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CONVERGENCE IN GENERALIZED SCHATTEN CLASSES OF OPERATORS

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In this paper we give definitions for several generalized Schatten classes of operators and prove that several properties from the classes S_p can be generalized to them. The main goal is characterization of convergence in these classes. We also will prove that they are quasi-Banach spaces.

1. Introduction.

Let H be a Hilbert space and consider $B(H, H)$, the algebra of bounded linear operators $T : H \to H$. Let $T \in B(H, H)$ be a compact operator. Then the numbers

$$\alpha_n(T) = \inf_{x_1,\dots,x_{n-1}\in H} \sup_{\|x\|=1,(x,x_1)=\dots=(x,x_{n-1})=0} \|Tx\|$$

and

$$\lambda_n(T) = \inf_{\text{rank}(S)<n} \|T - S\|$$

both coincide with the n-th s-number of T (which is the n-th eigenvalue of $(T^*T)^{\frac{1}{2}}$) ([3]). The Schatten class of operators S_p ($1 \le p < \infty$) is then defined (see [3]) by

$$S_p = \{T \in B(H, H) : \{\lambda_n(T)\}_{n=1}^{\infty} \in l_p\}.$$

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This is a Banach algebra with the norm

$$\|T\|_{S_p} = \|\{\lambda_n(T)\}_{n=1}^{\infty}\|_{l_p}.$$

In this paper we give definitions for several generalized Schatten classes of operators and prove that several properties from the classes S_p can be generalized to them. The main goal is characterization of convergence in these classes. We also will prove that they are quasi-Banach spaces.

2. Generalized Schatten classes of operators.

Let X, Y be two Banach spaces. Denote by $\boldsymbol{B}(X, Y)$ the space of bounded linear operators $T : X \to Y$, by $K(X, Y)$ the subspace of compact operators and by $F(X, Y)$ the closure of the space of finite rank bounded operators. Then $T \in F(X, Y)$ if and only if $\lim_{n\to\infty} \delta_n(T) = 0$, where

$$\delta_n(T) = \inf_{\mathrm{rank}(S)\leq n,\ S\in \boldsymbol{B}(X,Y)} \|T - S\|$$

is the n-th linear width of T (see [2]). On the other hand, if

$$d^n(T) = \inf_{L_1,\ldots,L_n\in X^*} \sup_{\|x\|=1, L_1x=\ldots=L_nx=0} \|Tx\|;\ d^0(T) = \|T\|$$

is the n-th width of T in the sense of Gelfand, it is well known that $T \in K(X, Y)$ if and only if $\lim_{n\to\infty} d^n(T) = 0$ (see [2]).

Observe that $d^n(T)$ generalizes $\alpha_{n+1}(T)$ (use the Riesz-Fischer representation theorem) and $\delta_n(T)$ generalizes $\lambda_{n+1}(T)$.

Definition 2.1. *Let X, Y be Banach spaces and let $\beta = \{b_n\}$ be a sequence of non negative real numbers. We define*

$$\textstyle\sum_p^{\beta}(X, Y) = \left\{T \in \boldsymbol{B}(X, Y) : \|T\|_{\Sigma_p^{\beta}(X,Y)} = \|\{b_n d^n(T)\}_{n=0}^{\infty}\|_{l_p} < \infty\right\}$$
$$S_p^{\beta}(X, Y) = \left\{T \in \boldsymbol{B}(X, Y) : \|T\|_{S_p^{\beta}(X,Y)} = \|\{b_n \delta_n(T)\}_{n=0}^{\infty}\|_{l_p} < \infty\right\}$$

Remark 2.1. *In what follows we will use the following notation: $\Sigma_p^{\beta}(X)$ denotes the set $\Sigma_p^{\beta}(X, X)$ and $S_p^{\beta}(X)$ denotes the set $S_p^{\beta}(X, X)$.*

Now we list several well known properties of the sequences $\{d^n(T)\}_{n=1}^{\infty}$ and $\{\delta_n(T)\}_{n=1}^{\infty}$:

Proposition 2.1. *Let $T \in \boldsymbol{B}(X,X)$. Then the sequence $\{d^n(T)\}_{n=1}^{\infty}$ is non-increasing. Furthermore $d^n(T) \leq \delta_n(T)$ for all n.*

Proof. See [2]. $\square$

Proposition 2.2. *Let $T_1, T_2 \in \boldsymbol{B}(X,Y)$. Then*

(1) $d^{n+m}(T_1+T_2) \leq d^n(T_1) + d^m(T_2)$

(2) $|d^n(T_1) - d^n(T_2)| \leq \|T_1 - T_2\|$

(3) $\delta_{n+m}(T_1+T_2) \leq \delta_n(T_1) + \delta_m(T_2)$

(4) $|\delta_n(T_1) - \delta_n(T_2)| \leq \|T_1 - T_2\|$

for all $n, m \in \mathbb{N}$. Furthermore, if $T_1 \in \boldsymbol{B}(Y,Z), T_2 \in \boldsymbol{B}(X,Y)$, and $T_3 \in \boldsymbol{B}(W,X)$ then

(5) $d^{n+m}(T_1T_2) \leq d^n(T_1)d^m(T_2)$

(6) $d^n(T_1T_2T_3) \leq \|T_1\|d^n(T_2)\|T_3\|$

(7) $\delta_n(T_1T_2T_3) \leq \|T_1\|\delta_n(T_2)\|T_3\|$

Proof. See [1]. $\square$

Corollary 2.3. *Let $\beta = (b_n) \subset (0,\infty)$ such that $\sup_{n\in\mathbb{N}}\{\frac{\max\{b_{2n},b_{2n+1}\}}{b_n}\} < \infty$. Then $\big(\Sigma_p^{\beta}(X,Y), \|\cdot\|_{\Sigma_p^{\beta}(X,Y)}\big)$ and $\big(S_p^{\beta}(X,Y), \|\cdot\|_{S_p^{\beta}(X,Y)}\big)$ are quasinormed spaces.*

Proof. Let $\sup_{n\in N}\{\frac{\max\{b_{2n},b_{2n+1}\}}{b_n}\} = C < \infty$. Then

$$\Big[\sum_{n=0}^{\infty} b_{2n}^p d^{2n}(T_1+T_2)^p\Big]^{\frac{1}{p}} \leq \Big[\sum_{n=0}^{\infty} C^p b_n^p (d^n(T_1)+d^n(T_2))^p\Big]^{\frac{1}{p}}$$

$$\leq C\Big[\Big(\sum_{n=0}^{\infty} b_n^p d^n(T_1)^p\Big)^{\frac{1}{p}} + \Big(\sum_{n=0}^{\infty} b_n^p d^n(T_2)^p\Big)^{\frac{1}{p}}\Big]$$

$$= C\Big(\|T_1\|_{\Sigma_p^{\beta}(X,Y)} + \|T_2\|_{\Sigma_p^{\beta}(X,Y)}\Big)$$

and

$$\Big[\sum_{n=0}^{\infty} b_{2n+1}^p d^{2n+1}(T_1+T_2)^p\Big]^{\frac{1}{p}} \leq C\Big(\|T_1\|_{\Sigma_p^{\beta}(X,Y)} + \|T_2\|_{\Sigma_p^{\beta}(X,Y)}\Big).$$

Hence

$$\|T_1+T_2\|_{\Sigma_p^{\beta}(X,Y)}^p \leq 2C^p\Big(\|T_1\|_{\Sigma_p^{\beta}(X,Y)} + \|T_2\|_{\Sigma_p^{\beta}(X,Y)}\Big)^p$$

and

$$\|T_1+T_2\|_{\Sigma_p^{\beta}(X,Y)} \leq 2^{\frac{1}{p}}C\Big(\|T_1\|_{\Sigma_p^{\beta}(X,Y)} + \|T_2\|_{\Sigma_p^{\beta}(X,Y)}\Big)$$

so that $\|\cdot\|_{\Sigma_p^\beta(X,Y)}$ is a quasinorm. The same arguments prove that $\|\cdot\|_{\Sigma_p^\beta(X,Y)}$ is also a quasinorm. □

