HÖLDER CONTINUITY FOR SECOND ORDER
NON VARIATIONAL PARABOLIC SYSTEMS

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Let \(Q\) be the cylinder \(\Omega \times (-T, 0)\), with \(T > 0\), we prove that if
\(u \in W^{2}(Q, \mathbb{R}^{N})\) \((N \text{ integer } \geq 1)\) is a solution in \(Q\) of the system

\[
a(X, H(u)) - \frac{\partial u}{\partial t} = 0,
\]

where \(X = (x, t)\), \(a(X, \xi)\) is a vector of \(\mathbb{R}^{N}\), measurable in \(X\), continuous in \(\xi\) and satisfying the conditions \(a(X, 0) = 0\) and \((A)\), then \(u\) and \(Du\) are
Hölder continuous in \(Q\), if \(n \leq 4\) and \(n = 2\), respectively.

We obtain similar results for the solutions in \(Q\) of the systems

\[
a(X, H(u)) - \frac{\partial u}{\partial t} = f(X)
\]
and

\[
a(X, H(u)) - \frac{\partial u}{\partial t} = b(X, u, Du),
\]

where \(f \in L^{2,\mu}(Q, \mathbb{R}^{N})\) and \(b(X, u, p)\) is a vector of \(\mathbb{R}^{N}\) with linear growth.

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1. Introduction.

Let $\Omega$ be an open bounded set of $\mathbb{R}^n$, $n \geq 2$, of generic point $x = (x_1, x_2, \ldots, x_n)$, let $T$ be a real positive number, we denote the cylinder $\Omega \times (-T, 0)$ by $Q$ and the point $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, by $X$.

If $u(X) : Q \to \mathbb{R}^N$, $N$ integer $\geq 1$, we set

$$
D_i u = \frac{\partial u}{\partial x_i},
$$

$$
Du = (D_1 u, D_2 u, \ldots, D_n u),
$$

$$
H(u) = \{D_i D_j u \mid i, j = 1, 2, \ldots, n; i \neq j\},
$$

$Du$ is an element of $\mathbb{R}^{nN}$ and $H(u)$ is an element of $\mathbb{R}^{n^2N}$.

We denote by

$$
W^p(Q, \mathbb{R}^k) = \left\{ u : u \in L^p(-T, 0, H^{2,p}(\Omega, \mathbb{R}^k)), \frac{\partial u}{\partial t} \in L^p(Q, \mathbb{R}^k) \right\}
$$

and

$$
W^p_0(Q, \mathbb{R}^k) = \{u \in W^p(Q, \mathbb{R}^k) : u \in L^p(-T, 0, H^{1,p}_0(\Omega, \mathbb{R}^k)), u(x, -T) = 0\},
$$

where $p \in [1, +\infty[$, $k$ is an integer $\geq 1$, $H^{2,p}(\Omega, \mathbb{R}^k)$ and $H^{1,p}_0(\Omega, \mathbb{R}^k)$ are the usual Sobolev spaces.

We consider the system

(1.1) $$
a(X, H(u)) - \frac{\partial u}{\partial t} = 0,
$$

where $a(X, \xi)$ is a vector in $\mathbb{R}^N$, measurable in $X$, continuous in $\xi$ and satisfying the conditions

(1.2) $$
a(X, 0) = 0;
$$

(A) there exist three positive constants $\alpha$, $\gamma$ and $\delta$, with $\gamma + \delta < 1$, such that

$$
\forall \tau, \xi \in \mathbb{R}^{nN} \text{ and for a.e. } X \in Q, \text{ it results}
$$

$$
\left\| \sum_{i=1}^n \tau_{ii} - \alpha[a(X, \tau + \xi) - a(X, \xi)] \right\|^2 \leq \gamma \|\tau\|^2 + \delta \sum_{i=1}^n \tau_{ii}^2.
$$
(B) there exists a bounded non-negative function $\omega(t)$, defined for $t > 0$, which is non-decreasing and goes to zero as $t \to 0^+$, such that $\forall X, Y \in Q$ and $\forall \xi \in \mathbb{R}^{\mathbb{R}^N}$

$$\|a(X, \xi) - a(Y, \xi)\|^2 \leq \omega(d(X, Y)) \|\xi\|^2,$$

where $d(X, Y) = \max \left\{ \|x - y\|, |\tau - \tau'|^{\frac{1}{2}} \right\}$. $X = (x, t)$ and $Y = (y, \tau)$.

