LE MATEMATICHE
Vol. LIII (1998) – Fasc. II, pp. 319–329

MINIMAL MODELS AND THE VIRTUAL DEGREE OF SEIFERT FIBERED SPACES

GABRIELA PUTINAR

We compute the minimal models (in the sense of Sullivan) of Seifert fibered spaces and show that they are classified by the virtual degree.

As a consequence, we reobtain the results of Neumann-Raymond (1978) on the virtual degree of Seifert fibered spaces.

1. Introduction.

The topology of Seifert fibered spaces has been completely understood since the work of Seifert; it is described by a set of integers, called the Seifert invariants (modulo a certain equivalence relation) ([6]). In particular, the homotopy types of these spaces are classified by an invariant γ, which is a rational number expressed in terms of the Seifert invariants (cf.§ 2.2 for the definitions).

This suggests that a rational homotopy approach should suffice for distinguishing, at least weakly, between Seifert fibered spaces. In this paper, we compute the minimal models of Seifert fibered spaces; we relate the invariants of these models with Seifert's invariant γ; and we discuss several known [4] results on γ from the point of view of rational homotopy.

As is well-known, rational homotopy is a weaker version of homotopy, in which the coefficients of all the homology (or homotopy etc) groups of the spaces are extended from $\mathbb{Z}$ to $\mathbb{Q}$ (and the torsion is deleted). The advantage

Entrato in Redazione il 29 giugno 1998.

of this reduction is that everything can be translated (dualized) into a purely algebraic framework. One such framework is that of minimal models and was developed by Sullivan [7]: a minimal model is a c.g.d.a (commutative-graded differential algebra) which is free as an algebra and satisfies a certain minimality condition.

For $K(G,1)$-spaces (where G is a group), the rational homotopy type is determined by the one-minimal model, which is the part of the minimal model generated by elements in degree one. An equivalent object, in terms of groups G, of the one-minimal model, is the Malcev completion $G \otimes \mathbb{Q}$, which "completes" G with respect to root extraction.

There are many applications of minimal models to topological or geometric questions. Among the first were: the computation (Serre) of the homotopy groups of the spheres up to torsion (the torsion case is still open); and the rational version of Bott's periodicity. Further examples may be found for instance in [3].

Because Seifert fibered spaces are (with one exception) $K(\pi,1)$-spaces, their rational homotopy is equivalent with the Malcev completion; we compute this completion in § 3, using Seifert's presentation for the fundamental group. An alternative computation is obtained in § 4, using fibrations. Next we reobtain (§ 5, Thm. 1) the result [4] that a Seifert fibered spaces is (rationally) not the link of an isolated C^*-singularity if and only if $\gamma = 0$. We show that this is equivalent with formality of the rational homotopy type.
Another theorem of [4] shows that γ transforms well under certain maps. We see (§ 6, Thm. 2) that in rational homotopic terms, this is precisely the equivalence of Hirsch extensions for Serre fibrations with fibre S^1.

Finally, γ is related with another invariant, the virtual degree. The first proof that these invariants are equal was needed for the study of the resolutions of isolated C^*-singularities [5]; the topological interpretation of the virtual degree became clear with [2], [4]. On the geometric side, [8] shows that the model of the geometry of a hyperbolic 3-orbifold is also determined by the genus of the base and the virtual degree; this is based on the construction of a rational connection underlying the virtual degree.

Acknowledgement. I wish to take this opportunity to express my warm thanks to Michael Freedman and Richard Stong for stimulating conversations on this subject.

2. Preliminaries.

2.1. *Rational homotopy.* We recall first the few facts of rational homotopy that we use in this section.

If G is a nilpotent group, localizing with respect to $\mathbb{Q}$ is the same as completing G with respect to (unique) extraction of roots, and defines the Malcev completion $G \longrightarrow G \otimes \mathbb{Q}$. Alternatively, $G \otimes \mathbb{Q}$ can be defined (roughly) by induction, tensoring with $\mathbb{Q}$ the central extensions associated to the quotient groups G/Γ_n, $n \geq 2$. Here Γ_n denotes the n-th term of the lower central series of G ($\Gamma_2 = [G, G]$, $\Gamma_n = [G, \Gamma_{n-1}]$, with $[\,,\,]$ denoting commutators).

