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DISCRETE MATHEMATICS, DISCRETE PHYSICS AND NUMERICAL METHODS

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Discrete mathematics has been neglected for a long time. It has been put in the shade by the striking success of continuous mathematics in the last two centuries, mainly because continuous models in physics proved very reliable, but also because of the greater difficulty in dealing with it. This perspective has been rapidly changing in the last years owing to the needs of the numerical analysis and, more recently, of the so called discrete physics. In this paper, starting from some sentences of Fichera about discrete and continuous world, we shall present some considerations about discrete phenomena which arise when designing numerical methods or discrete models for some classical physical problems.

1. Introduction

Our starting point is the reproduction of a few sentences of the great mathematician G. Fichera, which testify the uneasy sense of mathematicians regarding the discrete world, for a long time almost absolutely neglected.

> *"...I will not debate here whether the continuum truly exists or it is rather an illusion of our senses. However the continuum is a comfortable scheme and is surely very convenient to shape in it the*

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world of physics and geometry. This situation then arises: the real number is a mental abstraction, whose definition has often provoked, especially among the logicians, not few perplexities; however it can be approximated, by defect and excess, with particular rational numbers, the only ones that constitute concrete truth for human beings."

"The real number (in spite of its name!) is the aspiration, the ideal entity that allows to create an illusion of thoroughness and perfection."

"The decimal number is the truth that is allowed to us, that can approach the ideal, never catching up with it..."

"After all, isn't this the usual human condition?"

The last remark, considering that rational numbers are a slight minority with respect to real numbers, expresses more than the dismay of the scientist, forced to base his aspiration to knowledge on such (vanishing) bases.

The increasing interest in discrete mathematics essentially resides both in the rapid developments of numerical analysis in the computer age and in the increasing needs of *discreteness* in physical models.

Concerning numerical analysis, it uses discrete mathematics, but it is bound to remain in the furrow of continuous mathematics, having the approximation as objective. Said differently, numerical analyst is like a traveller who tries to get an idea of the immense regions crossed by observing the limited landscape visible from the window of his train.

Regarding physics, a debated (and not yet resolved) question of the last century was whether some aspects of physics could be better represented by discrete models, consequence of the assumption of non-continuous space and/or time. At the moment there is an increasing number of physicists which got interested in the so called discrete physics.

The efforts of both numerical analysts and discrete physicists, so far independent, seems to converge on a common field of interest in discrete mathematics. To quote a few examples, we remember the discrete approach operated by Caldirola [1] to the problem of the classical radiating electron. He showed that all the paradoxes existing in the continuous classical models of the radiating electron disappear when discrete equations are used instead. His arguments are very much resembling to the those used in numerical analysis when dealing with stability of numerical methods for differential equations (see [5]). A more recent example is the disclosure of the algebraic structure typical of quantum theories when non commutative calculus, naturally needed in discrete physics, is introduced [9].

In the following sections, with the help of some simple examples, we shall have a look to the new world which opens up when we give up to the ideal of continuity and we undertake the voyage into the richer and varied world of discrete mathematics.

2. Past and Future

The first question we shall introduce is the relation between past and future in conservative problems. The importance of this question resides in its strict relation with the causality principle, which is one of the basic stones of physics. For sake of simplicity, we shall refer to the harmonic oscillator problem, i.e.

$$\frac{d}{dt}\begin{pmatrix} y \\ x \end{pmatrix} = \omega J \begin{pmatrix} y \\ x \end{pmatrix} \tag{1}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is known that in the phase space the solution lies on a circle of radius dependent on the initial conditions. It is natural to consider that every point of the circle belongs to the same solution not only for a single value of time but for infinite values, the solution being periodic.