A solution of the system (1.1) is a function $u \in W^2(Q, \mathbb{R}^N)$ which satisfies (1.1) for a.e. $X \in Q$.

In [3] and [6] there are Hölder continuity results in $Q$ for the solutions of the basic system

$$a(H(u)) - \frac{\partial u}{\partial t} = 0;$$

these results are been obtained by S. Campanato in [3], using the Sobolev imbedding Theorem and, with different technique (using fundamental estimates for $H(Du)$, $\frac{\partial Du}{\partial t}$, $H(u)$ and $\frac{\partial u}{\partial t}$) by M. Marino and A. Maugeri in [6].

Furthermore, in [6] M. Marino and A. Maugeri obtained similar results for the following systems

$$a(H(u)) - \frac{\partial u}{\partial t} = f(X)$$

and

$$a(H(u)) - \frac{\partial u}{\partial t} = b(X, u, Du),$$

with $f \in L^{2+\mu}(Q, \mathbb{R}^N)$ and $b(X, u, p)$ vector in $\mathbb{R}^N$ with linear growth, i.e.

$$\|b(X, u, p)\| \leq c(1 + \|u\| + \|p\|)$$

for a.e. $X \in Q$, $\forall u \in \mathbb{R}^N$ and $\forall p \in \mathbb{R}^{\mathbb{R}^N}$.

In this paper, for system (1.1), thanks to the hypothesis (B), we obtain $L_{\text{loc}}^{2, 2 + (\sigma + 2)(1 - \frac{1}{2}) - \epsilon}$ regularity results for $H(u)$ and $\frac{\partial u}{\partial t}$ and then we prove that $u$ and $Du$ are Hölder continuous in $Q$, if $n \leq 4$ and $n = 2$, respectively.

In Section 3 we obtain similar results for the following system

$$a(X, H(u)) - \frac{\partial u}{\partial t} = f(X)$$

with $f \in L^{2+\mu}(Q, \mathbb{R}^N)$. At last in Section 4 we study the Hölder continuity in $Q$ for system

$$a(X, H(u)) - \frac{\partial u}{\partial t} = b(X, u, Du),$$

where $b(X, u, p)$ is a vector of $\mathbb{R}^N$ with linear growth.
2. Hölder continuity for systems of type (1.1).

Let \( u \in W^2(Q, \mathbb{R}^N) \) be a solution in \( Q \) of the following parabolic system

\[
a(X, H(u)) - \frac{\partial u}{\partial t} = 0.
\]

In this section we shall give some Hölder continuity results in \( Q \) for \( u \) and \( Du \).

**Theorem 2.1.** If the vector \( a(X, \xi) \) satisfies the hypothesis (1.2), (A) and (B), then, \( \forall \varepsilon > 0 \) and \( \forall q \in (2, \min(\tilde{q}, n + 2)) \) \(^{(1)} \), it results:

\[
H(u) \in L^{2,2+\min(2,\frac{2}{3})-\varepsilon}_{\text{loc}}(Q, \mathbb{R}^{\sigma N})
\]

and

\[
\frac{\partial u}{\partial t} \in L^{2,2+\min(2,\frac{2}{3})-\varepsilon}_{\text{loc}}(Q, \mathbb{R}^N).
\]

**Proof.** Fixed \( Q(X^0, \sigma) = Q(\sigma) \subset Q \) \(^{(2)} \), with \( 0 < \sigma < 1 \), let \( w \) be the solution of the Cauchy-Dirichlet problem (the existence and uniqueness are ensured by Theorem 1.2 in [6]):

\[
\begin{cases}
   w \in W^2_0(Q(\sigma), \mathbb{R}^N) \\
   a(X^0, H(u) + H(w)) - \frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} \
   \text{in } Q(\sigma).
\end{cases}
\]

For this solution, (1.8) of [6] ensures the following inequality

\[
\int_{Q(\sigma)} \left( \|H(w)\|^2 + \left\| \frac{\partial w}{\partial t} \right\|^2 \right) dX \leq c \int_{Q(\sigma)} \left\| \frac{\partial u}{\partial t} - a(X^0, H(u)) \right\|^2 dX.
\]

Set \( v = u + w \) in \( Q(\sigma) \), it results: \( v \in W^2(Q(\sigma), \mathbb{R}^N) \) and

\[
a(X^0, H(v)) - \frac{\partial v}{\partial t} = 0 \quad \text{in } Q(\sigma).
\]

\(^{(1)}\) \( \tilde{q} \) is the constant \( (> 2) \) that occurs in (2.19) of [6].