Dually, for nilpotent spaces X (i.e. spaces for which π_1 is nilpotent and acts nilpotently on $\pi_n, n \geq 2$) the Malcev completion corresponds by a functorial equivalence to the one-minimal model, which is a commutative-graded algebra (c.g.d.a), free as a commutative-graded algebra and which comes with a map to the PL-forms on the space, inducing an isomorphism on $H^1 \otimes \mathbb{Q}$ and an injection on $H^2 \otimes \mathbb{Q}$.

The one-minimal model contains the degree one elements of a larger c.g.d.a, the minimal model in the sense of Sullivan ([7], [3]); this is a free c.g.d.a with a map as above which induces an isomorphism on all of $H^* \otimes \mathbb{Q}$.
The minimal model is the algebraic equivalent of the (usual) homotopy of a certain space $X_{(0)}$, called the rationalisation of X. Therefore it contains the rational homotopy type of X.

If the space X is not nilpotent, we can still consider ([1]) a minimal model, filtered according to the lower central series of $\pi_1(X)$ and its action on $\pi_n(X)$, $n \geq 2$. The dual of this filtered minimal model is the so-called $\mathbb{Q}$-completion of X, which for nilpotent spaces coincides with the rational homotopy type.

In the case of Riemann surfaces, because (excepting S^2) these spaces are $K(\pi, 1)'s$ and because the lower central series is countable, the filtered model mentioned above is in fact the one-minimal model; therefore (except for S^2) the Malcev completion of Riemann surfaces is (dual to) their $\mathbb{Q}$-completion. The same discussion applies for Seifert fibered spaces.

In what follows "rational homotopy" is used as a paraphrase for $\mathbb{Q}$-completion. The cohomology is taken with rational coefficients throughout.

2.2. *Seifert invariants*. We recall next some standard facts about Seifert fibered spaces, ([6], [4]). A 3-manifold M is *Seifert fibered* if it admits a map $M \to S$ which locally over S is of the form:

$$D^2 \times [0, 1]/(x, 0) \sim (\phi(x), 1) \longrightarrow D^2 \tag{1}$$

where ϕ rotates D^2 by an angle of $2\pi\nu/\mu$ (with ν, μ relatively prime integers, $0 \leq \nu < \mu$).

If $\mu > 1$, the fibre is called *singular*. We assume M closed, connected and M and S orientable. Then S is a closed Riemann surface and the number of singular fibers is finite.

The topology of Seifert fibered spaces was determined by Seifert [6], up to orientation and fibre preserving homeomorphism, and is given by the set of Seifert invariants (modulo a certain equivalence relation [4], under which the number γ defined below is invariant). The Seifert invariants are:

$$(g; b; (\alpha_i, \beta_i), i = 1, \dots, r) \tag{2}$$

where g is the genus of S, r is the number of singular fibers. To define the remaining invariants, we delete small neighbourhoods of all the singular fibers, lying as in (1) over some disks $D_1, \dots, D_r$ in S. We do the same for a regular fibre, lying over a disk D_0. Then given arbitrary sections over ∂D_i, $0 \le i \le r$, there exists an extension of these to a section over $S_0 = S \setminus D_0 \cup \dots \cup D_r$. The invariant b is defined as the degree of a section relative to the regular fiber. In the same way, β_i, (when not normalized, as in [4]), is defined as the degree of the fibration restricted over ∂D_i, $i = 1, \dots, r$; while α_i is equal to the order of the isotropy subgroup at the i-th singular fiber. Note that these are related with the invariants defined in (1) by $\alpha_i = \mu_i$, $\beta_i \nu_i \equiv 1 \pmod{\alpha_i}$, $1 \le i \le r$.