In the continuous case of problem (1), the two functions

$$y_1(t) = e^{\omega J t} y_0, \qquad y_2(t) = e^{-\omega J t} y_0$$

are future and past solutions respectively. The same point may belong to the future ($t_f > 0$) or to the past($t_p < 0$), provided that

$$t_f + t_p = k\frac{2\pi}{\omega}, \qquad k \text{ integer.}$$

In the discrete world this is, in general, no longer true. To see this let us discretize (1): this is not a trivial step, being it possible to associate with (1) infinitely many discrete approximations. Among the simplest methods of discretization we discard those of *Euler* (both implicit and explicit), since they do not even provide bounded solution (for $t \to \pm\infty$). We also discard the *midpoint method* which, while providing a bounded solution, does not guarantee that it remains on a circle. We instead consider the trapezoidal method:

$$y_{n+1} = (I - \frac{h}{2}\omega J)^{-1}(I + \frac{h}{2}\omega J)y_n \equiv R(h\omega, J)y_n. \tag{2}$$

The properties of $R(h\omega, J)$ considerably reflect those of $e^{h\omega J}$. In fact, they are both orthogonal and symplectic matrices. The orthogonality guarantees that the solutions are on the same circle. The symplecticity guarantees the conservation of the areas in the phase plane. In the linear case both these properties guarantee the conservation of the energy.

What do the two maps differ for? Consider once again the past and future solutions passing through y_0. Set $R_1 = R(h\omega, J)$ and $R_2 = R(-h\omega, J)$. We look for N such that

$$R_1^N y_0 = R_2^N y_0. \tag{3}$$

Obviously we are looking for periodic solutions. It should not be difficult, considering that in the continuous case all of them are periodic. By posing

$$\sin\phi = \frac{h\omega}{1+\left(\frac{h\omega}{2}\right)^2},$$

and observing that condition (3) is equivalent to

$$R_1^{2N} y_0 = y_0,$$

where

$$R_1^{2N} = \begin{pmatrix} \cos(2N\phi) & -\sin(2N\phi) \\ \sin(2N\phi) & \cos(2N\phi) \end{pmatrix},$$

we get

$$(2N)\phi = 2k\pi \Longrightarrow \phi_{kN} = \frac{k\pi}{N}, \quad k = 0, 1, \ldots, N.$$

Only for such values, will the corresponding points be both on the past and future solutions. For example, plugging $\omega = 1$, $\phi = \pi/2$, we have $h = 2$. This is one of the few values exactly representable on the computer, the other being irrational. Correspondingly, the solution of period 4 is obtained.

From the above discussion, the following results may be deduced:

- Fixing a point on the circumference, infinitely many values of k and N can be found such that his neighborhoods contain points of periodic solutions of different periods.

- Contrary to what happens in the continuous case, where fixing the period of oscillation T, there is a single frequency associated with it, here, fixing the period Nh, there are N possible frequencies. We will later see the consequences of this phenomenon.

- The presence of infinitely many periodic solutions which are dense in a region of the phase space, is one of the conditions characterizing a chaotic behavior. This would also require the topological transitivity condition and the exponential divergence of neighboring solutions. The map (2) satisfies the former condition but, due to linearity, it does not satisfy the sensitivity condition: we shall denote the present situation as *pre-chaotic* (this relation with the chaotic behavior will be exploited below).

- For a generic value of h and therefore of ϕ, a partition on the points of the circle is created:

  - points belonging to the future;
  - points belonging to the past.

  The latter points are obtained by computing the antecedents of y_0 by means of the inverse map $R^{-1}(h\omega, J)$ which, due to the symmetry of the trapezoidal method, coincides with $R(-h\omega, J)$. When the solution is not periodic, the two sets have no common points, but they *interlace*. This property is absent in the continuous case, where interlacing of solutions is possible only for higher dimensions (greater than 2).