\(^{(2)}\) If \( X^0 = (x^0, t^0) \in Q \) and if \( \rho > 0 \), the symbol \( Q(X^0, \rho) \) denotes the cylinder \( B(x^0, \rho) \times (t^0 - \rho^2, t^0) \).
Thanks to Theorem 3.2 in [6], for $v$ the following inequality holds

$$\int_{Q^{(t)}} \left( \|H(v)\|^2 + \left\| \frac{\partial v}{\partial t} \right\|^2 \right) dX \leq$$

$$\leq c t^{2+(\alpha + 2)(1 - \frac{1}{2})} \int_{Q^{(t)}} \left( \|H(v)\|^2 + \left\| \frac{\partial v}{\partial t} \right\|^2 \right) dX,$$

$\forall t \in (0, 1)$ and $\forall q \in (2, \min(\tilde{q}, n + 2))$.

Now, thanks to (2.2) and (2.3), since $u = v - w$ and is solution of system (1.1) and thanks to the hypothesis (B), it follows, $\forall t \in (0, 1)$ and $\forall q \in (2, \min(\tilde{q}, n + 2))$

$$\int_{Q^{(t)}} \left( \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX \leq$$

$$\leq 2 \int_{Q^{(t)}} \left( \|H(v)\|^2 + \left\| \frac{\partial v}{\partial t} \right\|^2 \right) dX + 2 \int_{Q^{(t)}} \left( \|H(w)\|^2 + \left\| \frac{\partial w}{\partial t} \right\|^2 \right) dX \leq$$

$$\leq c t^{2+(\alpha + 2)(1 - \frac{1}{2})} \int_{Q^{(t)}} \left( \|H(v)\|^2 + \left\| \frac{\partial v}{\partial t} \right\|^2 \right) dX +$$

$$+ 2 \int_{Q^{(t)}} \left( \|H(w)\|^2 + \left\| \frac{\partial w}{\partial t} \right\|^2 \right) dX \leq$$

$$\leq c t^{2+(\alpha + 2)(1 - \frac{1}{2})} \int_{Q^{(t)}} \left( \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX +$$

$$+ c \int_{Q^{(t)}} \left( \|H(w)\|^2 + \left\| \frac{\partial w}{\partial t} \right\|^2 \right) dX \leq$$

$$\leq c t^{2+(\alpha + 2)(1 - \frac{1}{2})} \int_{Q^{(t)}} \left( \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX +$$

$$+ c \int_{Q^{(t)}} \left\| \frac{\partial u}{\partial t} - a(X^0, H(u)) \right\|^2 dX \leq$$

$$\leq c t^{2+(\alpha + 2)(1 - \frac{1}{2})} \int_{Q^{(t)}} \left( \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX +$$

$$+ c \int_{Q^{(t)}} \|a(X, H(u)) - a(X^0, H(u))\|^2 dX \leq$$

$$\leq c t^{2+(\alpha + 2)(1 - \frac{1}{2})} \int_{Q^{(t)}} \left( \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX + c \sigma(\sigma) \int_{Q^{(t)}} \|H(u)\|^2 dX \leq$$
and hence

\[
(2.5) \quad \int_{Q_{r(\sigma)}} \left( \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) \, dX \leq \\
\leq c \left[ \tau^{2+(n+1)(1-\frac{2}{q}) + \omega(\sigma)} \int_{Q_{\tau(\sigma)}} \left( \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) \, dX, \right.
\]

From (2.5) and Lemma 1.III in [1] it deduces that \( \forall \varepsilon > 0 \) there exists \( \sigma_\varepsilon \in (0, 1) \) such that \( \forall \tau \in (0, 1) \) and \( \forall \sigma \in (0, \sigma_\varepsilon) \)

\[
(2.6) \quad \int_{Q_{r(\sigma)}} \left( \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) \, dX \leq \\
\leq c \tau^{2+(n+2)(1-\frac{2}{q}) - \varepsilon} \int_{Q_{\tau(\sigma)}} \left( \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) \, dX,
\]

\( \forall q \in (2, \min(\tilde{q}, n + 2)) \).

