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AN ABSTRACT ULTRAPARABOLIC INTEGRODIFFERENTIAL EQUATION

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We prove an existence and uniqueness result for a ultraparabolic integrodifferential equation in the strip $[0, T_1] \times [0, T_2]$ in the context of the spaces of continuous functions with values in a Banach space X and we give some applications to specific partial integrodifferential problems.

Introduction and statement of the main result.

This paper is concerned with the problem of determining a solution u depending on two variables t and s in an evolution integrodifferential equation involving the two *time* variables t and s (for the physical motivation see the appendix).

The present model may be interpreted as a generalization of the corresponding well-known model for thermic materials with memory to the case of memories related to several time variables. We note that, while the case of memory related to one time variable is well studied (cf. [1], [2], [3], [5], [6], [8], [9], [10], [13]), the case of multi-time memory problems does not seem, to the author's knowledge, to have been investigated not even from the point of

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view of a general mathematical theory. Consequently, this paper wants to give a contribution in this new field.

We can now pose our problem: *determine a function* $u : [0, T_1] \times [0, T_2] \times \Omega \to \mathbb{R}$ *such that*

$$
\begin{cases}
a_1(t)D_t u(t,s,x) + a_2(s)D_s u(t,s,x) - \mathcal{A}u(t,s,x) = \\
\quad = f(t,s,x) + \int_0^t d\tau \int_0^s h(t-\tau, s-\sigma)\mathcal{B}u(\tau,\sigma,x)\, d\sigma, \\
\qquad\qquad\qquad\qquad\qquad (t,s,x) \in [0,T_1] \times [0,T_2] \times \Omega, \\
u(t,0,x) = u_1(t,x), \qquad (t,x) \in [0,T_1] \times \overline{\Omega}, \\
u(0,s,x) = u_2(s,x), \qquad (s,x) \in [0,T_2] \times \overline{\Omega}, \\
u(t,s,x) = 0, \qquad (t,s,x) \in [0,T_1] \times [0,T_2] \times \partial\Omega.
\end{cases} \tag{0.1}
$$

Here Ω is a bounded and open set in $\mathbb{R}^n$ while $\mathcal{A}$ and $\mathcal{B}$ are linear second-order differential operators with variable coefficients depending on x only:

$$
(\mathcal{A}u)(x) = \sum_{i,j=1}^{n} c_{i,j}(x) D_i D_j u(x) + \sum_{j=1}^{n} c_j(x) D_j u(x) + c(x)u(x), \ \ x \in \overline{\Omega}; \tag{0.2}
$$

$$
(\mathcal{B}u)(x) = \sum_{i,j=1}^{n} d_{i,j}(x) D_i D_j u(x) + \sum_{j=1}^{n} d_j(x) D_j u(x) + d(x)u(x), \ \ x \in \overline{\Omega}. \tag{0.3}
$$

A basic requirement is that $\mathcal{A}$ should be *uniformly* elliptic.

Consequently, under our assumptions, the differential operator $a_1(t)D_t + a_2(s)D_s - \mathcal{A}$ turns out to be *ultraparabolic*.

We observe that the aim of this work consists in proving an existence, uniqueness and continuous dependence result *in the large*. Owing to the complexity of the problem under consideration it will be more convenient to deal with an abstract version of problem (0.1).

We now pose the corresponding problem related to a Banach space X: *determine a function* $u : [0, T_1] \times [0, T_2] \to X$ *such that*

$$(0.4)\qquad \begin{cases} a_1(t)D_t u(t,s) + a_2(s)D_s u(t,s) - \mathcal{A}u(t,s) = \\ \qquad f(t,s) + \displaystyle\int_0^t d\tau \int_0^s h(t-\tau, s-\sigma)\mathcal{B}u(\tau,\sigma)\, d\sigma, \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,s) \in [0,T_1]\times[0,T_2], \\ u(t,0) = u_1(t), \qquad\qquad\qquad t \in [0,T_1], \\ u(0,s) = u_2(s), \qquad\qquad\qquad s \in [0,T_2]. \end{cases}$$

To solve problem (0.4) we first consider the case where $h \equiv 0$ and we prove an existence, uniqueness and continuous dependence result in the context of the space of Hölder continuous functions.
Then, we consider the general case and show that problem (0.4) can be transformed into a fixed point problem in the space $C^\alpha([0, T_1] \times [0, T_2]; \mathcal{D}(A)) \cap C^{1+\alpha}([0, T_1] \times [0, T_2]; X)$.
We study this fixed point problem and prove that it is uniquely solvable in $C^\alpha([0, T_1] \times [0, T_2]; \mathcal{D}(A)) \cap C^{1+\alpha}([0, T_1] \times [0, T_2]; X)$.
In this section we limit ourselves to stating the main result related to be abstract problem (0.4). For this purpose we need the following functional spaces.

Definition 0.1. For any $(T_1, T_2) \in \mathbb{R}_+ \times \mathbb{R}_+$, any $k \in \mathbb{N} \cup \{0\}$, any $\alpha \in \mathbb{R}_+$ and any Banach space Y we set

$$B([0,T_1];Y) = \Big\{ f : [0,T_1] \to Y : \|f\|_{B([0,T_1];Y)} = \\ = \sup_{t\in[0,T_1]} \|f(t)\|_Y < +\infty \Big\};$$

$$C^k([0,T_1]\times[0,T_2];Y) = \Big\{ f : [0,T_1]\times[0,T_2] \to Y : \ f \text{ is continuously} \\ \text{differentiable up to the } k\text{-order with values in } Y \Big\}.$$

$C^k([0, T_1] \times [0, T_2]; Y)$ is normed by

$$\|f\|_{C^k([0,T_1]\times[0,T_2];Y)} = \sum_{|\beta|\le k} \|D^\beta f\|_{C([0,T_1]\times[0,T_2];Y)}.$$

$$C^\alpha([0,T_1]\times[0,T_2];Y) = \Big\{ f \in C^{[\alpha]}([0,T_1]\times[0,T_2];Y) : [D^\beta f]_{\alpha-[\alpha],\infty}$$

$$= \sup\Big\{\frac{\|D^\beta f(t_2,s_2) - D^\beta f(t_1,s_1)\|}{[|t_2-t_1|^2 + |s_2-s_1|^2]^{(\alpha-[\alpha])/2}},\ (t_1,s_1),\ (t_2,s_2) \in$$

$$[0,T_1]\times[0,T_2],\ (t_1,s_1)\neq(t_2,s_2)\Big\} < +\infty$$

$$\text{for any multindex } \beta \text{ such that } |\beta| = [\alpha]\Big\}.$$

$C^\alpha([0,T_1]\times[0,T_2];Y)$ is normed by

$$\|f\|_{C^\alpha([0,T_1]\times[0,T_2];Y)} = \sum_{0\le|\beta|\le[\alpha]} \|D^\beta f\|_{C([0,T_1]\times[0,T_2];Y)} + \sum_{|\beta|=[\alpha]} [D^\beta f]_{\alpha-[\alpha],\infty}.$$

We have used here the notation $[\alpha]$ = the largest integer not exceeding α.

Definition 0.2. Let X and $A : \mathcal{D}(A) \subset X \to X$ be a Banach space and an infinitesimal generator of an equibounded analytic semigroup in X, respectively. Then, for any $\alpha \in (0,1)$ and any $k \in \mathbb{N}\cup\{0\}$, the vector spaces $\mathcal{D}_A(k+\alpha,\infty)$ are defined by

$$\mathcal{D}_A(k+\alpha,\infty) = \Big\{x \in \mathcal{D}(A^k) : [A^k x]_{\mathcal{D}_A(\alpha,\infty)} =$$

$$= \sup_{t>0} \|t^{1-\alpha} A \exp(tA) A^k x\| < +\infty\Big\}.$$

$\mathcal{D}_A(k+\alpha,\infty)$ turns out to be a Banach space when endowed with the norm

$$\|x\|_{\mathcal{D}_A(k+\alpha,\infty)} = \sum_{j=0}^{k} \|A^j x\| + [A^k x]_{\mathcal{D}_A(\alpha,\infty)},\ \ \forall x \in \mathcal{D}_A(k+\alpha,\infty).$$

Finally, we denote by $E_1(T_1,T_2)$ and $E_2(T_1,T_2)$ the sets in $[0,T_1]\times[0,T_2]$ defined by:

(0.5) $E_1(T_1,T_2) = \Big\{(t,s) \in [0,T_1]\times[0,T_2] : s \le b_2^{-1}(b_1(t)) \quad \text{if } t \in [0,T_0]\Big\};$

(0.6) $E_2(T_1,T_2) = \Big\{(t,s) \in [0,T_0]\times[0,T_2] : s \ge b_2^{-1}(b_1(t))\Big\}.$

The functions $b_1 : [0,T_1] \to \mathbb{R}_+$, $b_2 : [0,T_2] \to \mathbb{R}_+$ are defined by

(0.7) $b_1(t) = \displaystyle\int_0^t \frac{1}{a_1(r)}\,dr,\ \forall t \in [0,T_1];\ b_2(s) = \int_0^s \frac{1}{a_2(r)}\,dr,\ \forall s \in [0,T_2].$

Moreover $[0,T_0] \subset [0,T_1]$ denotes the interval of definition of the function $b_2^{-1}\circ b_1$. We can state now our abstract result.