Further, a standard presentation for $\pi_1(M)$ in terms of the Seifert invariants is given by:

$$(3) \qquad \pi_1(M) = \langle A_1, B_1, \dots, A_g, B_g, Q_0, \dots, Q_r, H \,|\, [A_1, B_1] \dots$$
$$\dots [A_g, B_g] Q_0 \dots Q_r = 1, Q_0 H^b = 1, Q_j^{\alpha j} H^{\beta_j} = 1,\ j = 1, \dots, r,\ H \text{ central} \rangle.$$

Here H denotes any regular fiber, Q_j are sections above the boundaries of D_j, $j = 0, \dots, r$; while A_i, B_i, $i = 1, \dots g$ are standard generators for π_g.

3. Malcev completion.

Let M be a Seifert fibered as in § 2.2. Using the above presentation we will compute the Malcev completion $\pi_1(M) \otimes \mathbb{Q}$ of $\pi_1(M)$. We shall denote by R_g the Riemann surface of genus g and by π_g its fundamental group.

Proposition 1. *Let $M \to R_g$ be a Seifert fibered space with Seifert invariants $(g; b; (\alpha_i, \beta_i), i = 1, \dots, r)$ and let*

$$\gamma = -b - \sum_{j=1}^{r} \frac{\beta_j}{\alpha_j} \in \mathbb{Q}. \tag{4}$$

i) if $\gamma = 0$, then $\pi_1(M) \otimes \mathbb{Q} = G^0 \otimes \mathbb{Q}$, where $G^0 = \pi_g \times \mathbb{Z}$;

ii) if $\gamma \neq 0$*, then* $\pi_1(M) \otimes \mathbb{Q} = G^1 \otimes \mathbb{Q}$*, where*

(5) $G^1 = \langle A_1, B_1, \ldots A_h, B_g | \, [A_i, [A_1, B_1] \ldots [A_g, B_g]] = 1,$

$$[B_i, [A_1, B_1] \ldots [A_g, B_g]] = 1, 1 \leq i \leq g\rangle.$$

Proof. Let us denote for uniformity $\alpha_0 = 1$, $\beta_0 = b$. In the above presentation for $\pi_1(M)$, assuming root extraction is possible, we can write

$$Q_j = H^{-\beta_j/\alpha_j}, \quad j = 0, \ldots, r.$$

Therefore

$$[A_1, B_1] \ldots [A_g, B_g] H^{-\sum_{j=0}^{r} \beta_j/\alpha_j} = 1. \tag{6}$$

If $\gamma = 0$, the equality i) is therefore true mod Γ_n, for any n so it is true for the completions themselves. If $\gamma \neq 0$, then replacing H with $H^{-\gamma}$ gives a new presentation for $\pi_1(M) \otimes \mathbb{Q}$, in which H is redundant; eliminating H we obtain the presentation ii).

In conclusion we have the following:

Corollary 1. *The Malcev completions of* $\pi_1(M)$*,* M *any Seifert fibered space, are classified by the set*

$$\{(g, \gamma); \quad g \geq 0, \gamma = 0 \quad or\ 1\}$$

where g *is the genus of the base, and* γ *(defined by* (4) *above) is regarded mod* $\mathbb{Q}^*$.

4. Minimal models.

By dualizing the above Corollary, we see that, when the genus g of the base is fixed, the one-minimal models of Seifert fibered spaces $M \to R_g$ depend on γ (mod $\mathbb{Q}^*$). Alternatively, this follows using fibrations, as shown below.

Let $\mathscr{M}_X$ denote the Sullivan minimal model of the space X. The Serre fibration $S^1_{(0)} \to M_{(0)} \to R_{g(0)}$ is principal because the fibre is central, so $\mathscr{M}_M$ is an extension

$$\mathscr{M}_{R_g} \to \mathscr{M}_M \to \mathscr{M}_{S^1}.$$

Therefore $\mathscr{M}_M$ is obtained by adding to $\mathscr{M}_{R_g}$ and element h such that $dh \in H^2(R_g; \mathbb{Q}) = \mathbb{Q}$. By construction ([3]), $dh = k$, where k is the *k-invariant* at this stage of the minimal model. Because up to isomorphism of models, either $dh = 0$ or $dh =$ the fundamental class of R_g, we reobtain the above two cases of Proposition 1, but in terms of the k-invariant instead of γ. In particular it follows that $\gamma = k$ (mod $\mathbb{Q}^*$) (cf. § 6 for full equality). Threfore:

Proposition 2. *If $M \to R_g$ is a Seifert fibered space, then its minimal model $\mathscr{M}_M$ is given by:*
i) if $\gamma = 0$, $\mathscr{M}_M = \mathscr{M}_{R_g} \otimes (\Lambda(h), dh = 0)$,
ii) if $\gamma \neq 0$, $\mathscr{M}_M = \mathscr{M}_{R_g} \otimes_d \Lambda(h)$, where dh =generator of $H^2(R_g; \mathbb{Q})$.