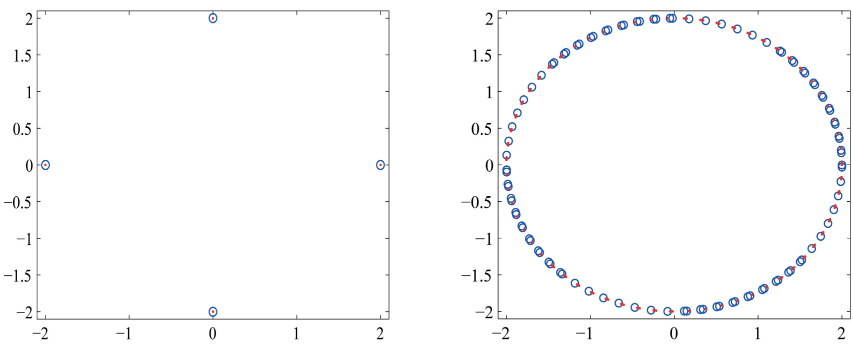

Figure 1: Period 4 solution (left, $h = 2, \omega = 1$), and a quasiperiodic solution(right, $h = 2.05, \omega = 1$) of (2). The circles and dots represent the future and the past solutions obtained by considering the first 100 forward and backward images of $y_0 = [2, 0]^T$, respectively. The right picture shows the interlacing phenomenon.

2.1. Relation to the logistic map and chaotic behavior

Similar considerations hold in one dimension, starting from the harmonic oscillator

$$y'' + \omega^2 y = 0,$$

and discretizing it with centered differences:

$$y_{m+2} - 2\gamma y_{m+1} + y_m = 0 \tag{4}$$

where

$$\gamma = 1 - \frac{\omega^2 h^2}{2}.$$

Stable solutions for $m \to \pm\infty$ are obtained for $\gamma \in (-1,1)$. Explicit forms of the solutions can be obtained in terms of Chebishev polynomials T_n. For $y_0 = 1$, $y_1 = \gamma$,

$$y_n = T_n(\gamma), \tag{5}$$

where γ is related to the initial point y_0.

As in the previous case, periodic solutions are obtained only for special values of γ (they are the same as above, except for a factor 2). Moreover, as before, by fixing the period Nh, there are N possible frequencies associated with it, while in the continuous case there is only one. It is interesting to observe that the continuous and discrete linear pendulums share the distribution function (see [6] for details)

$$P(x) = \frac{1}{\pi}(1-x^2)^{-1/2}.$$

Suppose now to take the first, the second, the fourth, ..., the 2^j-th points of the solution (5). The obtained subset of points are related to the solution of the famous logistic equation (the emblem of chaotic behavior!)

$$x_{n+1} = 4x_n(1-x_n).$$

After the transformation $x_n = \frac{1-z_n}{2}$, it becomes

$$z_{n+1} = 2z_n^2 - 1, \tag{6}$$

for which the interval $[-1,1]$ is invariant. The explicit solutions of (6) are known and expressible in the form

$$z_n = T_{2^n}(\gamma).$$

The solutions are therefore subsets of those of the discrete oscillator. It is interesting to observe that the probability distribution of such solution is, apart from a shift in the definition interval, identical to that of the harmonic oscillator (see [14, pag. 478]). Also in this case there are infinite periodic solutions obtained by varying the initial condition. The solutions are very sensitive to perturbations.

In order to see this, let us rewrite the map as

$$\begin{cases} y_{n+1} = 2y_n, & 0 \le y_n \le \frac{1}{2}, \\ y_{n+1} = 2(1-y_n), & \frac{1}{2} \le y_n \le 1, \end{cases} \tag{7}$$

obtained by means of the transformation $z = \cos y_n$. As expected, (7) has infinitely many periodic solutions, depending on the initial condition. For example there is an attractive period two solution for $y_0 = 0.05$.