Theorem A. *Suppose that the assumptions K1–K7 and the compatibility condition K8, K9 of Section 2 are fulfilled. Then, problem* (0.4) *is uniquely solvable in* $C^\alpha([0, T_1] \times [0, T_2];\ \mathcal{D}(A)) \cap C^{1+\alpha}([0, T_1] \times [0, T_2];\ X)$*. Moreover, u satisfies the following estimates*:

$$
\begin{aligned}
(0.8)\qquad & \|u\|_{C^\alpha([0,T_1]\times[0,T_2];\mathcal{D}(A))} \le C_1 g(C_2\|h\|_{C^1([0,T_1]\times[0,T_2])})\cdot \\
&\cdot\big(\|u_1\|_{C^\alpha([0,T_1];\mathcal{D}(A))} + \|u_2\|_{C^\alpha([0,T_2];\mathcal{D}(A))} \\
&+ \|f\|_{C^\alpha([0,T_1]\times[0,T_2];X)} + \|D_t f\|_{C^\alpha(E_1(T_1,T_2);X)} + \|D_s f\|_{C^\alpha(E_2(T_1,T_2);X)} \\
&+ \|Au_1 + f(\cdot,0)\|_{B([0,T_1];\mathcal{D}_A(\alpha,\infty))} + \|Au_2 + f(0,\cdot)\|_{B([0,T_2];\mathcal{D}_A(\alpha,\infty))}\big);
\end{aligned}
$$

$$
\begin{aligned}
(0.9)\qquad & \|u\|_{C^{1+\alpha}([0,T_1]\times[0,T_2];X)} \le \\
&\le \big(C_3\|h\|_{C^1([0,T_1]\times[0,T_2])} g(C_2\|h\|_{C^1([0,T_1]\times[0,T_2])}) + C_4\big)\cdot \\
&\cdot\big(\|u_1\|_{C^\alpha([0,T_1];\mathcal{D}(A))} + \|u_2\|_{C^\alpha([0,T_2];\mathcal{D}(A))} \\
&+ \|u_1'\|_{C([0,T_1];X)} + \|u_2'\|_{C([0,T_2];X)} + \|f\|_{C^\alpha([0,T_1]\times[0,T_2];X)} \\
&+ \|D_t f\|_{C^\alpha(E_1(T_1,T_2);X)} + \|D_s f\|_{C^\alpha(E_2(T_1,T_2);X)} \\
&+ \|Au_1 + f(\cdot,0)\|_{B([0,T_1];\mathcal{D}_A(\alpha,\infty))} + \|Au_2 + f(0,\cdot)\|_{B([0,T_2];\mathcal{D}_A(\alpha,\infty))}\big),
\end{aligned}
$$

where C_j $(j = 1, \dots, 4)$ *are positive constants depending on* α, $M = \max_{k=0,1,2} \sup_{t\in[0,\max(T_1,T_2)]} \|t^k A e^{tA}\|_{\mathcal{L}(X)}$, T_k, $\|a_k\|_{C^\alpha([0,T_k])} + \|1/a_k\|_{C^\alpha([0,T_k])} (K = 1, 2)$ *and* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *is an increasing and analytic function such that* $g(0) = 1$.

Theorem A is then applied (cf. Section 3) to the specific problem (0.1) in the cases where $X = L^p(\Omega)$ $(p \in (1, \infty))$, $X = L^\infty(\Omega)$ and $X = C(\overline{\Omega})$, Ω denoting any open and bounded set in $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class C^2.

1. The differential problem.

In this section we consider the following problem: *determine a function* $u : [0, T_1] \times [0, T_2] \to X$ *solution to the Cauchy problem*

$$
(1.1)\qquad
\begin{cases}
a_1(t)D_t u(t,s) + a_2(s)D_s u(t,s) - \\
\qquad - Au(t,s) = f(t,s), & (t,s) \in [0,T_1] \times [0,T_2], \\
u(t,0) = u_1(t), & t \in [0,T_1], \\
u(0,s) = u_2(s), & s \in [0,T_2].
\end{cases}
$$

We assume:

H1 $a_1 \in C^{\alpha}([0,T_1])$, $0 < m_1 \le a_1(t) \le M_1$ *for any* $t \in [0,T_1]$ *and some positive constants* m_1, M_1;

H2 $a_2 \in C^{\alpha}([0,T_2])$ $(\alpha \in (0,1))$ *(cf.(0.7))*, $m_1 \le a_2(s) \le M_1$ *for any* $s \in [0,T_2]$;

H3 $f \in C^{\alpha}([0,T_1]\times[0,T_2];X)$, $D_t f \in C^{\alpha}(E_1(T_1,T_2);X)$, $D_s f \in C^{\alpha}(E_2(T_1,T_2);X)$;

H4 $A : \mathcal{D}(A) \subset X \to X$ *is a generator of an analytic semigroup* $\{\exp(tA)\}_{t\ge 0}$ *in* X;

H5 $u_j \in C^{\alpha}([0,T_j];\ \mathcal{D}(A)) \cap C^{1+\alpha}([0,T_j];X)$, $u_j' \in B([0,T_j];\ \mathcal{D}_A(\alpha,\infty))$, $(j=1,2)$;

H6 $Au_1 + f(\cdot,0) \in B([0,T_1];\ \mathcal{D}_A(\alpha,\infty))$, $Au_2 + f(0,\cdot) \in B([0,T_2];\ \mathcal{D}_A(\alpha,\infty))$;

H7 $u_1(0) = u_2(0)$, $\quad a_1(0)u_1'(0) + a_2(0)u_2'(0) - Au_1(0) = f(0,0)$.

Here $E_j(T_1,T_2)$ $(j=1,2)$ are defined by (0.5) and (0.6).

1.1. *The uniqueness of the solution.*

Definition 1.1. For any $T_1, T_2 \in \mathbb{R}_+$ we define the sets $D_1(T_1,T_2)$ and $D_2(T_1,T_2)$ by

(1.2) $D_1(T_1,T_2) = \{(\tau,\sigma) \in [0,b_1(T_1)]\times[0,b_2(T_2)] : \tau+\sigma \le b_1(T_1)\}$;

(1.3) $D_2(T_1,T_2) = \{(\tau,\sigma) \in [0,b_1(T_1)]\times[0,b_2(T_2)] : \tau+\sigma \le b_2(T_2)\}$.

We now introduce the functions $g_1 : D_1(T_1,T_2) \to X$ and $g_2 : D_2(T_1,T_2) \to X$ defined by

$$g_1(\tau,\sigma) = f(b_1^{-1}(\tau+\sigma), b_2^{-1}(\sigma)), \quad \forall(\tau,\sigma) \in D_1(T_1,T_2); \tag{1.4}$$

$$g_2(\tau,\sigma) = f(b_1^{-1}(\tau), b_2^{-1}(\tau+\sigma)), \quad \forall(\tau,\sigma) \in D_2(T_1,T_2). \tag{1.5}$$

Some basic properties of g_1 and g_2 are listed in the following lemma.

Lemma 1.1. *Suppose that assumptions H1–H3 are fulfilled. Then, the functions* $g_1, D_\tau g_1$ *belong to* $C^{\alpha}(D_1(T_1,T_2);\ X)$, *while* g_2, $D_\sigma g_2$ *belong to* $C^{\alpha}(D_2(T_1,T_2);\ X)$. *Moreover, the following estimates hold:*

$$\|g_1\|_{C^{\alpha}(D_1(T_1,T_2);X)} \le \tag{1.6}$$
$$\le \max\left(1, (2\|a_1\|^2_{C([0,T_1])} + \|a_2\|^2_{C([0,T_2])})^{\alpha/2}\right)\|f\|_{C^{\alpha}(E_1(T_1,T_2);X)};$$

(1.7) $$\|g_2\|_{C^\alpha(D_2(T_1,T_2);X)} \le$$
$$\le \max\big(1, (\|a_1\|^2_{C([0,T_1])} + 2\|a_2\|^2_{C([0,T_2])})^{\alpha/2}\big)\|f\|_{C^\alpha(E_2(T_1,T_2);X)};$$

(1.8) $$\|D_\tau g_1\|_{C^\alpha(D_1(T_1,T_2);X)} \le \max\big(1, (2\|a_1\|^2_{C([0,T_1])} + \|a_2\|^2_{C([0,T_2])})^{\alpha/2}\big) \cdot\|a_1\|_{C^\alpha([0,T_1])}\|D_t f\|_{C^\alpha(E_1(T_1,T_2);X)};$$

(1.9) $$\|D_\sigma g_2\|_{C^\alpha(D_2(T_1,T_2);X)} \le \max\big(1, (\|a_1\|^2_{C([0,T_1])} + 2\|a_2\|^2_{C([0,T_2])})^{\alpha/2}\big) \cdot\|a_2\|_{C^\alpha([0,T_2])}\|D_s f\|_{C^\alpha(E_2(T_1,T_2);X)}.$$

Proof. We limit ourselves to proving estimates (1.6), (1.8), the derivation of (1.7), (1.9) being quite similar.

We observe that the function ψ_1 defined by $\psi_1(\tau,\sigma) = (b_1^{-1}(\tau+\sigma), b_2^{-1}(\sigma))$ for any $(\tau,\sigma) \in D_1(T_1,T_2)$ is a continuous and invertible map from $D_1(T_1,T_2)$ into $E_1(T_1,T_2)$ (cf. (0.5), (0.6)). Then, for any $(\tau_j,\sigma_j) \in D_1(T_1,T_2)$ $(j=1,2)$,

(1.10) $$\|g_1(\tau_2,\sigma_2) - g_1(\tau_1,\sigma_1)\| \le$$
$$\le [f]_{C^\alpha(E_1(T_1,T_2);X)}(|b_1^{-1}(\tau_2+\sigma_2) - b_1^{-1}(\tau_1+\sigma_1)|^2 + |b_2^{-1}(\sigma_2) - b_2^{-1}(\sigma_1)|^2)^{\alpha/2}$$
$$\le (2\|a_1\|^2_{C([0,T_1])} + \|a_2\|^2_{C([0,T_2])})^{\alpha/2}[f]_{C^\alpha(E_1(T_1,T_2);X)}(|\tau_2-\tau_1|^2 + |\sigma_2-\sigma_1|^2)^{\alpha/2}.$$

Moreover,

(1.11) $$\|g_1\|_{C(D_1(T_1,T_2);X)} = \|f\|_{C(E_1(T_1,T_2);X)}.$$

Taking advantage of (1.10) and (1.11), we easily deduce that g_1 belongs to $C^\alpha(D_1(T_1,T_2);X)$ and satisfies estimate (1.6).