Recall [7] that the minimal model $\mathscr{M}_g$ of the Riemann surface R_g is *formal* i.e. that it is determined by the cohomology ring. The model of $\mathscr{M}_g$ begins with

$$\bigwedge(a_1, b_1, \dots a_g, b_g) \otimes \bigwedge(\xi_1, \dots \xi_{g-1}), \tag{7}$$

where $d\xi_i = a_i \wedge b_i - a_{i+1} \wedge b_{i+1}$, $1 \leq i < g$. We get the first two stages if we add elements in degree one that kill all the 2-cocycles $a_i b_j$, $a_i a_j$, $b_i b_j$, $i \neq j$. After the first two stages the model is built by adding elements to kill all the new generating 2-cocycles.

Remark. If the base R_g is fixed, then by varying the k-invariant (a rational number) in $H^2(R_g; \mathbb{Q}) = \mathbb{Q}$ we obtain all the possible S^1 principal fibrations, i.e. all the homotopy types of Seifert fibered spaces over R_g.
Note that this regards $1 \in \mathbb{Q}$ as fixed. By contrast, to obtain the rational homotopy types of Seifert fibered spaces of base R_g, we must allow c.g.d.a.-s isomorphism, i.e. we must allow the base 1 of $\mathbb{Q}^*$ to be changed freely. This distinction will be important in § 6.

In conclusion we can state:

Corollary 2. *The homotopy (resp. rational homotopy) types of Seifert fibered spaces are classified by the set*

$$\{(g, k);\, g \geq 0,\, k \in \mathbb{Q}\},$$

where g is the genus of the base, k is the k-invariant of the S^1-fibration (respectively k regarded mod $\mathbb{Q}^*$).

The above corollaries are also obvious from Seifert's description, since by a (rational) homotopy, the singular fibers can be added to form a single singular fiber.

5. Massey products.

Since (triple) Massey products are defined in terms of the differential algebra of forms on M, they can be determined from the minimal model.

Case $\gamma = 0$. In this case all (triple) Massey products are 0. The following lemma is a version of the formality of the tensor product of formal models:

Lemma 1. *If $\mathscr{M}$ and $\mathscr{N}$ are (one)-minimal models for which all the Massey products are* 0*, then the same holds for $\mathscr{M} \otimes \mathscr{N}$.*

Proof. Indeed, let $\langle \alpha, \beta, \gamma \rangle = \alpha v + u\gamma$, be a Massey product of 1-forms in the model $\mathscr{M} \otimes \mathscr{N}$, where $\alpha \wedge \beta = du$, $\beta \wedge \gamma = dv$. Since Massey products are multilinear, we may assume $\alpha = \alpha^{(1,0)} + \alpha^{(0,1)}$, with $\alpha^{(1,0)} \in \mathscr{M} \otimes \mathbb{Q}$, $\alpha^{(0,1)} \in \mathbb{Q} \otimes \mathscr{N}$, and similarly for β etc. From this decomposition it is clear that du has only pure terms indexed (2,0) and (0,2), so the (1,1) term in $\alpha \wedge \beta$ is 0, therefore:

$$\alpha^{(1,0)} \otimes \beta^{(0,1)} = \beta^{(1,0)} \otimes \alpha^{(0,1)}.$$

This implies $\alpha \in \mathbb{Q}\beta$ unless $\beta = 0$; similarly for γ. So for all β we get $\langle \alpha, \beta\gamma \rangle \in \mathbb{Q}\langle \beta, \beta, \beta \rangle = 0$, and this ends the proof.