To the above dynamical system we associate the equivalent (in infinite precision) system, where α is any scalar parameter,

$$\begin{cases} y_{n+1} = 2y_n, & 0 \le y_n \le \frac{1}{2}, \\ y_{n+1} = 2\dfrac{(\alpha - \alpha y_n)}{\alpha}, & \frac{1}{2} \le y_n \le 1. \end{cases} \tag{8}$$

It is obvious that the both (7) and (8) are identical in infinite precision, but they differ crucially in finite precision. Here we do not intend to discuss about the numeration system. We rather intend to use the finite precision to generate solutions whose initial points are very close to the precise ones, in order to show the sensitivity of the dynamical system to slight variations of data. The variation here is obtained by imposing the computer to execute operations in a different order considering that finite precision arithmetic is not commutative. In Figure 2.1 we report the two solution obtained starting from the same point but with different values of α. While the solutions are independent on such parameter in

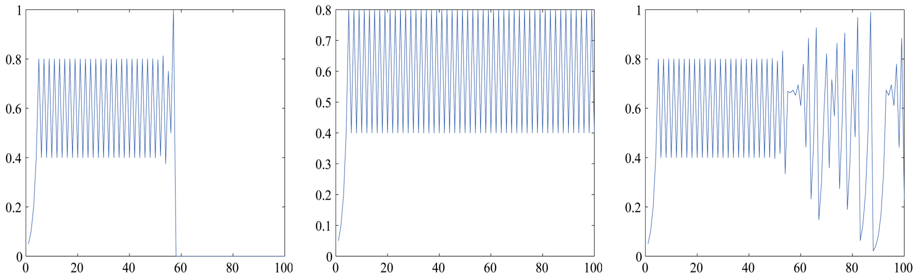

Figure 2: Due to the use of finite arithmetic, different choices of the parameter α in (8) lead to different asymptotic behavior. The solution tends to the equilibrium (left picture, $\alpha = 1,\ y_0 = 0.05$), or it tends to period 2 (central picture, $\alpha = \pi,\ y_0 = 0.05$), or it is aperiodic (right picture, $\alpha = 0.19,\ y_0 = 0.05$)

infinite precision, they differ in finite precision (paradoxically the exact solution is that corresponding to $\alpha = \pi$). Different solutions are obtained for different values of α. The sensitivity with respect to the initial data is typical of chaotic regime, which in the continuous world appears at higher dimension. In the discrete case it may appear in one dimension, as is well known and as shown by the previous example. The reason is that the uniqueness theorem, which holds true in both cases, *in the discrete case does not prevent two distinct solutions to* *interlace without having common points*. This make possible the existence, in any neighborhood of a point of a solution the coexistence of infinite points of different solutions. While this seems not important from a numerical point of view, it becomes important in physics owing to the unexpected new phenomena that arise both in the linear case and in the nonlinear case. We shall see some of them in the next section.

3. Physical consequences

3.1. Causality principle

In the continuous case there is some ambiguity in defining Green's functions, thus creating problems with the causality principle.

Instances are:

- advanced and retarded potentials in electromagnetism;
- advanced and retarded propagators in field theory;
- Dirac equation for the dynamic of the classical (radiating) electron.

In the discrete case such ambiguity disappears. In order to clarify this point we shall refer once again to the harmonic oscillator, following the exposition in [15, p. 351-356]. The response to an impulsive force of the linear oscillator with friction, i.e. the solution of

$$y'' + \beta y' + \omega_0 y = -\delta(t - t'), \quad \beta > 0,$$

where $\delta(t - t')$ is the Dirac impulsive function, is given by

$$G(t - t') = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\omega(t-t')}}{\omega^2 - \omega_0^2 - i\beta\omega} d\omega.$$

The integral in the above formula can be easily evaluated by the residue theorem, by taking into account the poles in the complex plane of the integrand function. The poles, in this case, are well positioned in the upper part of the complex plane. This implies that $G(t - t')$ is zero for $t < t'$, according to the causality principle. This is not surprising since in the continuous case dissipation naturally defines a partition between past and future on each trajectory. The problem arises in the case $\beta = 0$ since the evaluation of the integral is not unambiguous, being the poles positioned on the real axis. In fact, in this limit case one usually applies the residue theorem moving slightly upwards or downwards the poles. This double possibility allows the existence of both causal and anti-causal solutions. But, moving upward or downward the poles is equivalent to the introduction of a small dissipation ($\beta > 0$) or creation of energy ($\beta < 0$) thus breaking the symmetry and introducing a partition between past and future.