Corollary 2.4. *$\left(\Sigma_p^\beta(X), \|\cdot\|_{\Sigma_p^\beta(X)}\right)$ and $\left(S_p^\beta(X), \|\cdot\|_{S_p^\beta(X)}\right)$ are algebras. Furthermore, if $b_0 > 0$,*

$$\|T_1T_2\|_{S_p^\beta(X)} \le \frac{1}{b_0}\|T_1\|_{S_p^\beta(X)}\|T_2\|_{S_p^\beta(X)}$$

$$\|T_1T_2\|_{\Sigma_p^\beta(X)} \le \frac{1}{b_0}\|T_1\|_{\Sigma_p^\beta(X)}\|T_2\|_{\Sigma_p^\beta(X)}.$$

Proof. Let $T_1, T_2 \in S_p^\beta(X)$. Then

$$\|T_1T_2\|_{S_p^\beta(X)} = \|\{b_n\delta_n(T_1T_2)\}_{n=0}^\infty\|_{l_p}$$
$$\le \|\{b_n\|T_1\|\delta_n(T_2)\}_{n=0}^\infty\|_{l_p} = \|T_1\|\,\|\{b_n\delta_n(T_2)\}_{n=0}^\infty\|_{l_p}$$
$$\le \frac{1}{b_0}\|T_1\|_{S_p^\beta(X)}\|T_2\|_{S_p^\beta(X)}.$$

The same arguments are valid on $\left(\Sigma_p^\beta(X), \|\cdot\|_{\Sigma_p^\beta(X)}\right)$. □

Corollary 2.5. *Let $T_1 \in \Sigma_{p_1}^\beta(X)$ and $T_2 \in \Sigma_{p_2}^{\beta^*}(X)$, where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\beta = \{b_n\}_{n=0}^\infty \subset (0,\infty)$, $\beta^* = \{\frac{1}{b_n}\}_{n=0}^\infty$ and set $\delta = \{1\}_{n\in\mathbb{N}}$. Then*

$$\|T_1T_2\|_{\Sigma_p^\delta(X)} \le 2^{\frac{1}{p}}\|T_1\|_{\Sigma_{p_1}^\beta(X)}\|T_2\|_{\Sigma_{p_2}^{\beta^*}(X)}.$$

Proof. We have that

$$\left[\sum_{n=0}^\infty d^{2n}(T_1T_2)^p\right]^{\frac{1}{p}} \le \left[\sum_{n=0}^\infty d^n(T_1)^p d^n(T_2)^p\right]^{\frac{1}{p}}$$
$$= \left[\sum_{n=0}^\infty b_n^p d^n(T_1)^p (\frac{1}{b_n})^p d^n(T_2)^p\right]^{\frac{1}{p}}$$
$$\le \left[\sum_{n=0}^\infty b_n^{p_1} d^n(T_1)^{p_1}\right]^{\frac{1}{p_1}}\left[\sum_{n=0}^\infty (\frac{1}{b_n})^{p_2} d^n(T_2)^{p_2}\right]^{\frac{1}{p_2}}$$
$$= \|T_1\|_{\Sigma_{p_1}^\beta(X)}\|T_2\|_{\Sigma_{p_2}^{\beta^*}(X)}$$

and, by an analogous argument,

$$\left[\sum_{n=0}^{\infty} d^{2n+1}(T_1T_2)^p\right]^{\frac{1}{p}} \leq \|T_1\|_{\Sigma_{p_1}^{\beta}(X)}\|T_2\|_{\Sigma_{p_2}^{\beta^*}(X)}$$

also holds. Hence

$$\|T_1T_2\|^p_{\Sigma_p^{\delta}(X)} \leq 2\Big[\|T_1\|_{\Sigma_{p_1}^{\beta}(X)}\|T_2\|_{\Sigma_{p_2}^{\beta^*}(X)}\Big]^p$$