From (2.6) the thesis follows. \( \square \)

Thanks to Theorem 2.1 and Lemma 2.II in [4] it deduces that

\[
(2.7) \quad Du \in L^{2,4+(n+2)(1-\frac{2}{q})-\varepsilon}_{\text{loc}}(Q, \mathbb{R}^N)
\]

\( \forall \varepsilon > 0 \) and \( \forall q \in (2, \min(\tilde{q}, n + 2)) \).

If \( n < 2\tilde{q} - 2 \) (and in particular if \( n = 2 \)), there exist \( \varepsilon > 0 \) and \( q \in (2, \min(\tilde{q}, n + 2)) \) such that \( 4 + (n+2) \left( 1 - \frac{2}{q} \right) - \varepsilon > n + 2 \) and (2.7) holds (with this choice of \( \varepsilon \) and \( q \)), and then

\[ Du \text{ is Hölder continuous in } Q. \]

On the other hand, from Lemma 2.I in [4] and from conditions (2.1) and (2.7), it follows:

\[ u \in L^{2,6+(n+2)(1-\frac{2}{q})-\varepsilon}_{\text{loc}}(Q, \mathbb{R}^N), \]

\( \forall \varepsilon > 0 \) and \( \forall q \in (2, \min(\tilde{q}, n + 2)) \), from which we have that, if \( n < 3\tilde{q} - 2 \) (and in particular if \( n \leq 4 \)), the vector

\[ u \text{ is Hölder continuous in } Q. \]
3. Hölder continuity for system of type (1.3).

Let \( f : Q \to \mathbb{R}^N \) be a function of class \( L^2(Q, \mathbb{R}^N) \) and let \( u \in W^2(Q, \mathbb{R}^N) \) be a solution in \( Q \) of the parabolic system

\[
(3.1) \quad a(X, H(u)) - \frac{\partial u}{\partial t} = f(X),
\]

with \( a(X, \xi) : Q \times \mathbb{R}^{n^2} \to \mathbb{R}^N \) measurable in \( X \), continuous in \( \xi \) and satisfying conditions (1.2), (A) and (B).

**Lemma 3.1.** For every cylinder \( Q(X^0, \sigma) = Q(\sigma) \subset Q \), with \( \sigma < 1 \), \( \forall \tau \in (0, 1) \) and \( \forall q \in (2, \min(q, n + 2)) \), one has:

\[
(3.2) \quad \int_{Q(\tau \sigma)} \left( \|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 \right) dX \leq
\]

\[
\leq c \left[ \tau^{2+ (n+2)(1-\frac{2}{\beta})} + \omega(\sigma) \right] \int_{Q(\sigma)} \left( \|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 \right) dX +
\]

\[
+ c \int_{Q(\sigma)} \|f\|^2 dX,
\]

where the constant \( c \) does not depend on \( \sigma \) and \( \tau \).

**Proof.** Fixed \( Q(X^0, \sigma) = Q(\sigma) \subset Q \), with \( \sigma < 1 \), let us denote, as in the proof of Theorem 2.1, by \( w \) the solution of the Cauchy-Dirichlet problem

\[
\begin{cases}
 w \in W^2_0(Q(X^0, \sigma), \mathbb{R}^N) \\
 a(X^0, H(u) + H(w)) - \frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} \quad \text{in} \quad Q(\sigma).
\end{cases}
\]

Set \( v = w + u \) in \( Q(\sigma) \), it results: \( v \in W^2(Q(\sigma), \mathbb{R}^N) \) and

\[
a(X^0, H(v)) - \frac{\partial v}{\partial t} = 0 \quad \text{in} \quad Q(\sigma).
\]

For \( w \) and \( v \) the estimates (2.2) and (2.3) hold, respectively. Making use of these estimates and proceeding as in the proof of Theorem 2.1, one gets, \( \forall \tau \in (0, 1) \) and \( \forall q \in (2, \min(q, n + 2)) \)

\[
(3.3) \quad \int_{Q(\tau \sigma)} \left( \|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 \right) dX \leq
\]

\[
\leq c \tau^{2+ (n+2)(1-\frac{2}{\beta})} \int_{Q(\sigma)} \left( \|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 \right) dX +
\]

\[
+ c \int_{Q(\sigma)} \left\|\frac{\partial u}{\partial t} - a(X^0, H(u))\right\|^2 dX.
\]
Now let us estimate the last integral in the right hand side of (3.3).