This ambiguity disappears in the discrete case, where, as we have seen, there is always (except for a vanishing set of points) a distinction between past and future solutions.

This has important consequences. For example it explains the success of the discrete model of the classical radiating electron, as proposed by Caldirola ([1],[2],[5])

3.2. Soliton-like solutions

In this section we introduce a different application of the result presented in Section 2. It concerns the soliton-like solution of the discrete wave equation. To begin with, let us discretize, with respect to both variables, the continuous wave equation

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = 0. \tag{9}$$

We obtain

$$u_{n,j+1} - 2u_{n,j} + u_{n,j-1} = q^2(u_{n+1,j} - 2u_{n,j} + u_{n-1,j}), \tag{10}$$

where $q = v\frac{\tau}{\sigma}$, being τ e σ the time and space elements, respectively. Index n refers to space and j to time. By separation of variables, i.e. by posing $u(n, j) = \xi_n \eta_j$, we obtain

$$\frac{1}{q^2} \frac{\eta_{j+1} - 2\eta_j + \eta_{j-1}}{\eta_j} = \frac{\xi_{n+1} - 2\xi_n + \xi_{n-1}}{\xi_j} = -2\gamma^2,$$

and hence

$$\begin{aligned} \eta_{j+1} - 2(1-\gamma^2 q^2)\eta_j + \eta_{j-1} &= 0, \qquad (11) \\ \xi_{n+1} - 2(1-\gamma^2)\xi_n + \xi_{n-1}\xi_j &= 0. \qquad (12) \end{aligned}$$

Both equations are of the form already considered in Section 2. In order to have bounded solutions both in space and time, it is necessary to assume

$$|1 - q^2\gamma^2| < 1, \qquad |1-\gamma^2| < 1,$$

which implies that $q \leq 1$. By posing

$$1 - q^2\gamma^2 = \cos\omega\tau \quad 1 - \gamma^2 = \cos k\sigma.$$

It follows that the solutions are combination of plane waves such as

$$Ae^{i(\pm kn\sigma \pm \omega j\tau)},$$

provided that (dispersion relations)

$$\left|\sin\frac{\omega\tau}{2}\right| = q\left|\sin\frac{k\sigma}{2}\right|$$

It is worth nothing that in the continuous case dispersion relations only appears in higher dimensions.

The case $q > 1$ is to be rejected since it does not allow the existence of solutions for generic initial spatial condition ([6]). We then assume $q \leq 1$. Accordingly, the dispersion relations assume the form

$$\omega(k) = \frac{2}{\tau}\sin^{-1}\left(q\left|\sin\frac{k\sigma}{2}\right|\right),$$

from which the phase and the group velocities are derived;

$$\begin{aligned} u_{ph} &= \left|\frac{\omega}{k}\right| = \frac{\sigma}{\tau}\left|\frac{\sin^{-1} q\sin(k\sigma/2)}{k\sigma/2}\right| \\ u_{gr} &= \left|\frac{d\omega}{dk}\right| = \frac{\sigma}{\tau}q\frac{\cos(k\sigma/2)}{\sqrt{1-q^2\sin^2(k\sigma/2)}}. \end{aligned}$$

Both velocities assume a common value only when $q = 1$, i.e. $\frac{\sigma}{\tau}$, which is the maximum allowed. In correspondence of such value there is not dispersion.