and the proof follows. □

It is clear that if $\|\{b_n\}_{n=1}^{\infty}\|_{l_p} < \infty$ then

$$\|\{b_n d^n(T)\}_{n=1}^{\infty}\|_{l_p} \leq \|\{b_n\|T\|\}_{n=1}^{\infty}\|_{l_p} = \|T\|\|\{b_n\}_{n=1}^{\infty}\|_{l_p} < \infty$$

and

$$\|\{b_n\delta_n(T)\}_{n=1}^{\infty}\|_{l_p} \leq \|\{b_n\|T\|\}_{n=1}^{\infty}\|_{l_p} = \|T\|\|\{b_n\}_{n=1}^{\infty}\|_{l_p} < \infty$$

for all $T \in B(X, Y)$. Hence we have the following.

Proposition 2.6. *If* $\|\{b_n\}_{n=1}^{\infty}\|_{l_p} < \infty$ *then* $\Sigma_p^{\beta}(X, Y) = S_p^{\beta}(X, Y) = \boldsymbol{B}(X, Y)$.

Definition 2.2. *We say that* $\Sigma_p^{\beta}(X, Y)(S_p^{\beta}(X, Y)$, *respectively) is a proper generalized Schatten Class of Operators if* $\|\{b_n\}_{n=1}^{\infty}\|_{l_p} = \infty$.

Proposition 2.7. *If* $\|\{b_n\}_{n=1}^{\infty}\|_{l_p} = \infty$ *then* $\Sigma_p^{\beta}(X, Y) \subseteq K(X, Y)$ *and* $S_p^{\beta}(X, Y) \subseteq F(X, Y)$.

Proof. Let $T \in \Sigma_p^{\beta}(X, Y)$. If $T \notin K(X, Y)$ then the sequence $\{d^n(T)\}$ does not converge to zero. Hence there exists some $c > 0$ such that $d^n(T) \geq c$ for all n. Hence

$$\|T\|_{\Sigma_p^{\beta}(X,Y)} = \|\{b_n d^n(T)\}_{n=0}^{\infty}\|_{l_p} \geq c\|\{b_n\}_{n=1}^{\infty}\|_{l_p} = \infty$$

which is in contradiction with $T \in \Sigma_p^{\beta}(X, Y)$.

The second claim has an analogous proof. □

Proposition 2.8. *For all compact operator* $T \in K(X, Y)$ *there exists some sequence* $\beta = \{b_n\} \subset [0, \infty[$ *such that* $\|\{b_n\}_{n=1}^{\infty}\|_{l_p} = \infty$ *for all* $p \geq 1$ *and* $T \in \Sigma_p^{\beta}(X, Y)$.

Proof. $T \in K(X,Y)$ implies that $\lim_{n\to\infty} d^n(T) = 0$. Hence we may choose a sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $n_k < n_{k+1}$ and $d^{n_k}(T) < 2^{-k}$ for all k. We set $\beta = \{b_n\}$, where

$$b_n = \begin{cases} 1 & \text{if } \exists k, n = n_k \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\{b_n\}_{n=1}^{\infty}\|_{l_p} = \infty$ for all $p \geq 1$ and

$$\|T\|_{\Sigma_p^{\beta}(X,Y)} = \|\{b_n d^n(T)\}_{n=0}^{\infty}\|_{l_p} = \left\{\sum_{k=0}^{\infty} d^{n_k}(T)^p\right\}^{\frac{1}{p}} < \infty. \qquad \square$$

The same arguments are valid to prove the following.

Proposition 2.9. *For all $T \in F(X,Y)$ there exists some sequence $\beta = \{b_n\} \subset [0,\infty[$ such that $T \in S_p^{\beta}(X,Y)$, $\|\{b_n\}_{n=1}^{\infty}\|_{l_p} = \infty$ for all $p \geq 1$.*

From the fact that for any subspace X_n of dimension $\leq n$ of a Banach space X there exists a projection $P_n : X \to X$ on X_n of norm $\leq \sqrt{n} + 1$, it can be deduced that (see [2]).

