \frac{2\pi}{(\log R + C)^2} \int_0^R \frac{1}{r}\, dr = +\infty.$$

Remark II. If hypothesis (10) is replaced by

$$\text{(42)} \qquad |H(x, s, \zeta)| \le \gamma |\zeta|^N,$$

the statement of Theorem 3.1 holds true with the same assumption on f and g.

For example if (42) holds, $\alpha = \gamma = 1$ and $g = 0$, then Theorem 3.1 states that the problem:

$$\text{(43)} \qquad \begin{cases} -\operatorname{div}(a(x, u, Du)) = H(x, u, Du) + f \text{ in } \mathcal{D}'(\Omega) \\ u \in W_0^{1,N}(\Omega) \end{cases}$$

admits a solution if f satisfies :

$$\|f\|_{L(\log L)^{N-1}} < (N-1)^{N-1} N^N C_N. \tag{44}$$

Under hypothesis (42) the existence of a bounded solution of problem (43) has been studied, for example, in [10] and [11]. In this last paper it is shown that if $f \in L^1(\Omega)$ and moreover f is such that

$$\Big[\int_0^{|\Omega|} \frac{1}{s}\Big(\int_0^s f^*(t)dt\Big)^{\frac{1}{N-1}} ds\Big]^{N-1} < (N-1)^{N-1} N^N C_N, \tag{45}$$

then problem (43) admits a solution in $W_0^{1,N}(\Omega) \cap L^\infty(\Omega)$.

We firstly observe that for $N = 2$ conditions (44) and (45) are the same.

If $N \geq 3$ assumption (45) is stronger than (44). Indeed it is easy to show that if the left hand side of (45) is finite then $\|f\|_{L(\log L)^{N-1}}$ is finite too and more precisely it holds

$$\|f\|_{L(\log L)^{N-1}} \leq (N-1)^{N-1}\Big[\int_0^{|\Omega|} \frac{1}{s}\Big(\int_0^s f^*(t)\,dt\Big)^{\frac{1}{N-1}} ds\Big]^{N-1}. \tag{46}$$

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