In order to show a consequence of this result, we add to the equation the boundary conditions

$$u_{0,j} = u_{N,j} = 0.$$

The boundary value problem has the following complete set of eigenfunction

$$u_{n,j}^{\ell} = Ae^{i\omega_\ell j\tau}\sin\frac{\ell n\pi}{2N}, \quad \ell = 1,2,\dots N-1,$$

by means of which all the solutions of the problem can be represented. For example, we can get the solutions of period $M\tau$

$$\omega_\ell\tau = \frac{2s\pi}{M}, \quad s = 1,2,\dots,S = \begin{cases} \mathrm{int}\frac{M}{2} & M \quad \text{odd} \\ \frac{M}{2}-1 & M \quad \text{even} \end{cases}$$

with dispersion relations

$$\sin^2\frac{s\pi}{M} = q^2\sin^2\frac{\ell\pi}{2N}.$$

Assuming $q = 1$, $M = 2N$, $s = \ell$, every solution has period $2N\tau$. An initial combination of the eigenfunctions, reproducing a pulse, will move forward and backward maintaining his form because of the absence of dispersion. By continuity when $q \simeq 1$ the dispersion phenomenon is almost absent and the solutions posses the typical features of solitons. This is shown in Figure 3.2. and one

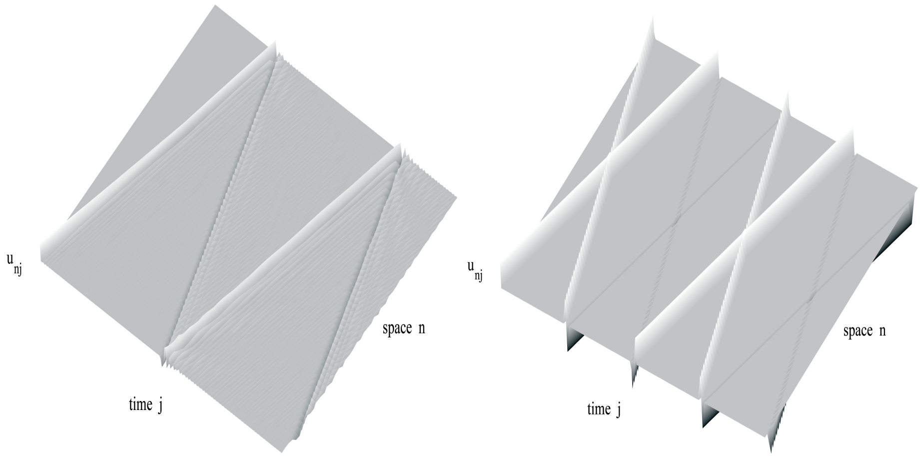

Figure 3: Left picture: a pulse (soliton-like solution) traveling forward and backward in a field with a small dispersion factor ($q = .99$). Right picture: collision between two pulses in absence of dispersion ($q = 1$).

gets $u_{ph} = u_{gr} = \sigma/\tau$, for $q = 1$. This limit case is interesting in many respects. As $q \to 1$, all the waves assume the same velocity σ/τ, which turns out to be the maximum allowed in that medium (the existence of a finite limit speed is a direct consequence of the discrete nature of the model).

The relation $q = 1$ in correspondence of which the dispersion disappears, induces interesting considerations of physical nature. For example, if we consider any two values of σ and τ such that $\frac{\sigma}{\tau} = c$, then $q = 1$ implies that $v = c$. It is interesting to observe that the Planck universal constants, $\sigma = \hbar/Mc$, $\tau = \hbar/Mc^2$, with $M = (\hbar c/G)^{1/2}$, where G is the gravitational constant, are two of such values.

3.3. Relations to Numerical Analysis

We have said in the introduction that the physical observations made above and other similar which can be derived for other kind of equations, for example the diffusion equations including the Schrödinger equation [9] have a lot in common with the theory of stability and convergence of numerical methods. As matter of fact, in such context the ratio $q = v\frac{\sigma^2}{\tau^2}$ ($q = D\frac{\tau}{\sigma^2}$, D the diffusion constant, in the parabolic case) are the key parameters for the convergence of the discrete solution to the continuous one.