Case $\gamma \neq 0$. In this case, the opposite is true:

Lemma 2. *If $\gamma \neq 0$ and $g \geq 1$ then the Massey products generate $H^2(M; \mathbb{Q})$ as a $\mathbb{Q}$-vector space.*

Proof. Indeed, we know by (7) that $dh = a_1 \wedge b_1$, and we may choose (on M), for $i \neq j$, $a_i \wedge a_j = d0$ and similarly for b. Then, if $i \neq 1$, we have $\langle a_1, b_1, a_i \rangle = h \wedge a_i + a_1 \wedge 0 = h \wedge a_i$; while if $i = 1$ then $\langle a_1, b_1, a_i \rangle = h \wedge a_1 + a_1 \wedge (-h) = 2h \wedge a_1$. And similarly replacing a_i with b_i. This shows that (disregarding for a moment the indeterminacy) the Massey products generate the vector space $V = \bigoplus_{i=1}^{g} \mathbb{Q}(h \wedge a_i) \oplus \bigoplus_{i=1}^{g} \mathbb{Q}(h \wedge b_i)$.

We claim that $V = H^2(M)$. Indeed, the forms $h \wedge a_i$, $h \wedge b_i$ are $\mathbb{Q}$-independent, since h is added in the last stage. Also, these elements are $\mathbb{Q}$-independent in $H^2(M)$ since $H^2(\mathscr{M}) \hookrightarrow H^2(M)$. Moreover they generate $H^2(M)$ because by (3) we see that $H^2(M) = H_1(M) = Ab\ \pi_1(M)$ has $2g$ generators $a_i, b_i, i = 1, \ldots, g$.

To end the proof we note that the indeterminacy in the Massey products is 0. The indeterminacy of say $\langle a_1, b_1, a_i \rangle$ is $a_1.H^1(M) + a_i.H^1(M)$. It is enough to check on the generators a_j, b_j of $H^1(M)$. This is clear because $a_1 \wedge b_1 = dh$.

Theorem 1. *If M is (rationally homotopic to) a Seifert fibered space with $g \geq 1$ then the following are equivalent:*

i) $\gamma = 0$;

ii) all (triple) Massey products on M are 0*;*

iii) the Massey products do not generate $H^2(M; \mathbb{Q})$;

iv) the minimal model of M is formal;

v) M is not the link of an isolated singularity with $\mathbb{C}^*$-action.

Proof. The equivalences *i) – iv)* follow from the above lemmas, Proposition 2 and the formality of Riemann surfaces.

i) $\Rightarrow$ *v)*: a computation (as above or as in [6) shows that γ is the determinant of the intersection matrix; but for an isolated C^*-singularity this matrix must be negative-definite.

v) $\Rightarrow$ *i)*: By contradiction, since (up to iso of models) dh = the fundamental class of R_g, the space M is obtained as the boundary of the 4D-plumbing of $D_1 \times D_2$ (where D_1 and D_2 are 2-disks) with $(R_g \setminus D_2) \times D_1$, under which the subsets $D_1 \times S^1$ and (respectively) $(R_g \setminus D_2) \times S^1$ are glued by a zero-framed Dehn surgery. This plumbing can be made S^1-equivariant and can be chosen analytical (because the degree is > 0); next, since after plumbing, the above S^1's become $\partial D_i'$s, $(i = 1, 2)$ this S^1-action extends to a C^*-action which is locally analytic and glues to an analytic C^*-action. As is well-known the blow-down of the central curves can be made equivariant, giving the link structure on M.

Remark. The statement *i)* $\Leftrightarrow$ *v)* above (for any g) can be easily strengthened to a statement about the homeomorphy class of M. Then the equivalence becomes i') $\gamma < 0 \Leftrightarrow$ non *v)*, which is a theorem of [4]. Indeed, the $\mathbb{Q}$-completion of a Seifert fibered space coincides with its homotopy type and is given by γ, while the surgeries which build the space up to homeomorphism can be glued analytically if $\gamma < 0$. (If $\gamma \neq 0$ then by changing the orientation on M we may assume $\gamma < 0$.)