Since \( u \) is solution of system (3.1), one has:

\[
\frac{\partial u}{\partial t} - a(X^0, H(u)) = a(X, H(u)) - a(X^0, H(u)) - f(X) \quad \text{in} \quad Q(\sigma),
\]

and then, thanks to hypothesis (B)

\[
(3.4) \quad \int_{Q(\sigma)} \left\| \frac{\partial u}{\partial t} - a(X^0, H(u)) \right\|^2 dX \leq \leq 2 \int_{Q(\sigma)} \|a(X, H(u)) - a(X^0, H(u))\|^2 dX + 2 \int_{Q(\sigma)} \|f(X)\|^2 dX \leq \leq c\omega(\sigma) \int_{Q(\sigma)} \|H(u)\|^2 dX + 2 \int_{Q(\sigma)} \|f(X)\|^2 dX.
\]

(3.2) is a trivial consequence of inequalities (3.3) and (3.4). \( \square \)

The next theorem follows easily from Lemma 3.1:

**Theorem 3.1.** If \( u \in W^2(Q, \mathbb{R}^N) \) is a solution of system (3.1), if the vector \( a(X, \xi) \) satisfies conditions (1.2), (A) and (B) and if \( f \in L^{2,\mu}(Q, \mathbb{R}^N) \), \( 0 < \mu < \lambda = \min \left\{ 2 + (n + 2)(1 - \frac{2}{q}), n + 2 \right\} \), then

\[
(3.5) \quad Du \in L^{2,\mu+2}_{\text{loc}}(Q, \mathbb{R}^n)
\]

and

\[
(3.6) \quad u \in L^{2,\mu+4}_{\text{loc}}(Q, \mathbb{R}^N).
\]

**Proof.** The proof is similar to that of Theorem 4.1 in [6]. Fixed \( Q(\sigma) \subset Q \), with \( \sigma < 1 \), \( \forall \tau \in (0, 1) \) and for every \( q \in (2, \min(q, n + 2)) \), in virtue of Lemma 3.1 and from the hypothesis \( f \in L^{2,\mu}(Q, \mathbb{R}^N) \), one has:

\[
(3.7) \quad \int_{Q(\tau(\sigma))} \left( \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX \leq \leq c \left[ \tau^{2+\mu(1-\frac{1}{q})} + \omega(\sigma) \right] \int_{Q(\sigma)} \left( \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX + c\sigma^\mu \|f\|^2_{L^{2,\mu}(Q, \mathbb{R}^N)}.
\]

From (3.7) the thesis follows proceeding as in the proof of Theorem 4.1 in [6] and using Lemma 2.VII in [2], instead of Lemma 1.1 in [1]. \( \square \)
From (3.5) it follows, as in [6], that, if \( n < 2\tilde{q} - 2 \) (and in particular if \( n = 2 \)) and if \( f \in \mathcal{L}^{2,\mu}(Q, \mathbb{R}^N) \), with \( \mu \in (n, \lambda) \),
\[
Du \in \mathcal{L}^{2,\mu+2}_{\text{loc}}(Q, \mathbb{R}^n),
\]
with \( \mu + 2 > n + 2 \), and then

\( Du \) is Hölder continuous in \( Q \).

If \( n < 3\tilde{q} - 2 \) (and in particular if \( n \leq 4 \)) and if \( f \in \mathcal{L}^{2,\mu}(Q, \mathbb{R}^N) \), with \( \mu \in (n - 2, \lambda) \), for (3.6) one has
\[
u \in \mathcal{L}^{2,\mu+4}_{\text{loc}}(Q, \mathbb{R}^N),
\]
with \( \mu + 4 > n + 2 \), and then

\( u \) is Hölder continuous in \( Q \).