4. Nonlinear case

The examples discussed in the previous section, except for the logistic equation, were linear. They already showed substantial differences between continuous and discrete solutions. As expected, for nonlinear equations these differences become are even more profound. This is confirmed by the analysis of the nonlinear pendulum, described by

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = J\begin{pmatrix} \sin(x) \\ y \end{pmatrix} \tag{13}$$

It is known that the energy $H_c(x,y) = \frac{1}{2}y^2 + 1 - \cos(x)$ is a constant of the motion. In fact, given a solution $(x(t),y(t))^T$ of (13), multiplying both sides of (13) by $(\sin(x(t)),y(t))$ and observing that $z^t Jz = 0$ for any vector z, yield

$$0 = (\sin(x(t)),y(t))\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \dot{H}_c(x(t),y(t)).$$

Defining methods which guarantee that the energy function of a Hamiltonian (or conservative) problem is preserved in the discrete case continues to be one of most challenging problems of Numerical Analysis. We do not examine here this issue. Rather, following the analogous steps done in the continuous case, we briefly discuss what happens when the discretization is carried out by the trapezoidal method which, as we have seen, has done a very good job in the linear case. The discrete nonlinear pendulum equation assumes the form

$$\begin{pmatrix} x_{n+1} - x_n \\ y_{n+1} - y_n \end{pmatrix} = \frac{h}{2}J\begin{pmatrix} \sin(x_n) + \sin(x_{n+1}) \\ y_n + y_{n+1} \end{pmatrix}$$

Multiplying on the left by $(\sin(x_n) + \sin(x_{n+1}), y_n + y_{n+1})$ gives

$$(\sin(x_n) + \sin(x_{n+1}), y_n + y_{n+1})\begin{pmatrix} x_{n+1} - x_n \\ y_{n+1} - y_n \end{pmatrix} = 0,$$

which, after posing $\sigma_{n,n+1} = x_{n+1}\sin x_n - x_n \sin x_{n+1}$, becomes

$$\frac{1}{2}y_{n+1}^2 + \frac{1}{2}x_{n+1}\sin x_{n+1} + \frac{1}{2}\sigma_{n,n+1} = \frac{1}{2}y_n^2 + \frac{1}{2}x_n \sin x_n. \tag{14}$$

There is no conservation in the classical sense, since $\sigma_{n,n+1} \neq 0$ unless we take the linear approximation $\sin x_n \simeq x_n$. In order to interpret (14) we introduce the *basic* Hamiltonian function

$$H_b(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x\sin(x) \equiv \frac{1}{2}y^2 + \frac{1}{2}x^2\,\mathrm{sinc}(x), \quad \mathrm{sinc}(x) = \frac{\sin(x)}{x}, \tag{15}$$

which depends only on the point (x,y) in the phase plane, and the *volatile* term $R_n = 1/2\sum_{k=0}^{n-1}(\sigma_{k,k+1})$ which instead depends on the history of the system, from time t_0 to time t_n. The *discrete* Hamiltonian function is then

$$H_d(x_n,y_n) = H_b(x_n,y_n) + R_n.$$

In the paper [7] it has been shown that R_n is related to the impossibility to extend univocally to discrete world the usual continuous vector calculus for generic mesh sizes. As a matter of fact, it can be proved that, on solutions of period N (whose existence has been proven in [8]), R_n is in turn periodic and vanishes after each period.

However, periodic solutions only appear for exceptional values of the parameter h (see [8]): as for the continuous case, the corresponding points are simultaneously points of the past and of the future. For generic values of h, future solutions and past solutions have no common points apart, of course, the initial one.

Considering that on a solution of period N the volatile part of the Hamiltonian vanishes after each period, it follows that

$$H_d(x_{kN},y_{kN}) = H_b(x_{kN},y_{kN}) = \frac{1}{2}(y_0^2 + x_0\sin x_0) \equiv E$$

hence the discrete energy function remains constant after each period. Since it can be easily proved that $H_b(x,y) \leq H_c(x,y)$, this implies that the discrete periodic solutions have less energy than the corresponding continuous ones.