We now observe that g_1 is differentiable in $D_1(T_1,T_2)$ and

$$D_\tau g_1(\tau,\sigma) = a_1(b_1^{-1}(\tau+\sigma))D_t f(b_1^{-1}(\tau+\sigma), b_2^{-1}(\sigma)).$$

Consequently, for any $(\tau_j,\sigma_j) \in D_1(T_1,T_2)$ $(j=1,2)$, we get

(1.12) $$\|D_\tau g_1(\tau_2,\sigma_2) - D_\tau g_1(\tau_1,\sigma_1)\| \le$$
$$\le |a_1(b_1^{-1}(\tau_2+\sigma_2)) - a_1(b_1^{-1}(\tau_1+\sigma_1))|\|D_t f(b_1^{-1}(\tau_2+\sigma_2), b_2^{-1}(\sigma_2))\|$$
$$+|a_1(b_1^{-1}(\tau_1+\sigma_1))|\|D_t f(b_1^{-1}(\tau_2+\sigma_2), b_2^{-1}(\sigma_2)) - D_t f(b_1^{-1}(\tau_1+\sigma_1), b_2^{-1}(\sigma_1))\|$$
$$\le 2^{\alpha/2}[a_1]_{C^\alpha([0,T_1])}\|a_1\|^\alpha_{C([0,T_1])}\|D_t f\|_{C(E_1(T_1,T_2);X)}(|\tau_2-\tau_1|^2 + |\sigma_2-\sigma_1|^2)^{\alpha/2}$$
$$+(2\|a_1\|^2_{C([0,T_1])} + \|a_2\|^2_{C([0,T_2])})^{\alpha/2}\|a_1\|_{C([0,T_1])}[D_t f]_{C^\alpha(E_1(T_1,T_2);X)}(|\tau_2-\tau_1|^2 + |\sigma_2-\sigma_1|^2)^{\alpha/2}.$$

Moreover,

$$\|D_\tau g_1\|_{C(D_1(T_1,T_2);X)} \leq \|a_1\|_{C([0,T_1])}\|D_t f\|_{C(E_1(T_1,T_2);X)}. \tag{1.13}$$

From (1.12) and (1.13) we immediately get (1.8). □

Lemma 1.2. *Suppose that* $u_1 \in C^\alpha([0,T_1];\mathcal{D}(A)) \cap C^{1+\alpha}([0,T_1];X)$, $u_1' \in B([0,T_1];\ \mathcal{D}_A(\alpha,\infty))$, $u_2 \in C^\alpha([0,T_2];\ \mathcal{D}(A)) \cap C^{1+\alpha}([0,T_2];X)$, $u_2' \in B([0,T_2];\ \mathcal{D}_A(\alpha,\infty))$. *Then, the function* $\tilde{u}_j$ *defined by*

$$\tilde{u}_j(t) = u_j(b_j^{-1}(t)), \quad t \in [0, b_j(T_j)], \quad (j=1,2),$$

belongs to $C^\alpha([0,b_j(T_j)];\mathcal{D}(A)) \cap C^{1+\alpha}([0,b_j(T_j)];X)$ *and* $\tilde{u}_j' \in B([0,b_j(T_j)];\ \mathcal{D}_A(\alpha,\infty))$. *Moreover, the following estimates hold:*

$$\|\tilde{u}_j\|_{C^\alpha([0,b_j(T_j)];\mathcal{D}(A))} \leq \leq \max(1, \|a_j\|^\alpha_{C([0,T_j])})\|u_j\|_{C^\alpha([0,T_j];\mathcal{D}(A))}, \quad (j=1,2); \tag{1.14}$$

$$\|\tilde{u}_j\|_{C^{1+\alpha}([0,b_j(T_j)];X)} \leq \leq \max(1, \|a_j\|^{1+\alpha}_{C^\alpha([0,T_j])})\|u_j\|_{C^{1+\alpha}([0,T_j];X)}, \quad (j=1,2); \tag{1.15}$$

$$\|\tilde{u_j'}\|_{B([0,b_j(T_j)];\mathcal{D}_A(\alpha,\infty))} \leq \leq \|a_j\|_{C([0,T_j])}\|u_j'\|_{B([0,T_j];\mathcal{D}_A(\alpha,\infty))}, \quad (j=1,2). \tag{1.16}$$

Proof. We limit ourselves to showing (1.14)–(1.16) when $j=1$, since the derivation of (1.14)–(1.16) when $j=2$ is quite similar. We begin by proving that $\tilde{u}_1 \in C^\alpha([0,b_1(T_1)];\ \mathcal{D}(A))$. For this purpose we fix t_1, t_2 in $[0, b_1(T_1)]$ and observe that

$$\|\tilde{u}_1(t_2) - \tilde{u}_1(t_1)\| \leq [u_1]_{C^\alpha([0,T_1];X)}|b_1^{-1}(t_2) - b_1^{-1}(t_1)|^\alpha \leq \leq [u_1]_{C^\alpha([0,T_1],X)}\|a_1\|^\alpha_{C([0,T_1])}|t_2 - t_1|^\alpha. \tag{1.17}$$

Moreover,

$$\|\tilde{u}_1\|_{C([0,b_1(T_1)];X)} = \|u_1\|_{C([0,T_1];X)}. \tag{1.18}$$

Consequently,

$$\|\tilde{u}_1\|_{C^\alpha([0,b_1(T_1)];X)} \le \max(1, \|a_1\|^\alpha_{C([0,T_1])})\|u_1\|_{C^\alpha([0,T_1];X)}. \tag{1.19}$$

By the same technique, we can prove that $A\tilde{u}_1$ belongs to $C^\alpha([0, b_1(T_1)]; X)$ and satisfies the same estimate as $\tilde{u}_1$ with u_1 replaced by Au_1. Hence (1.14) follows.

We now observe that $\tilde{u}_1$ is differentiable in $[0, b_1(T_1)]$ and

$$\tilde{u}_1'(t) = a_1(b_1^{-1}(t))u_1'(b_1^{-1}(t)), \quad \forall t \in [0, b_1(T_1)].$$

Therefore

$$\|\tilde{u_1'}(t)\| \le \|a_1\|_{C([0,T_1])}\|u_1'\|_{C([0,T_1];X)}, \quad \forall t \in [0, b_1(T_1)]. \tag{1.20}$$

Moreover, for any $t_1, t_2 \in [0, b_1(T_1)]$, we get

$$\begin{aligned} \|\tilde{u_1'}(t_2) - \tilde{u_1'}(t_1)\| &\le |a_1(b_1^{-1}(t_2)) - a_1(b_1^{-1}(t_1))|\|u_1'\|_{C([0,T_1];X)} \\ &\quad + \|a_1\|_{C([0,T_1])}\|u_1'(b_1^{-1}(t_2)) - u_1'(b_1^{-1}(t_1))\| \\ &\le [a_1]_{C^\alpha([0,T_1])}\|a_1\|^\alpha_{C([0,T_1])}\|u_1'\|_{C([0,T_1];X)}|t_2 - t_1|^\alpha \\ &\quad + \|a_1\|^{1+\alpha}_{C([0,T_1])}[u_1']_{C^\alpha([0,T_1];X)}|t_2 - t_1|^\alpha. \end{aligned} \tag{1.21}$$

From (1.20) and (1.21) we derive the following estimate:

$$\|\tilde{u_1'}\|_{C^\alpha([0,b_1(T_1)];X)} \le \max(1, \|a_1\|^\alpha_{C([0,T_1])})\|a_1\|_{C^\alpha([0,T_1])}\|u_1'\|_{C^\alpha([0,T_1];X)}. \tag{1.22}$$

From (1.18) and (1.22) we get (1.15). Then, it is an easy task to prove (1.16). □

Theorem 1.1. *Under assumptions H1–H7 problem* (1.1) *admits at most an unique solution* $u \in C^\alpha([0, T_1] \times [0, T_2]; \mathcal{D}(A)) \cap C^{1+\alpha}([0, T_1] \times [0, T_2]; X)$ *represented by the following formulae:*

$$\begin{aligned} u(t, s) &= \exp[b_2(s)A]u_1(b_1^{-1}(b_1(t) - b_2(s))) \\ &\quad + \int_0^{b_2(s)} \exp[(b_2(s) - \xi)A] f(b_1^{-1}(b_1(t) - b_2(s) + \xi), b_2^{-1}(\xi))\, d\xi, \\ &\qquad (t, s) \in E_1(T_1, T_2); \end{aligned} \tag{1.23}$$

$$\begin{aligned} u(t, s) &= \exp[b_1(t)A]u_2(b_2^{-1}(b_2(s) - b_1(t))) \\ &\quad + \int_0^{b_1(t)} \exp[(b_1(t) - \xi)A] f(b_1^{-1}(\xi), b_2^{-1}(b_2(s) - b_1(t) + \xi))\, d\xi, \\ &\qquad (t, s) \in E_2(T_1, T_2). \end{aligned} \tag{1.24}$$

Proof. Let us suppose that $u \in C^{\alpha}([0,T_1]\times[0,T_2];\mathcal{D}(A)) \cap C^{1+\alpha}([0,T_1]\times[0,T_2];X)$ solves problem (1.1). Then the functions $v_j : D_j(T_1,T_2) \to X$ $(j = 1,2)$ defined by

$$v_1(\tau,\sigma) = u(b_1^{-1}(\tau+\sigma), b_2^{-1}(\sigma)), \quad (\tau,\sigma) \in D_1(T_1,T_2);$$

$$v_2(\tau,\sigma) = u(b_1^{-1}(\tau), b_2^{-1}(\tau+\sigma)), \quad (\tau,\sigma) \in D_2(T_1,T_2),$$

turn out be solutions, respectively, to the Cauchy problems

$$\text{(1.25)} \qquad \begin{cases} D_\sigma v_1(\tau,\sigma) - Av_1(\tau,\sigma) = \\ \qquad = f(b_1^{-1}(\tau+\sigma), b_2^{-1}(\sigma)), & (\tau,\sigma) \in D_1(T_1,T_2), \\ v_1(\tau,0) = u_1(b_1^{-1}(\tau)), & \tau \in [0, b_1(T_1)]; \end{cases}$$

$$\text{(1.26)} \qquad \begin{cases} D_\tau v_2(\tau,\sigma) - Av_2(\tau,\sigma) = \\ \qquad = f(b_1^{-1}(\tau), b_2^{-1}(\tau+\sigma)), & (\tau,\sigma) \in D_2(T_1,T_2), \\ v_2(0,\sigma) = u_2(b_2^{-1}(\sigma)), & \sigma \in [0, b_2(T_2)]. \end{cases}$$