Proposition 2.10. *For all $T \in \boldsymbol{B}(X,Y)$ and all n,*

$$\delta_n(T) \leq (1 + \sqrt{n}) d^n(T).$$

Hence we have the following

Corollary 2.11.

$$S_p^{(b_n)}(X,Y) \subseteq \Sigma_p^{(b_n)}(X,Y) \subseteq S_p^{(\frac{b_n}{1+\sqrt{n}})}(X,Y).$$

Corollary 2.12. *Let us assume that $\left\|\left\{\frac{b_n}{1+\sqrt{n}}\right\}_{n=0}^{\infty}\right\|_{l_p} = \infty$. Then*

$$\Sigma_p^{(b_n)}(X,Y) \subseteq F(X,Y).$$

Proposition 2.13. *Let X be an infinite dimensional Banach space. Let $\{\varepsilon_n\}$ be a non-increasing sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$. Then there exists an operator (with no finite rank) $T \in F(X,X)$ such that $\delta_n(T) \leq \varepsilon_n$ (hence, $d^n(T) \leq \varepsilon_n$) for all n.*

Proof. We choose a sequence $\{a_n\}$ such that $\sum_{k=n}^{\infty} a_k = \varepsilon_n$ for all n and let $\{x_n\}_{n=1}^{\infty}$ be a free subset of X. Set $X_n = \operatorname{span}\{x_k\}_{k=1}^{n}$ for $n \geq 1$ and $X_0 = \{0\}$.

Denote by P_n a projection of X on X_n. Then the series $\sum_{n=1}^{\infty} a_n \frac{1}{\|P_n\|} P_n$ is absolutely convergent, so that it converges to some operator

$$T = \sum_{n=1}^{\infty} a_n \frac{1}{\|P_n\|} P_n.$$

Now it is clear that $\operatorname{rank}(S_n) \leq n$ for all n, where $S_0 = P_0$ and

$$S_n = \sum_{k=1}^{n} a_k \frac{1}{\|P_k\|} P_k$$

since $X_0 \subset \ldots \subset X_{n-1} \subset X_n \subset \ldots$. Hence

$$\delta_n(T) \leq \|T - S_n| \leq \sum_{k=n+1}^{\infty} a_k = \varepsilon_n \text{ for all } n. \qquad \square$$

Corollary 2.14. *Let X be a Banach space. Then for all $\beta = \{b_n\} \subset [0, \infty)$ there are $T \in \Sigma_p^{\beta}(X, X)$ and $S \in S_p^{\beta}(X, X)$, operators with no finite rank.*

A normed linear space Y has the extension property (cf. [2]) if for all normed linear space X and all M linear subspace of X and $T \in \boldsymbol{B}(M, Y)$, there exists an extension $\overline{T} \in \boldsymbol{B}(X, Y)$ with $\|\overline{T}\| = \|T\|$. It is well known that if Y has the extension property then all $T \in \boldsymbol{B}(X, Y)$ satisfy the relation $d^n(T) = \delta_n(T)$ for all n. Hence we have the following

Proposition 2.15. *If Y has the extension property then $\Sigma_p^{\beta}(X, Y) = S_p^{\beta}(X, Y)$ for all X and β.*

3. Convergence in generalized Schatten classes of operators.

In what follows we will assume that $\{b_n\} \subset (0, \infty)$ and

$$\sup_{n \in \mathbb{N}} \left\{ \frac{\max\{b_{2n}, b_{2n+1}\}}{b_n} \right\} = C < \infty,$$

to be sure hat the generalized Schatten classes are quasinormed spaces.