4. Hölder continuity for systems of type (1.4).

Let \( u \in W^2(Q, \mathbb{R}^N) \) be a solution in \( Q \) of the system
\[
(4.1) \quad a(X, H(u)) - \frac{\partial u}{\partial t} = b(X, u, Du),
\]
where \( a(X, \xi) \) is a vector in \( \mathbb{R}^N \), measurable in \( X \), continuous in \( \xi \) and satisfying conditions \((1.2), (A) \) and \((B) \).

Let \( b(X, u, p) \) be a vector in \( \mathbb{R}^N \), measurable in \( X \), continuous in \( (u, p) \) and satisfying the condition
\[
(4.2) \quad \|b(X, u, p)\| \leq c(1 + \|u\| + \|p\|)
\]
for a.e. \( X \in Q \), \( \forall u \in \mathbb{R}^N \) and \( \forall p \in \mathbb{R}^n \).

Theorem 3.1 in [5] ensures that there exists \( \tilde{q} \in \left( 2, \frac{2(n+2)}{n} \right) \) such that, \( \forall q \in (2, \tilde{q}) \),
\[
u \in W^q_{\text{loc}}(Q, \mathbb{R}^N),
\]
hence, thanks to Lemma 2.11 in [4], one has, \( \forall Q(\sigma) \subset \subset Q \) and \( \forall q \in (2, \tilde{q}) \):
\[
\int_{Q(\sigma)} \|Du - (Du)_{Q(\sigma)}\|^2 \, dX \leq c\sigma^2 \int_{Q(\sigma)} \left( \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) \, dX \leq \frac{1}{\tilde{q}} \int_{Q(\sigma)} \left( \|H(u)\|^q + \left\| \frac{\partial u}{\partial t} \right\|^q \right) \, dX ,
\]

\[
\int_{Q(\sigma)} \|Du - (Du)_{Q(\sigma)}\|^2 \, dX \leq c\sigma^2 \int_{Q(\sigma)} \left( \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) \, dX \leq \frac{1}{\tilde{q}} \int_{Q(\sigma)} \left( \|H(u)\|^q + \left\| \frac{\partial u}{\partial t} \right\|^q \right) \, dX ,
\]

\[
l^2 \leq \frac{1}{\tilde{q}} \int_{Q(\sigma)} \left( \|H(u)\|^q + \left\| \frac{\partial u}{\partial t} \right\|^q \right) \, dX .
\]
from which

\[ Du \in \mathcal{L}^{2, 2 + (n + 2)(1 - \frac{2}{q})}(Q, \mathbb{R}^N), \]

\[ \forall q \in (2, \bar{q}). \]

On the other hand, making use of Lemma 2.1 in [5] (with \( p = 2 \)), it
deduces

\[ u \in \mathcal{L}^{2, 4 + (n + 2)(1 - \frac{2}{q})}(Q, \mathbb{R}^N), \]

\[ \forall q \in (2, \bar{q}). \]

Then \( u \) and \( D_i u, \ i = 1, 2, \ldots, n \), belong to the space \( \mathcal{L}^{2, \mu}(Q^*, \mathbb{R}^N) \),
\( \forall \mu \in \left( 0, 2 + (n + 2) \left( 1 - \frac{2}{q} \right) \right) \) and \( \forall Q^* \subset \subset Q \), from which, for condition

\[ (4.2) \], it follows that the vector \( f(X) = b(X, u, Du) \in \mathcal{L}^{2, \mu}(Q^*, \mathbb{R}^N) \), \( \forall \mu \in \left( 0, 2 + (n + 2) \left( 1 - \frac{2}{q} \right) \right) \) and \( \forall Q^* \subset \subset Q \).

For Theorem 3.1 one has

\[ Du \in \mathcal{L}^{2, \mu + 2}(Q, \mathbb{R}^N), \]

\[ u \in \mathcal{L}^{2, \mu + 4}(Q, \mathbb{R}^N), \]

\[ \forall \mu \in (0, \lambda^*), \] where \( \lambda^* = \min \left\{ 2 + (n + 2) \left( 1 - \frac{2}{q^*} \right), n + 2 \right\}, q^* = \min(\bar{q}, \bar{q}). \)

Now reasoning exactly as in Section 5 of [6] it follows that

\[ Du \text{ is Hölder continuous in } Q \text{ if } n < 2q^* - 2 \]

and

\[ u \text{ is H"older continuous in } Q \text{ if } n < 3q^* - 2. \]

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