Thanks to Lemmata 1.1, 1.2 and Theorem 4.3.1 in [7] we deduce that problems (1.25) and (1.26) admit the unique solutions v_1 and v_2, respectively, given by the following formulae:

$$\text{(1.27)} \qquad v_1(\tau,\sigma) = \exp(\sigma A)u_1(b_1^{-1}(\tau)) + \\ + \int_0^\sigma \exp[(\sigma-\xi)A] f(b_1^{-1}(\tau+\xi), b_2^{-1}(\xi))\, d\xi;$$

$$\text{(1.28)} \qquad v_2(\tau,\sigma) = \exp(\tau A)u_2(b_2^{-1}(\sigma)) + \\ + \int_0^\tau \exp[(\tau-\xi)A] f(b_1^{-1}(\xi), b_2^{-1}(\sigma+\xi))\, d\xi.$$

Finally, the assertion easily follows from the representation formula

$$\text{(1.29)} \qquad u(t,s) = \begin{cases} v_1(b_1(t) - b_2(s), b_2(s)), & (t,s) \in E_1(T_1,T_2), \\ v_2(b_1(t), b_2(s) - b_1(t)), & (t,s) \in [0,T_0]\times[0,T_2], \\ & 0 \le b_2^{-1} \circ b_1(t) < s. \end{cases} \qquad \square$$

1.2. *Existence of the solution.*

By virtue of Theorem 1.1 we need only to prove that (1.23), (1.24) define a solution to problem (1.1) belonging to $C^{\alpha}([0,T_1]\times[0,T_2];\ \mathcal{D}(A))\cap C^{1+\alpha}([0,T_1]\times[0,T_2];X)$.
We begin by proving a lemma.

Lemma 1.3. *Suppose that f fulfills assumption H3. Then the functions g_1 and g_2 defined by* (1.4) *and* (1.5) *satisfy the inequalities*

$$(1.30)\qquad \begin{aligned}&\|g_1(\tau_2,\sigma_2)-g_1(\tau_2,\sigma_1)-g_1(\tau_1,\sigma_2)+g_1(\tau_1,\sigma_1)\|\le\\ &\le(2\|a_1\|^2_{C([0,T_1])}+\|a_2\|^2_{C([0,T_2])})^{\alpha/2}(2[f]_{C^{\alpha}(E_1(T_1,T_2);X)}+\\ &+\|a_1\|_{C^{\alpha}([0,T_1])}\|D_tf\|_{C^{\alpha}(E_1(T_1,T_2);X)})|\tau_2-\tau_1|^{\alpha}|\sigma_2-\sigma_1|^{\alpha},\end{aligned}$$

for any (τ_1,σ_1), (τ_1,σ_2), (τ_2,σ_1), $(\tau_2,\sigma_2)\in D_1(T_1,T_2)$;

$$(1.31)\qquad \begin{aligned}&\|g_2(\tau_2,\sigma_2)-g_2(\tau_2,\sigma_1)-g_2(\tau_1,\sigma_2)+g_2(\tau_1,\sigma_1)\|\le\\ &\le(\|a_1\|^2_{C([0,T_1])}+2\|a_2\|^2_{C([0,T_2])})^{\alpha/2}(2[f]_{C^{\alpha}(E_2(T_1,T_2);X)}+\\ &+\|a_2\|_{C^{\alpha}([0,T_2])}\|D_sf\|_{C^{\alpha}(E_2(T_1,T_2);X)})|\tau_2-\tau_1|^{\alpha}|\sigma_2-\sigma_1|^{\alpha},\end{aligned}$$

for any (τ_1,σ_1), (τ_1,σ_2), (τ_2,σ_1), $(\tau_2,\sigma_2)\in D_2(T_1,T_2)$,

Proof. We limit ourselves to proving estimate (1.30), the derivation of (1.31) being quite similar.
To derive inequality (1.30) we observe that, for any (τ_1,σ_1), (τ_1,σ_2), (τ_2,σ_1), $(\tau_2,\sigma_2)\in D_1(T_1,T_2)$ we can deduce (cf. (1.6), (1.8)) the following estimates according as $|\tau_2-\tau_1|\le 1$ or $|\tau_2-\tau_1|>1$:

$$\begin{aligned}&\|g_1(\tau_2,\sigma_2)-g_1(\tau_2,\sigma_1)-g_1(\tau_1,\sigma_2)+g_1(\tau_1,\sigma_1)\|=\\ &=\Big\|\int_{\tau_1}^{\tau_2}[D_\tau g_1(r,\sigma_2)-D_\tau g_1(r,\sigma_1)]dr\Big\|\\ &\le[D_\tau g_1]_{C^{\alpha}(D_1(T_1,T_2);X)}|\tau_2-\tau_1||\sigma_2-\sigma_1|^{\alpha}\\ &\le(2\|a_1\|^2_{C([0,T_1])}+\|a_2\|^2_{C([0,T_2])})^{\alpha/2}\|a_1\|_{C^{\alpha}([0,T_1])}\\ &\quad\cdot\|D_tf\|_{C^{\alpha}(E_1(T_1,T_2);X)}|\tau_2-\tau_1||\sigma_2-\sigma_1|^{\alpha}\\ &\le(2\|a_1\|^2_{C([0,T_1])}+\|a_2\|^2_{C([0,T_2])})^{\alpha/2}\|a_1\|_{C^{\alpha}([0,T_1])}\\ &\quad\cdot\|D_tf\|_{C^{\alpha}(E_1(T_1,T_2);X)}|\tau_2-\tau_1|^{\alpha}|\sigma_2-\sigma_1|^{\alpha};\end{aligned}$$

$$\|g_1(\tau_2,\sigma_2)-g_1(\tau_2,\sigma_1)-g_1(\tau_1,\sigma_2)+g_1(\tau_1,\sigma_1)\| \le 2[g_1]_{C^\alpha(D_1(T_1,T_2);X)}|\tau_2-\tau_1|^\alpha$$
$$\le 2(2\|a_1\|^2_{C([0,T_1])}+\|a_2\|^2_{C([0,T_2])})^{\alpha/2}[f]_{C^\alpha(E_1(T_1,T_2);X)}|\tau_2-\tau_1|^\alpha|\sigma_2-\sigma_1|^\alpha.$$

□

Taking Lemmata 1.1, 1.2, 1.3 into account, we can prove the following theorem.

Theorem 1.2. *Under assumptions H1–H7 the functions* v_1 *and* v_2 *defined by* (1.27) *and* (1.28) *belong to* $C^\alpha(D_1(T_1,T_2);\mathcal{D}(A))\cap C^{1+\alpha}(D_1(T_1,T_2);X)$ *and* $C^\alpha(D_2(T_1,T_2);\mathcal{D}(A))\cap C^{1+\alpha}(D_2(T_1,T_2);X)$*, respectively. Moreover, they solve the Cauchy problems* (1.25) *and* (1.26)*, respectively, and satisfy the following estimates (cf. Definition* 0.1*):*

$$\begin{aligned}(1.32)\qquad &\|v_1\|_{C^\alpha(D_1(T_1,T_2);\mathcal{D}(A))}\le\\ &\le C_1(\|u_1\|_{C^\alpha([0,T_1];\mathcal{D}(A))}+\|D_t f\|_{C^\alpha(E_1(T_1,T_2);X)}\\ &+\|f\|_{C^\alpha(E_1(T_1,T_2);X)}+\|Au_1+f(\cdot,0)\|_{B([0,T_1];\mathcal{D}_A(\alpha,\infty))});\end{aligned}$$

$$\begin{aligned}(1.33)\qquad &\|v_1\|_{C^{1+\alpha}(D_1(T_1,T_2);X)}\le\\ &\le C_2(\|u_1\|_{C^\alpha([0,T_1];\mathcal{D}(A))}+\|u_1'\|_{C^\alpha([0,T_1];X)}+\|D_t f\|_{C^\alpha(E_1(T_1,T_2);X)}\\ &+\|f\|_{C^\alpha(E_1(T_1,T_2);X)}+\|Au_1+f(\cdot,0)\|_{B([0,T_1];\mathcal{D}_A(\alpha,\infty))});\end{aligned}$$

$$\begin{aligned}(1.34)\qquad &\|v_2\|_{C^\alpha(D_2(T_1,T_2);\mathcal{D}(A))}\le\\ &\le C_3(\|u_2\|_{C^\alpha([0,T_2];\mathcal{D}(A))}+\|D_s f\|_{C^\alpha(E_2(T_1,T_2);X)}\\ &+\|f\|_{C^\alpha(E_2(T_1,T_2);X)}+\|Au_2+f(0,\cdot)\|_{B([0,T_2];\mathcal{D}_A(\alpha,\infty))});\end{aligned}$$

$$\begin{aligned}(1.35)\qquad &\|v_2\|_{C^\alpha(D_2(T_1,T_2);X)}\le\\ &\le C_4(\|u_2\|_{C^\alpha([0,T_2];\mathcal{D}(A))}+\|u_2'\|_{C^\alpha([0,T_2];X)}+\|D_s f\|_{C^\alpha(E_2(T_1,T_2);X)}\\ &+\|f\|_{C^\alpha(E_2(T_1,T_2);X)}+\|Au_2+f(0,\cdot)\|_{B([0,T_2];\mathcal{D}_A(\alpha,\infty))}),\end{aligned}$$

where C_j $(j=1,\dots,4)$ *are positive functions depending on* α, T_l, $\|a_l\|_{C^\alpha([0,T_l])}+\|1/a_l\|_{C([0,T_l])}$ $(l=1,2)$*, and on* $M=\max_{k=0,1,2}\sup_{t\in[0,\max(T_1,T_2)]}\|t^kA^ke^{tA}\|_{\mathcal{L}(X)}$.