Theorem 3.1. *The sequence of operators* $\{T_m\}$ *converges to* T *in the topology of* $S_p^\beta(X)$ *if and only if* $\|T_m - T\| \to 0$ *and the family of series* $\left\{\|\{b_n\delta_n(T_m)\}_{n=0}^\infty\|_{l_p}^p\right\}_{m\in\mathbb{N}}$ *is equiconvergent.*

Proof. ($\Rightarrow$) $\|T_m - T\|_{S_p^\beta(X)} \to 0$ implies $\|T_m - T\| \to 0$, since $\|T_m - T\| \leq \frac{1}{b_0}\|T_m - T\|_{S_p^\beta(X)}$.

Let $\varepsilon > 0$. Then there exists some $n_0 \in \mathbb{N}$ such that $\|T_m - T\|_{S_p^\beta(X)} < \varepsilon^{\frac{1}{p}}$ for all $m \geq n_0$. On the other hand $\|T_m - T\|_{S_p^\beta(X)} \leq \infty$ for all $m < n_0$. Hence there exists some $k_0 \in \mathbb{N}$ such that

$$\left[\sum_{k=k_0}^\infty b_k^p \delta_k(T_m - T)^p\right]^{\frac{1}{p}} < \varepsilon^{\frac{1}{p}} \quad \text{for all} \quad m \in \mathbb{N}$$

and

$$\left[\sum_{k=k_0}^\infty b_k^p \delta_k(T)^p\right]^{\frac{1}{p}} < \varepsilon^{\frac{1}{p}}.$$

But

$$\begin{aligned}
&\Big[\sum_{k=k_0}^\infty b_{2k}^p (T_m)^p\Big]^{\frac{1}{p}} \\
&\leq C\Big[(\sum_{k=k_0}^\infty b_k^p \delta_k(T_m - T)^p)^{\frac{1}{p}} + (\sum_{k=k_0}^\infty b_k^p \delta_k(T)^p)^{\frac{1}{p}}\Big] \\
&\leq 2C\varepsilon^{\frac{1}{p}}
\end{aligned}$$

and

$$\begin{aligned}
&\Big[\sum_{k=k_0}^\infty b_{2k+1}^p (T_m)^p\Big]^{\frac{1}{p}} \\
&\leq C\Big[(\sum_{k=k_0}^\infty b_k^p \delta_k(T_m - T)^p)^{\frac{1}{p}} + (\sum_{k=k_0}^\infty b_k^p \delta_k(T)^p)^{\frac{1}{p}}\Big] \\
&\leq 2C\varepsilon^{\frac{1}{p}}.
\end{aligned}$$

Hence

$$\sum_{k=2k_0}^\infty b_k^p \delta_k(T_m)^p \leq (2C)^p \varepsilon$$

and the family of series $\left\{\|\{b_n\delta_n(T_m)\}_{n=0}^{\infty}\|_{l_p}^p\right\}_{m\in\mathbb{N}}$ is equiconvergent.

($\Leftarrow$) Let $\{T_m\}\cup\{T\}\subset S_p^{\beta}(X)$ such that $\|T_m-T\|\to 0$ and the family of series $\left\{\|\{b_n\delta_n(T_m)\}_{n=0}^{\infty}\|_{l_p}^p\right\}_{m\in\mathbb{N}}$ is equiconvergent, and let $\varepsilon>0$. Then there exists some $k_0\in\mathbb{N}$ such that

$$\sum_{k=k_0}^{\infty} b_k^p\delta_k(T_m)^p\le\varepsilon \quad \text{for all } m\in\mathbb{N}$$

and

$$\sum_{k=k_0}^{\infty} b_k^p\delta_k(T)^p\le\varepsilon.$$

Hence

$$\begin{aligned}&\left[\sum_{k=k_0}^{\infty} b_{2k}^p\delta_{2k}(T_m-T)^p\right]^{\frac{1}{p}}\\&\le C\left[\Big(\sum_{k=k_0}^{\infty} b_k^p\delta_k(T_m)^p\Big)^{\frac{1}{p}}+\Big(\sum_{k=k_0}^{\infty} b_k^p\delta_k(T)^p)^{\frac{1}{p}}\right]\\&\le 2C\varepsilon^{\frac{1}{p}}\end{aligned}$$