6. Virtual degree.

Throughout this section we fix $1 \in \mathbb{Q}$ as in the remark in § 4 above. In particular the k-invariant at the last stage in the minimal model of a Seifert fibered space is a fixed rational number. We prove that the equivalence of Hirsch extensions ([3]) implies the following theorem of [4]:

Theorem 2. *Let M and M' be Seifert fibered over S and S' respectively and let $f : M \longrightarrow M'$ be an orientation and fiber preserving homeomorphism. Let the degree of the induced map on a typical fiber be n and the degree of the induced map $\overline{f} : S \longrightarrow S'$ be m. Then*

$$\gamma(M) = \frac{m}{n}\gamma(M'). \tag{8}$$

Proof. *Step I.* Let us note first that this is true if we replace γ by the k-invariant k. Indeed, since the minimal model of a Seifert fibered space is a Hirsch extension (i.e. one stage) over the model of a Riemann surface, the maps in the statement of Theorem 2 dualize to an equivalence of Hirsch extensions with isomorphic bases (by definition). Further, as in [3], this translates to a commutative diagram of transgressions:

$$\begin{array}{ccc} h\cdot\mathbb{Q} & \xrightarrow{d} & H^2(S;\mathbb{Q}) \\ \uparrow{\scriptstyle \cdot n} & & \uparrow{\scriptstyle \cdot m} \\ h'\cdot\mathbb{Q} & \xrightarrow{d'} & H^2(S';\mathbb{Q}) \end{array}$$

Because in general $k(M) = d(h)$, we have $mk(M') = k(M)n$, i.e. the theorem for k.

Step II. To end the proof, we will show below that $\gamma = k$. We will see first (Lemma 3) that $k = e$ by a classifying space argument, where e is the rational Euler number. Next, the fact that $e = \gamma$ follows easily by localizing at the singular fibers and comparing degrees. For the sake of completeness we also discuss the virtual degree and include the proofs, which are more or less implicit in [4]:

Definition. *The rational Euler class (resp. number) of a Seifert fibration is by definition the Euler class (resp. number) of the bundle obtained by replacing the fibre S^1 with the rationalized fibre $S^1_{(0)}$.*

Lemma 3. *The rational Euler class is equal to the k-invariant:*

$$e = k \in H^2(R_g;\mathbb{Q}).$$

Proof. If the fibration is a genuine S^1-bundle, the k-invariant

$$k \in H^2(R_g;\mathbb{Z}) = [R_g, K(\mathbb{Z},2)].$$

It happens that $K(\mathbb{Z},2) = \mathbb{C}P^\infty = Grassm_1(\mathbb{C}^\infty) = BU_1 = BS^1$. So the k-invariant $k \in [R_g, BS^1]$ is the classifying map of the bundle, i.e., via the identifications above, it is its Euler class. In the general case, we can make the fibration into a genuine bundle by rationalizing the fibers to $S^1_{(0)}$. Then the above argument goes through rationally.

Next we discuss the relation with the virtual degree. This invariant is defined using the existence of a regular branched cover of the base, of automorphism group a certain group G, via which the Seifert fibration pulls back to a genuine bundle σ.

Definition. *The virtual degree d of a Seifert fibration is the quotient $e(\sigma)/order$ (G), where $e(\sigma)$ is the Euler number of σ.*

Theorem 3. (cf. [2], [4], [5]). *For a Seifert fibered space, the following invariants are equal:*

i) $\gamma = -b - \Sigma_{i=1}^{r} \beta_i/\alpha_i$;
ii) $k =$ the k-invariant of the fibration;
iii) $e =$ the Euler number of the fibration;
iv) $d =$ the virtual degree.

Proof. To prove $\gamma = d$, note that this true for genuine bundles, and also for Seifert fibered tori, by definition; because both invariants can be localized (at the singular fibres) we are done. Similarly to prove $\gamma = e$: for a fibered torus, by replacing S^1 with the rationalized $S^1_{(0)}$, the i-th fibre becomes divisible by α_i, and its rational Euler number is minus its previous degree β_i, divided by α_i. Finally, $k = e$ is Lemma 3.

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Department of Mathematics,
University of California,
Santa Barbara, CA-93106 (USA)