Proof. We limit ourselves to proving the assertion for the function v_1, the other case being quite similar. We begin by nothing that $v_1 \in C^\alpha(D_1(T_1, T_2); \mathcal{D}(A))$ if and only if $v_1(\cdot, \sigma) \in C^\alpha(D_1^\sigma(T_1, T_2); \mathcal{D}(A))$ and $v_1(\tau, \cdot) \in C^\alpha(\widetilde{D}_1^\tau(T_1, T_2); \mathcal{D}(A))$ for any $\sigma \in [0, b_1(T_1)]$ and any $\tau \in [0, \min(b_1(T_1), b_2(T_2))]$ with Hölder norms and seminorms being independent of σ and τ, respectively. Here $D_1^\sigma(T_1, T_2) = \{\tau \in [0, b_1(T_1)] : (\tau, \sigma) \in D_1(T_1, T_2)\}$ and $\widetilde{D}_1^\tau(T_1, T_2) = \{\sigma \in [0, b_2(T_2)] : (\tau, \sigma) \in D_1(T_1, T_2)\}$. Moreover,

$$\|v_1\|_{C^\alpha(D_1(T_1,T_2);X)} \leq \sup_{\tau\in[0,b_1(T_1)]} \|v_1(\tau, \cdot)\|_{C(\widetilde{D}_1^\tau(T_1,T_2);X)}$$
$$+ \sup_{\tau\in[0,b_1(T_1)]} [v_1(\tau, \cdot)]_{C^\alpha(\widetilde{D}_1^\tau(T_1,T_2);X)} + \sup_{\sigma\in[0,b_2(T_2)]} [v_1(\cdot, \sigma)]_{C^\alpha(D_1^\sigma(T_1,T_2);X)}.$$

Then, we observe that, thanks to assumptions H1–H6 and Theorem 4.3.1 in [7], $v_1(\tau, \cdot)$ belongs to $C^{1+\alpha}(\widetilde{D}_1^\tau(T_1, T_2); X) \cap C^\alpha(\widetilde{D}_1^\tau(T_1, T_2); \mathcal{D}(A))$ for any $\tau \in [0, b_1(T_1)]$ and

$$D_\sigma v_1(\tau, \sigma) = Av_1(\tau, \sigma) + f(b_1^{-1}(\tau + \sigma), b_2^{-1}(\sigma)), \quad (\tau, \sigma) \in D_1(T_1, T_2). \tag{1.36}$$

Moreover, there exists a positive function C depending on $M, T_1, T_2, \alpha, \|1/a_j\|_{C([0,T_j])}$ $(j = 1, 2)$ such that (cf. (1.6))

$$\sup_{\tau\in[0,b_1(T_1)]} \big[\|v_1(\tau, \cdot)\|_{C^{1+\alpha}(\widetilde{D}_1^\tau(T_1,T_2);X)} + \|v_1(\tau, \cdot)\|_{C^\alpha(\widetilde{D}_1^\tau(T_1,T_2);\mathcal{D}(A))}\big] \leq$$
$$\leq C(\|u_1\|_{C([0,T_1];\mathcal{D}(A))}$$
$$+ \max\big(1, (2\|a_1\|^2_{C([0,T_1])} + \|a_2\|^2_{C([0,T_2])})^{\alpha/2}\big)\|f\|_{C^\alpha(E_1(T_1,T_2);X)}$$
$$+ \|Au_1 + f(\cdot, 0)\|_{B([0,T_1];\mathcal{D}_A(\alpha,\infty))}). \tag{1.37}$$

Therefore, we can limit ourselves to proving that $v_1(\cdot, \sigma)$ belongs to

$$C^{1+\alpha}(D_1^\sigma(T_1, T_2); X) \cap C^\alpha(D_1^\sigma(T_1, T_2); \mathcal{D}(A)),$$

for any $\sigma \in [0, b_1(T_1)]$.
Taking Lemmata 1.1, 1.2 into account, it is easy to show that, for any $\sigma \in [0, b_1(T_1)]$, $v_1(\cdot, \sigma) \in C^\alpha(D_1^\sigma(T_1, T_2); X)$ and

$$\sup_{\sigma\in[0,b_2(T_2)]} [v_1(\cdot, \sigma)]_{C^\alpha(D_1^\sigma(T_1,T_2);X)} \leq M\|a_1\|^\alpha_{C([0,T_1])}[u_1]_{C^\alpha([0,T_1];X)}$$
$$+ Mb_2(T_2)\max(1, (2\|a_1\|^2_{C([0,T_1])} + \|a_2\|^2_{C([0,T_2])})^{\alpha/2})[f]_{C^\alpha(E_1(T_1,T_2);X)}. \tag{1.38}$$

Thanks to Theorem 4.1 in [12], we get

$$
\begin{aligned}
(1.39)\quad Av_1(\tau,\sigma) = {} & \exp(\sigma A)[Au_1(b_1^{-1}(\tau)) + f(b_1^{-1}(\tau+\sigma), b_2^{-1}(\sigma))] \\
& - f(b_1^{-1}(\tau+\sigma), b_2^{-1}(\sigma)) \\
& + \int_0^\sigma A\exp((\sigma-\xi)A)[f(b_1^{-1}(\tau+\xi), b_2^{-1}(\xi)) - f(b_1^{-1}(\tau+\sigma), b_2^{-1}(\sigma))]d\xi.
\end{aligned}
$$

Therefore, for any $\tau_1, \tau_2 \in D_1^\sigma(T_1,T_2)$, $\tau_1 \leq \tau_2$ we have (cf. Lemma 1.3)

$$
\begin{aligned}
(1.40)\quad & \|Av_1(\tau_2,\sigma) - Av_1(\tau_1,\sigma)\| \leq \\
& \leq \|\exp(\sigma A)[Au_1(b_1^{-1}(\tau_2)) + f(b_1^{-1}(\tau_2+\sigma), b_2^{-1}(\sigma)) - Au_1(b_1^{-1}(\tau_1)) \\
& \quad - f(b_1^{-1}(\sigma+\tau_1), b_2^{-1}(\sigma))]\| \\
& + \|f(b_1^{-1}(\tau_2+\sigma), b_2^{-1}(\sigma)) - f(b_1^{-1}(\tau_1+\sigma), b_2^{-1}(\sigma))\| \\
& + \Big\| \int_0^\sigma A\exp(\xi A)[f(b_1^{-1}(\tau_2+\sigma-\xi), b_2^{-1}(\sigma-\xi)) - f(b_1^{-1}(\tau_2+\sigma), b_2^{-1}(\sigma)) \\
& \quad - f(b_1^{-1}(\tau_1+\sigma-\xi), b_2^{-1}(\sigma-\xi)) + f(b_1^{-1}(\tau_1+\sigma), b_2^{-1}(\sigma))]\,d\xi \Big\| \\
& \leq M\|a_1\|^\alpha_{C([0,T_1])}[Au_1]_{C^\alpha([0,T_1];X)}|\tau_2-\tau_1|^\alpha \\
& + (M+1)(2\|a_1\|^2_{C([0,T_1])} + \|a_2\|^2_{C([0,T_2])})^{\alpha/2}[f]_{C^\alpha(E_1(T_1,T_2);X)}|\tau_2-\tau_1|^\alpha \\
& + M\alpha^{-1}(b_2(T_2))^\alpha(2\|a_1\|^2_{C([0,T_1])} + \|a_2\|^2_{C([0,T_2])})^{\alpha/2} \\
& \cdot(2[f]_{C^\alpha(E_1(T_1,T_2);X)} + \|a_1\|_{C^\alpha([0,T_1])}\|D_t f\|_{C^\alpha(E_1(T_1,T_2);X)})|\tau_2-\tau_1|^\alpha.
\end{aligned}
$$

Consequently, from (1.37), (1.38), (1.40) we deduce that there exists a positive function C_1 depending on $M, \alpha, T_k, \|a_k\|_{C^\alpha([0,T_{2-k}])} + \|1/a_k\|_{C([0,T_k])} (k=1,2)$ for which (1.32) holds. Then, (1.6), (1.32) and (1.36) imply that $D_\sigma v_1 \in C^\alpha(D_1(T_1,T_2);X)$ and there exists a positive function $\widetilde{C}_1$ such that (1.33) holds. To conclude the proof we must show that v_1 is differentiable with respect to variable τ in $D_1(T_1,T_2)$ and $D_\tau v_1$ satisfies estimate (1.33). First we observe that $D_\tau v_1$ exists for any $(\tau,\sigma) \in D_1(T_1,T_2)$ and

$$
\begin{aligned}
(1.41)\quad D_\tau v_1(\tau,\sigma) = {} & \exp(\sigma A)D_\tau[u_1(b_1^{-1}(\tau))] + \\
& + \int_0^\sigma \exp(\xi A)D_\tau[f(b_1^{-1}(\tau+\sigma-\xi), b_2^{-1}(\sigma-\xi))]\,d\xi.
\end{aligned}
$$

Then, an easy computation shows that $D_\tau v_1(\tau,\cdot) \in C^\alpha(\widetilde{D}_1^\tau(T_1,T_2);X)$ for any $\tau \in [0, b_1(T_1)]$ and $D_\tau v_1(\cdot,\sigma) \in C^\alpha(D_1^\alpha(T_1,T_2);X)$ for any $\sigma \in [0, b_2(T_2)]$.