and

$$\begin{aligned}&\left[\sum_{k=k_0}^{\infty} b_{2k+1}^p\delta_{2k+1}(T_m-T)^p\right]^{\frac{1}{p}}\\&\le C\left[\Big(\sum_{k=k_0}^{\infty} b_k^p\delta_k(T_m)^p\Big)^{\frac{1}{p}}+\Big(\sum_{k=k_0}^{\infty} b_k^p\delta_k(T)^p\Big)^{\frac{1}{p}}\right]\\&\le 2C\varepsilon^{\frac{1}{p}}.\end{aligned}$$

Hence

$$\sum_{k=2k_0}^{\infty} b_k^p\delta_k(T_m-T)^p\le(2C)^p\varepsilon$$

and

$$\|T_m - T\|^p_{S_p^\beta(X)} = \sum_{k=0}^{2k_0-1} b_k^p \delta_k (T_m - T)^p + \sum_{k=2k_0}^{\infty} b_k^p \delta_k (T_m - T)^p$$

$$\leq \left[\sum_{k=0}^{2k_0-1} b_k^p \right] \|T_m - T\|^p + \sum_{k=2k_0}^{\infty} b_k^p \delta_k (T_m - T)^p$$

$$\leq \left[\sum_{k=0}^{2k_0-1} b_k^p \right] \|T_m - T\|^p + (2C)^p \varepsilon$$

which approaches to zero since $\|T_m - T\| \to 0$. □

Theorem 3.2. *The sequence of operators $\{T_m\}$ converges to T in the topology of $\Sigma_p^\beta(X)$ if and only if $\|T_m - T\| \to 0$ and the family of series $\left\{ \|\{b_n d^n(T_m)\}_{n=0}^\infty\|_{l_p}^p \right\}_{m\in\mathbb{N}}$ is equiconvergent.*

Proof. The proof follows the same arguments as in Theorem 3.1. □

Proposition 3.3. *Let $T \in S_p^\beta(X)$ and set $S(t) = \exp(tT) - I$ for all $t \geq 0$. Then $S(t) \in S_p^\beta(X)$ and*

$$\lim_{m\to\infty} \|S(t) - S_m(t)\|_{S_p^\beta(X)} = 0$$

where

$$S_m(t) = \sum_{k=1}^{m} \frac{t^k T^k}{k!}.$$

Proof. $\|S(t) - S_m(t)\| \to 0$ is clear. To prove that $\left\{ \|\{b_n \delta_n (S_m(t))\}_{n=0}^\infty\|_{l_P}^p \right\}_{m\in\mathbb{N}}$ is equiconvergent, we note that if $\operatorname{rank}(R) \leq n$ with $R \in \boldsymbol{B}(X, X)$ then

$TR \in \boldsymbol{B}(X, X)$ and $\operatorname{rank}(TR) \leq n$ for any $T \in \boldsymbol{B}(X, X)$. Hence

$$\begin{aligned}
\delta_n(S_m(t)) &\leq \inf_{\operatorname{rank}(R)\leq n} \left\| \sum_{k=1}^{m} \frac{t^k}{k!} T^k - \sum_{k=1}^{m} \frac{t^k}{k!} T^{k-1} R \right\| \\
&\leq \inf_{\operatorname{rank}(R)\leq n} \left\| \sum_{k=1}^{m} \frac{t^k}{k!} T^{k-1}(T - R) \right\| \\
&\leq \sum_{k=1}^{m} \frac{t^k}{k!} \|T\|^{k-1} \inf_{\operatorname{rank}(R)\leq n} \|T - R\| \\
&= \left[\sum_{k=1}^{m} \frac{t^k}{k!} \|T\|^{k-1} \right] \delta_n(T) \\
&\leq t e^{t\|T\|} \delta_n(T)
\end{aligned}$$

and

$$\sum_{k=k_0}^{\infty} b_k^p \delta_k(S_m(t))^p \leq t^p e^{t\|T\|p} \sum_{k=k_0}^{\infty} b_k^p \delta_k(T)^p$$

and the equiconvergence follows, since $T \in S_p^\beta(X)$.