The quantity E is the constant value of the energy of the system on periodic solutions, assumed at times $t_k = kN$. If the initial value x_0 of x is zero, this is also the constant value of the continuous energy. In the following, in order to better understand how the continuous and discrete systems are related, let us assume $x_0 = 0$. From what said above one has

$$\begin{aligned} &H_b(x_n,y_n) = E - R_n, \\ &H_c(x_n,y_n) - E + E - H_b(x_n,y_n) \geq 0, \end{aligned}$$

and hence

$$E - H_c(x_n,y_n) \leq R_n,$$

from which it follows that the difference between the constant value E and the values assumed by the continuous Hamiltonian function on the discrete solution is bounded by R_n. Denoting by $(x(t_n),y(t_n))$ the point of the continuous solution at time t_n, the above formula can be written as

$$H_c(x(t_n),y(t_n)) - H_c(x_n,y_n) \leq R_n. \tag{16}$$

Let us consider a periodic solution with period N. As we have seen, $R_n = 0$ for $n = 0, N, 2N, \dots$, and on such points, the discrete Hamiltonian function assumes the same value as the continuous one. In the other points, the excess of energy is stored in the volatile part of the Hamiltonian and can be released or absorbed. The radiation or absorption occurs only by discrete quantities, quanta of energy, defined by the terms $\sigma_{n,n+1}$ which can be either positive or negative. The cumulative term R_n takes care of the balance of the energy radiated or absorbed. This quantity may become large and, summed to the basic energy, may yield a overall energy larger than the energy threshold under which the orbits remain closed.

In Figure 4 we report the solutions for two different values of h, one ($h = 2.09$) smaller and the other ($h = 2.1$) larger than the value corresponding to the orbit of period 4. The expression of the basic energy given in (15) looks like the

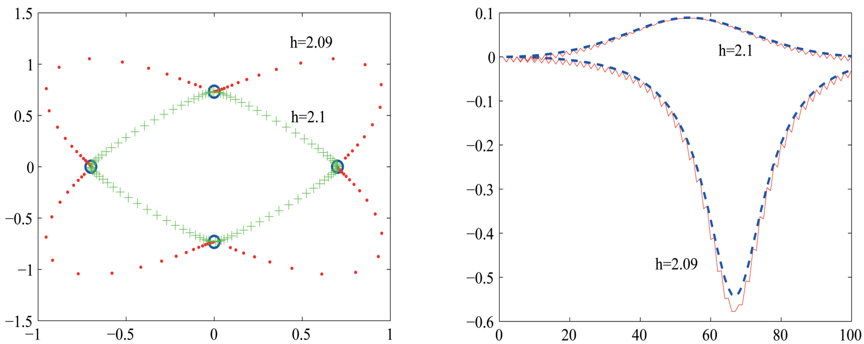

Figure 4: Left plot: orbit of period 4 (circles) and two neighboring solutions (dots and plus) obtained by a slight change of the stepsize h. Right plot: $E - H_c(x_n, y_n)$ (solid line) and R_n (dash line).

continuous one. Actually it is the same when $\operatorname{sinc}(x) = 1$, which is true when $x = 0$. The effect of the nonlinearity is then to multiply the space variable by the factor $\operatorname{sinc}(x)$ which is less than one for bounded solutions. The curve defined by $H_b(x, y) = \text{const}$ are closed but differ from circles.

We finally emphasize that in deriving the above physical interpretation of the discrete model, a crucial role was assumed by the discretization method. If instead of the trapezoidal method we had chosen the implicit midpoint method, which apart from being symmetric is also symplectic, we would have derived the following analogue of (14):

$$\frac{1}{2}y_{n+1}^2 + \sin\left(\frac{x_n + x_{n+1}}{2}\right)(x_{n+1} - x_n) = \frac{1}{2}y_n^2,$$

for which the basic and volatile parts of the Hamiltonian function are no longer explicitly accessible.

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