Moreover, the following estimates hold:

$$(1.42)\qquad \sup_{\tau\in[0,b_1(T_1)]} \|D_\tau v_1(\tau,\cdot)\|_{C^\alpha(\widetilde{D}_1^\tau(T_1,T_2);X)} \le$$
$$\le (1/\alpha)\|a_1\|_{C([0,T_1])}\|u_1'\|_{B([0,T_1];\mathcal{D}_A(\alpha,\infty))}$$
$$+ 2M\max(b_2(T_2)^{1-\alpha}, b_2(T_2))\max(1, (2\|a_1\|^2_{C([0,T_1])} + \|a_2\|^2_{C([0,T_2])})^{\alpha/2})$$
$$\cdot\|a_1\|_{C^\alpha([0,T_1])}\|D_t f\|_{C^\alpha(E_1(T_1,T_2);X)};$$

$$(1.43)\qquad \sup_{\sigma\in[0,b_2(T_2)]} [D_\tau v_1(\cdot,\sigma)]_{C^\alpha(D_1^\sigma(T_1,T_2);X)} \le M\|a_1\|^{1+\alpha}_{C^\alpha([0,T_1])}\|u_1'\|_{C^\alpha([0,T_1];X)}$$
$$+Mb_2(T_2)(2\|a_1\|^2_{C([0,T_1])} + \|a_2\|^2_{C([0,T_2])})^{\alpha/2}\|a_1\|_{C^\alpha([0,T_1])}\|D_t f\|_{C^\alpha(E_1(T_1,T_2);X)}.$$

Taking advantage of (1.42), (1.43) and the previous results, we deduce that v_1 belongs to $C^{1+\alpha}(D_1(T_1,T_2);X)$ and satisfies estimate (1.33). □

We can now prove the following existence theorem.

Theorem 1.3. *Suppose that assumptions H1–H7 are fulfilled. Then, the function u defined by* (1.23) *and* (1.24) *belongs to* $C^\alpha([0,T_1]\times[0,T_2];\mathcal{D}(A))\cap C^{1+\alpha}([0,T_1]\times[0,T_2];X)$. *In particular, u is the unique solution to the Cauchy problem* (1.1) *and satisfies the following estimates:*

$$(1.44)\qquad \|u\|_{C^\alpha([0,T_1]\times[0,T_2];\mathcal{D}(A))} \le C_5(\|u_1\|_{C^\alpha([0,T_1];\mathcal{D}(A))} + \|u_2\|_{C^\alpha([0,T_2];\mathcal{D}(A))}$$
$$+ \|f\|_{C^\alpha([0,T_1]\times[0,T_2];X)} + \|D_t f\|_{C^\alpha(E_1(T_1,T_2);X)} + \|D_s f\|_{C^\alpha(E_2(T_1,T_2);X)}$$
$$+ \|Au_1 + f(\cdot,0)\|_{B([0,T_1];\mathcal{D}_A(\alpha,\infty))} + \|Au_2 + f(0,\cdot)\|_{B([0,T_2];\mathcal{D}_A(\alpha,\infty))});$$

$$(1.45)\qquad \|u\|_{C^{1+\alpha}([0,T_1]\times[0,T_2];X)} \le C_6(\|u_1\|_{C^{1+\alpha}([0,T_1];\mathcal{D}(A))} + \|u_2\|_{C^\alpha([0,T_2];\mathcal{D}(A))}$$
$$+ \|u_1'\|_{C^\alpha([0,T_1];X)} + \|u_2'\|_{C^\alpha([0,T_2];X)} + \|f\|_{C^\alpha([0,T_1]\times[0,T_2];X)}$$
$$+ \|D_t f\|_{C^\alpha(E_1(T_1,T_2);X)} + \|D_s f\|_{C^\alpha(E_2(T_1,T_2);X)}$$
$$+ \|Au_1 + f(\cdot,0)\|_{B([0,T_1];\mathcal{D}_A(\alpha,\infty))} + \|Au_2 + f(0,\cdot)\|_{B([0,T_2];\mathcal{D}_A(\alpha,\infty))}),$$

C_5 and C_6 being positive functions depending on $M,\alpha,T_j,\|a_j\|_{C^\alpha([0,T_j])} + \|1/a_j\|_{C^\alpha([0,T_j])}$ $(j=1,2)$.

Proof. We begin by noting that for any Banach space Y and any $g : [0, T_1] \times [0, T_2] \to Y$ such that $g \in C^\alpha(E_1(T_1, T_2); Y) \cap C^\alpha(E_2(T_1, T_2); Y)$, g belongs to $C^\alpha([0, T_1] \times [0, T_2]; Y)$ and satisfies the following estimate:

$$\|g\|_{C^\alpha([0,T_1]\times[0,T_2];Y)} \le 2^{1-\alpha}(\|g\|_{C^\alpha(E_1(T_1,T_2);Y)} + \|g\|_{C^\alpha(E_2(T_1,T_2);Y)}). \tag{1.46}$$

In fact, suppose that $(t_j, s_j) \in E_j(T_1, T_2)$ $(j = 1, 2)$ and let (τ, σ) be the unique point on the straight line joining (t_1, s_1) and (t_2, s_2) such that $b_1(\tau) = b_2(\sigma)$. Then, we have

$$\begin{aligned} \|g(t_2, s_2) - g(t_1, s_1)\| &\le \|g(t_2, s_2) - g(\tau, \sigma)\| + \|g(\tau, \sigma) - g(t_1, s_1)\| \\ &\le [g]_{C^\alpha(E_1(T_1,T_2);X)}(|t_2 - \tau|^2 + |s_2 - \sigma|^2)^{\alpha/2} \\ &\quad + [g]_{C^\alpha(E_2(T_1,T_2);X)}(|\tau - t_1|^2 + |\sigma - s_1|^2)^{\alpha/2} \\ &\le 2^{1-\alpha} \max([g]_{C^\alpha(E_1(T_1,T_2);X)}, [g]_{C^\alpha(E_2(T_1,T_2);X)}) \\ &\qquad \cdot[|t_2 - t_1|^2 + |s_2 - s_1|^2]^{\alpha/2}. \end{aligned} \tag{1.47}$$

Moreover,

$$\|g\|_{C([0,T_1]\times[0,T_2];Y)} \le \max(\|g\|_{C(E_1(T_1,T_2);Y)}, \|g\|_{C(E_2(T_1,T_2);Y)}). \tag{1.48}$$

From (1.47) and (1.48) we easily get (1.46).
Thanks to (1.46) we can limit ourselves to showing that u belongs to $C^\alpha(E_1(T_1, T_2); \mathcal{D}(A)) \cap C^{1+\alpha}(E_1(T_1, T_2); X)$ and to $C^\alpha(E_2(T_1, T_2); \mathcal{D}(A)) \cap C^{1+\alpha}(E_2(T_1, T_2); X)$. Let us now prove that $u \in C^\alpha(E_1(T_1, T_2); \mathcal{D}(A))$. We first observe that, thanks to H7, u is continuous at the points $(t, b_2^{-1} \circ b_1(t))$ for any $t \in [0, T_1]$. Then, for any $(t_j, s_j) \in E_1(T_1, T_2)$ $(j = 1, 2)$, we have

$$\begin{aligned} &\|u(t_2, s_2) - u(t_1, s_1)\| \le \\ &\le \|v_1(b_1(t_2) - b_2(s_2), b_2(s_2)) - v_1(b_1(t_1) - b_2(s_1), b_2(s_1))\| \\ &\le \max(2^{\alpha/2}\|1/a_1\|^\alpha_{C([0,T_1])}, 3^{\alpha/2}\|1/a_2\|^\alpha_{C([0,T_2])})[v_1]_{C^\alpha(D_1(T_1,T_2);X)} \\ &\qquad \cdot(|t_2 - t_1|^2 + |s_2 - s_1|^2)^{\alpha/2}. \end{aligned} \tag{1.49}$$

Moreover,

$$\|u\|_{C(E_1(T_1,T_2);X)} = \|v_1\|_{C(D_1(T_1,T_2);X)}. \tag{1.50}$$

From (1.49) and (1.50) we deduce that u belongs to $C^\alpha(E_1(T_1, T_2); X)$ and satisfies

$$\begin{aligned} &\|u\|_{C^\alpha(E_1(T_1,T_2);X)} \le \\ &\le \max(1, 2^{\alpha/2}\|1/a_1\|^\alpha_{C([0,T_1])}, 3^{\alpha/2}\|1/a_2\|^\alpha_{C([0,T_2])})\|v_1\|_{C^\alpha(D_1(T_1,T_2);X)}. \end{aligned} \tag{1.51}$$

Reasoning likewise, we can easily prove that the function Au belongs to $C^\alpha(E_1(T_1, T_2); X)$ and fulfills (1.51) with v_1 replaced by Av_1. Therefore, $u \in C^\alpha(E_1(T_1, T_2); \mathcal{D}(A))$ and

$$\begin{aligned}(1.52) \qquad & \|u\|_{C^\alpha(E_1(T_1,T_2);\mathcal{D}(A))} \leq \\ & \leq \max(1, 2^{\alpha/2}\|1/a_1\|^\alpha_{C([0,T_1])}, 3^{\alpha/2}\|1/a_2\|^\alpha_{C([0,T_2])})\|v_1\|_{C^\alpha(D_1(T_1,T_2);\mathcal{D}(A))}.\end{aligned}$$

Analogously, we can prove that $u \in C^\alpha(E_2(T_1, T_2); \mathcal{D}(A))$ and

$$(1.53) \qquad \|u\|_{C(E_2(T_1,T_2);X)} = \|v_2\|_{C(D_2(T_1,T_2);X)};$$

$$\begin{aligned}(1.54) \qquad & \|u\|_{C^\alpha(E_2(T_1,T_2);\mathcal{D}(A))} \leq \\ & \leq \max(1, 3^{\alpha/2}\|1/a_1\|^\alpha_{C([0,T_1])}, 2^{\alpha/2}\|1/a_2\|^\alpha_{C([0,T_2])})\|v_2\|_{C^\alpha(D_2(T_1,T_2);\mathcal{D}(A))}.\end{aligned}$$