To finalize the proof we must show that $S(t) \in S_p^\beta(X)$. To do this, we observe that the inequality

$$|\delta_n(S_m(t)) - \delta_n(S(t))| \leq \|S_m(t) - S(t)\|$$

implies

$$\delta_n(S(t)) = \lim_{m\to\infty} \delta_n(S_m(t)) \leq \sup_{m\in\mathbb{N}} \delta_n(S_m(t)) \leq t e^{t\|T\|} \delta_n(T)$$

for all $n \in \mathbb{N}$ and now it is clear that $T \in S_p^\beta(X)$ implies $S(t) \in S_p^\beta(X)$. □

Theorem 3.4. *$(\Sigma_p^\beta(X), \|\cdot\|_{\Sigma_p^\beta(X)})$ and $(S_p^\beta(X), \|\cdot\|_{S_p^\beta(X)})$ are complete.*

Proof. Let $\{T_m\}$ be a Cauchy sequence in $S_p^\beta(X)$ and suppose (without loss of generality) that $\sup_{m\in\mathbb{N}} \|T_m\|_{S_p^\beta(X)} \leq M < \infty$. Then $\{T_m\}$ is also Cauchy in $\boldsymbol{B}(X, X)$ because of the inequality $\|T_{m_1} - T_{m_2}\| \leq \frac{1}{b_0} \|T_{m_1} - T_{m_2}\|_{S_p^\beta(X)}$. Since $\boldsymbol{B}(X, X)$ is complete, there exists an operator $T \in \boldsymbol{B}(X, X)$ such that

$$\lim_{m\to\infty} \|T - T_m\| = 0.$$

Let $N \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrarily chosen. Then there exists some $n = n(\varepsilon, N) \in \mathbb{N}$ such that

$$\left[\sum_{k=0}^{N} b_k^p\right] \|T - T_n\|^p < \varepsilon.$$

Hence

$$\begin{aligned}\sum_{k=0}^{N} b_{2k}^p \delta_{2k}(T)^p &\leq C^p \sum_{k=0}^{N} b_k^p (\delta_k(T - T_n) + \delta_k(T_n))^p \\ &\leq 2^{p-1} C^p \sum_{k=0}^{N} b_k^p \delta_k(T - T_n)^p + 2^{p-1} C^p \sum_{k=0}^{N} b_k^p \delta_k(T_n)^p \\ &\leq 2^{p-1} C^p \left[\sum_{k=0}^{N} b_k^p\right] \|T - T_n\|^p + 2^{p-1} C^p \sum_{k=0}^{N} b_k^p \delta_k(T_n)^p \\ &\leq 2^{p-1} C^p (\varepsilon + M^p).\end{aligned}$$

On the other hand

$$\sum_{k=0}^{N} b_{2k+1}^p \delta_{2k+1}(T)^p \leq 2^{p-1} C^p (\varepsilon + M^p)$$

has an analogous proof. It follows that

$$\sum_{k=0}^{2N+1} b_k^p \delta_k(T)^p \leq 2^p C^p (\varepsilon + M^p)$$

for all N and ε. Hence

$$\|T\|_{S_p^\beta(X)} \leq 2CM$$

and $T \in S_p^\beta(X)$. Let $\varepsilon > 0$. Then there exists an $m_0 \in \mathbb{N}$ such that $\|T_k - T_m\|_{S_p^\beta(X)} \leq \varepsilon$ for all $k, m \geq m_0$. Using the same arguments as above and noticing that $T - T_m = T - T_k + T_k - T_m$, it is clear that for all $m \geq m_0$,

$$\|T - T_m\|_{S_p^\beta(X)} \leq 2C\varepsilon.$$

This proves that $S_p^\beta(X)$ is complete. The proof for $\Sigma_p^\beta(X)$ is analogous. $\square$

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