Taking (1.32), (1.34) and (1.46) into account, from (1.52), (1.54) we deduce that $u \in C^\alpha([0, T_1] \times [0, T_2]; \mathcal{D}(A))$ and satisfies estimate (1.44).
We now prove that u is differentiable in $[0, T_1] \times [0, T_2]$. From Theorem 1.2, we deduce that u is differentiable at any point $(t, s) \in F = [0, T_1] \times [0, T_2] \setminus \{(t, b_2^{-1} \circ b_1(t)) : t \in [0, T_0]\}$ (cf. (0.5), (0.6)) and solves the Cauchy problem (1.1) in F. Let us now prove that u is differentiable also at the points $(t, b_2^{-1} \circ b_1(t))$. Since Au is continuous in $[0, T] \times [0, b_2^{-1} \circ b_1(T)]$ and $a_2(t) \geq m_1$, for any $t \in [0, T_1]$, we have only to show that u is differentiable with respect to t at the points $(t, b_2^{-1} \circ b_2(t))$, $(t \in [0, T_0])$. We note also that

$$(1.55) \quad D_t u(t, s) = \frac{1}{a_1(t)} D_\tau v_1(b_1(t) - b_2(s), b_2(s)), \quad (t, s) \in E_1(T_1, T_2);$$

$$\begin{aligned}(1.56) \qquad D_t u(t, s) = \frac{1}{a_1(t)}[& D_\tau v_2(b_1(t), b_2(s) - b_1(t)) - \\ & - D_\sigma v_2(b_1(t), b_2(s) - b_1(t))], \quad 0 \leq b_2^{-1} \circ b_1(t) < s.\end{aligned}$$

Therefore, u is differentiable with respect to t at $(t, b_2^{-1} \circ b_1(t))$, if and only if $D_\tau v_1(0, b_1(t)) - D_\tau v_2(b_1(t), 0) + D_\sigma v_2(b_1(t), 0) = 0$. We observe that (cf. Theorem 4.1 in [12])

$$\begin{aligned}(1.57) \qquad & D_\tau v_1(0, b_1(t)) - D_\tau v_2(b_1(t), 0) + D_\sigma v_2(b_1(t), 0) = \\ & = \exp(b_1(t)A)[a_1(0)u_1'(0) + a_2(0)u_2'(0) - Au_2(0)]\end{aligned}$$

$$+\int_0^{b_1(t)} a_1(b_1^{-1}(\xi))\exp((b_1(t)-\xi)A)D_t f(b_1^{-1}(\xi), b_2^{-1}(\xi))\,d\xi$$
$$+\int_0^{b_1(t)} a_2(b_2^{-1}(\xi))\exp((b_1(t)-\xi)A)D_s f(b_1^{-1}(\xi), b_2^{-1}(\xi))\,d\xi$$
$$-A\int_0^{b_1(t)} \exp((b_1(t)-\xi)A) f(b_1^{-1}(\xi), b_2^{-1}(\xi))d\xi - f(t, b_2^{-1}\circ b_1(t))$$
$$=\exp(b_1(t)A)[a_1(0)u_1'(0)+a_2(0)u_2'(0)-Au_2(0)-f(t,b_2^{-1}\circ b_1(t))]$$
$$+\int_0^{b_1(t)} \exp((b_1(t)-\xi)A)D_\xi f(b_1^{-1}(\xi), b_2^{-1}(\xi))\,d\xi$$
$$-\int_0^{b_1(t)} A\exp((b_1(t)-\xi)A)[f(b_1^{-1}(\xi), b_2^{-1}(\xi))-f(t,b_2^{-1}\circ b_1(t))]\,d\xi$$
$$=\exp(b_1(t)A)[a_1(0)u_1'(0)+a_2(0)u_2'(0)-Au_2(0)-f(0,0)].$$

Therefore, thanks to assumption H7,

$$D_\tau v_1(0,b_1(t)) - D_\sigma v_2(b_1(t),0) + D_\sigma v_2(b_1(t),0) = 0.$$

Hence, u is differentiable with respect to t at $(t, b_2^{-1}\circ b_1(t))$.
Let us now prove that $D_t u \in C^\alpha(E_1(T_1,T_2);X)$. Fix $(t_j,s_j)\in E_1(T_1,T_2)$ $(j=1,2)$ and observe that

$$\|D_t u(t_2,s_2) - D_t u(t_1,s_1)\| \le \tag{1.58}$$
$$\le \|D_\tau v_1(b_1(t_2)-b_2(s_2), b_2(s_2)) - D_\tau v_1(b_1(t_1)-b_2(s_1), b_2(s_1))\|\left|\frac{1}{a_1(t_2)}\right|$$
$$+\left|\frac{1}{a_1(t_2)}-\frac{1}{a_1(t_1)}\right| \|D_\tau v_1(b_1(t_1)-b_2(s_1), b_2(s_1))\|$$
$$\le \max(2^{\alpha/2}\|1/a_1\|^\alpha_{C([0,T_1])}, 3^{\alpha/2}\|1/a_2\|^\alpha_{C([0,T_2])})\|1/a_1\|_{C([0,T_1])}$$
$$\cdot[D_\tau v_1]_{C^\alpha(D_1(T_1,T_2);X)}(|t_2-t_1|^2+|s_2-s_1|^2)^{\alpha/2}$$
$$+[1/a_1]_{C^\alpha([0,T_1])}\|D_\tau v_1\|_{C(D_1(T_1,T_2);X)}(|t_2-t_1|^2+|s_2-s_1|^2)^{\alpha/2}.$$

Moreover,

$$\|D_t u\|_{C(E_1(T_1,T_2);X)} \le \|1/a_1\|_{C([0,T_1])}\|D_\tau v_1\|_{C(D_1(T_1,T_2);X)}. \tag{1.59}$$

Therefore,

$$(1.60)\quad \|D_t u\|_{C^\alpha(E_1(T_1,T_2);X)} \le \max(1, 2^{\alpha/2}\|1/a_1\|^\alpha_{C([0,T_1])}, 3^{\alpha/2}\|1/a_2\|^\alpha_{C([0,T_2])}) \cdot \|1/a_1\|_{C^\alpha([0,T_1])}\|D_\tau v_1\|_{C^\alpha(D_1(T_1,T_2);X)}.$$

By the same technique, we can prove that $D_t u \in C^\alpha(E_2(T_1, T_2); X)$ and

$$(1.61)\quad \|D_t u\|_{C^\alpha(E_2(T_1,T_2);X)} \le \max(1, 3^{\alpha/2}\|1/a_1\|^\alpha_{C([0,T_1])}, 2^{\alpha/2}\|1/a_2\|^\alpha_{C([0,T_2])}) \cdot \|1/a_1\|_{C^\alpha([0,T_1])}\big(\|D_\tau v_2\|_{C^\alpha(D_2(T_1,T_2);X)} + \|D_\sigma v_2\|_{C^\alpha(D_2(T_1,T_2);X)}\big).$$

Since $D_s u(t, s) = (a_2(s))^{-1}(Au(t, s)+f(t, s)-a_1(t)D_t u(t, s))$ for any $(t, s) \in [0, T_1] \times [0, T_2]$, we deduce that $D_s u \in C^\alpha([0, T_1] \times [0, T_2]; X)$. Moreover, reasoning as in the case of $D_t u$, we deduce that

$$(1.62)\quad \|D_s u\|_{C^\alpha(E_1(T_1,T_2);X)} \le \max(1, 2^{\alpha/2}\|1/a_1\|^\alpha_{C([0,T_1])}, 3^{\alpha/2}\|1/a_2\|^\alpha_{C([0,T_2])}) \cdot \|1/a_2\|_{C^\alpha([0,T_2])}(\|D_\tau v_1\|_{C^\alpha(D_1(T_1,T_2);X)} + \|D_\sigma v_1\|_{C^\alpha(D_1(T_1,T_2);X)});$$

$$(1.63)\quad \|D_s u\|_{C^\alpha(E_2(T_1,T_2);X)} \le \max(1, 3^{\alpha/2}\|1/a_1\|^\alpha_{C([0,T_1])}, 2^{\alpha/2}\|1/a_2\|^\alpha_{C([0,T_2])}) \cdot \|1/a_2\|_{C^\alpha([0,T_2])}\|D_\sigma v_2\|_{C^\alpha(D_2(T_1,T_2);X)}.$$

Taking (1.33), (1.35), (1.46) into account, from (1.50), (1.53), (1.60)–(1.63) we get (1.45). □

2. The integrodifferential problem.

In this section we consider the following integrodifferential Cauchy problem related to the set $[0, T_1] \times [0, T_2]$ $(T_1, T_2 \in \mathbb{R}_+)$: *determine a function* $u \in C^\alpha([0, T_1] \times [0, T_2]; \mathcal{D}(A)) \cap C^1([0, T_1] \times [0, T_2]; X)$ *solution to the problem*:

$$(2.1)\quad \begin{cases} a_1(t)D_t u(t, s) + a_2(s)D_s u(t, s) - Au(t, s) = \\ \qquad = f(t, s) + \displaystyle\int_0^t d\tau \int_0^s h(t-\tau, s-\sigma)Bu(\tau, \sigma)\, d\sigma, \\ \qquad\qquad (t, s) \in [0, T_1] \times [0, T_2], \\ u(t, 0) = u_1(t), \qquad t \in [0, T_1], \\ u(0, s) = u_2(s), \qquad s \in [0, T_2]. \end{cases}$$

under the following assumptions:

K1 $a_1 \in C^\alpha([0, T_1])$, $(\alpha \in (0, 1))$, $m_1 \leq a_1(t) \leq M_1$, $t \in [0, T_1]$ $(0 < m_1 < M_1)$;
K2 $a_2 \in C^\alpha([0, T_2])$, $(\alpha \in (0, 1))$ *(cf.* (0.7)*)*, $m_1 \leq a_2(s) \leq M_1$, $s \in [0, T_2]$;
K3 $f \in C^\alpha([0, T_1] \times [0, T_2]; X)$, $D_t f \in C^\alpha(E_1(T_1, T_2); X)$, $D_s f \in C^\alpha(E_2(T_1, T_2); X)$;
K4 $h \in C^1([0, T_1] \times [0, T_2]; \mathbb{R})$;
K5 $A : \mathcal{D}(A) \subset X \to X$ *is an invertible closed linear operator whose resolvent set* $\rho(A)$ *contains the angle* $\Sigma_\phi = \{\lambda \in \mathbb{C} : |\arg\lambda| < \phi\} \cup \{0\}$ *for some* $\phi \in (\pi/2, \pi)$*. Moreover, the resolvent operator* $(\lambda I - A)^{-1}$ *satisfies* $\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq M_0|\lambda|^{-1}$ *for any* $\lambda \in \Sigma_\phi$ *and some positive constant* M_0;
K6 $B : \mathcal{D}(B) \subset X \to X$ *is a closed linear operator with* $\mathcal{D}(B) \supset \mathcal{D}(A)$;
K7 $u_j \in C^\alpha([0, T_j]; \mathcal{D}(A)) \cap C^{1+\alpha}([0, T_j]; X)$; $u_j' \in B([0, T_j]; \mathcal{D}_A(\alpha, \infty))$ $(j = 1, 2)$;
K8 $Au_1 + f(\cdot, 0) \in B([0, T_1]; \mathcal{D}_A(\alpha, \infty))$, $Au_2 + f(0, \cdot) \in B([0, T_2]; \mathcal{D}_A(\alpha, \infty))$;
K9 $u_1(0) = u_2(0)$, $\quad a_1(0)u_1'(0) + a_2(0)u_2'(0) - Au_1(0) = f(0, 0)$.

Remark 2.1. According to the inclusion $\mathcal{D}(A) \subset \mathcal{D}(B)$ we can define $\Psi = BA^{-1}$ in the whole of X. Since it is a closed operator, according to the closed graph theorem, we get $BA^{-1} \in \mathcal{L}(X)$. This fact will be used throughout the remaining of this paper.

Remark 2.2. The assumption $0 \in \rho(A)$ is not restrictive as can be seen by replacing the unknown u and the datum (f, h, u_1, u_2), respectively, by $\bar{u}$ and $(\bar{f}, \bar{h}, \bar{u}_1, \bar{u}_2)$, where $\bar{u}(t, s) = e^{-\lambda k(t,s)}u(t, s)$, $\bar{f}(t, s) = e^{-\lambda k(t,s)}f(t, s)$, $\bar{h}(t, s) = e^{-\lambda k(t,s)}h(t, s)$, $\bar{u}_1(t) = e^{-\lambda k(t,0)}$, $\bar{u}_2(s) = e^{-\lambda k(0,s)}u_2(s)$. Here $k(t, s) = \int_0^t (1/a_1)(\tau)d\tau + \int_0^s (1/a_2)(\sigma)d\sigma$ for any $(t, s) \in [0, T_1] \times [0, T_2]$ and 2λ is a suitable positive constant contained in the resolvent of A.

We now prove the following two technical lemmata.

Lemma 2.1. *For any Banach space* Y *and any* $f \in C^\alpha([0, T_1] \times [0, T_2]; Y)$*, the function* $\varphi_\alpha : [0, T_1] \times [0, T_2] \to \mathbb{R}_+$ *defined by*

$$\varphi_\alpha(t, s) = \|f\|_{C^\alpha([0,t]\times[0,s];Y)}, \tag{2.2}$$

is a real, bounded and measurable function.

Proof. First of all we remark that φ is measurable in $[0, T_1] \times [0, T_2]$ if and only if the function $\varphi_\alpha : [0, T_1] \times [0, T_2] \to \mathbb{R}$ defined by $\varphi_\alpha(t, s) = [f]_{C^\alpha([0,t]\times[0,s];Y)}$ is measurable. In fact, the function $(t, s) \to \|g\|_{C([0,t]\times[0,s];Y)}$

is continuous in $[0, T_1] \times [0, T_2]$ for any $g \in C([0, T_1] \times [0, T_2]; Y)$.
To prove the measurability of function φ_α we proceed in two steps.

Step 1. Here we prove that for any $\theta \in [0, \alpha)$ the function φ_θ is continuous in $[0, T_1] \times [0, T_2]$ (observe that $f \in C^\theta([0, T_1] \times [0, T_2])$ for any $0 \le \theta \le \alpha$). Define the function $g_\theta : [0, T_1]^2 \times [0, T_2]^2 \to Y$ by

$$g_\theta(t_1, t_2, s_1, s_2) = \begin{cases} \dfrac{\|f(t_2, s_2) - f(t_1, s_1)\|}{(|t_2 - t_1|^2 + |s_2 - s_1|^2)^{\theta/2}}, & \text{if } (t_1, s_1) \neq (t_2, s_2), \\ 0, & \text{if } (t_1, s_1) = (t_2, s_2). \end{cases}$$

As is easily seen

$$(2.3) \quad |g_\theta(t_1, t_2, s_1, s_2)| \le [f]_{C^\alpha([0,T_1]\times[0,T_2];Y)}(|t_2 - t_1|^2 + |s_2 - s_1|^2)^{(\alpha-\theta)/2},$$

for any $(t_1, t_2, s_1, s_2) \in [0, T_1]^2 \times [0, T_2]^2$. Hence, $g_\theta \in C([0, T_1]^2 \times [0, T_2]^2)$. Obviously,

$$(2.4) \quad \varphi_\theta(t, s) := [f]_{C^\theta([0,t]\times[0,s];Y)} = \|g_\theta\|_{C([0,t]^2\times[0,s]^2)}, \qquad \forall (t, s) \in [0, T_1] \times [0, T_2].$$

Hence, $\varphi_\theta \in C([0, T_1] \times [0, T_2])$.

Step 2. From (2.3) we easily deduce that

$$(2.5) \quad \varphi_\theta(t, s) \le (T_1^2 + T_2^2)^{(\alpha-\theta)/2}\varphi_\alpha(t, s), \qquad \forall (t, s) \in [0, T_1] \times [0, T_2], \quad \forall \theta \in [0, \alpha).$$

Let us consider an increasing sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ converging to α as $n \to +\infty$. Then,

$$(2.6) \qquad \limsup_{n\to+\infty} \varphi_{\alpha_n}(t, s) = \varphi_\alpha(t, s), \quad \forall (t, s) \in [0, T_1] \times [0, T_2].$$

From (2.6) we immediately deduce that φ_α is measurable in $[0, T_1] \times [0, T_2]$. To prove (2.6) we observe that, for any $(t, s) \in [0, T_1] \times [0, T_2]$ and any $(t_j, s_j) \in [0, t] \times [0, s]$ $(j = 1, 2)$ we have

$$\frac{\|f(t_2, s_2) - f(t_1, s_1)\|}{(|t_2 - t_1|^2 + |s_2 - s_1|^2)^{\alpha_n/2}} \le \varphi_{\alpha_n}(t, s), \quad \text{if } (t_1, s_1) \neq (t_2, s_2).$$

Therefore,

$$\varphi_\alpha(t, s) \le \limsup_{n\to+\infty} \varphi_{\alpha_n}(t, s).$$

From (2.5), with θ replaced by α_n, we deduce

$$\varphi_\alpha(t,s) \geq \limsup_{n\to+\infty} \varphi_{\alpha_n}(t,s).$$

(2.6) is so proved. □

Lemma 2.2. *Suppose that assumptions K4–K6 hold. Then, for any $u \in C^\alpha([0,T_1]\times[0,T_2];\mathcal{D}(A))$, the function $F(u):[0,T_1]\times[0,T_2]\to X$ defined by*

$$F(u)(t,s) = \int_0^t d\tau \int_0^s h(t-\tau, s-\sigma)Bu(\tau,\sigma)\,d\sigma, \quad \forall (t,s)\in[0,T_1]\times[0,T_2],$$

belongs to $C^{1+\alpha}([0,T_1]\times[0,T_2];X)$ and, for any $(t,s)\in[0,T_1]\times[0,T_2]$, satisfies the following estimate:

$$\begin{aligned}(2.7)\quad \|F(u)\|_{C^{1+\alpha}([0,t]\times[0,s];X)} &\leq 2(1+2^{-\alpha/2})\max(1,T_1^\alpha,T_2^\alpha,(T_1T_2)^\alpha)\|\Psi\|_{\mathcal{L}(X)}\\ &\cdot\|h\|_{C^1([0,T_1]\times[0,T_2])}\Big[\Big(\int_0^t d\tau\int_0^s \|u\|^{1/(1-\alpha)}_{C^\alpha([0,\tau]\times[0,\sigma];\mathcal{D}(A))}\,d\sigma\Big)^{1-\alpha}\\ &+\Big(\int_0^t \|u\|^{1/(1-\alpha)}_{C^\alpha([0,\tau]\times[0,s];\mathcal{D}(A))}\,d\tau\Big)^{1-\alpha}+\Big(\int_0^s \|u\|^{1/(1-\alpha)}_{C^\alpha([0,t]\times[0,\sigma];\mathcal{D}(A))}\,d\sigma\Big)^{1-\alpha}\Big].\end{aligned}$$

Proof. We will prove that, for any $(t,s)\in[0,T_1]\times[0,T_2]$, $F(u)\in C^{1+\alpha}([0,t]\times[0,s];X)$. We first observe that $Bu\in C^\alpha([0,T_1]\times[0,T_2];X)$. In fact, $Bu = \Psi Au$ and, owing to Remark 2.1, ψ is a bounded linear operator mapping X into itself. Taking advantage of the Hölder inequality, we deduce that

$$\begin{aligned}(2.8)\qquad \|F(u)(t,s)\| &\leq \Big(\int_0^t d\tau\int_0^s |h(\tau,\sigma)|^{1/\alpha}\,d\sigma\Big)^\alpha \|\Psi\|_{\mathcal{L}(X)}\\ &\cdot\Big(\int_0^t d\tau\int_0^s \|Au(\tau,\sigma)\|^{1/(1-\alpha)}d\sigma\Big)^{1-\alpha}\\ &\leq (T_1T_2)^\alpha\|\Psi\|_{\mathcal{L}(X)}\|h\|_{C([0,T_1]\times[0,T_2])}\Big(\int_0^t d\tau\int_0^s \|Au\|^{1/(1-\alpha)}_{C([0,\tau]\times[0,\sigma];X)}\,d\sigma\Big)^{1-\alpha},\end{aligned}$$

for any $(t,s)\in[0,T_1]\times[0,T_2]$. Moreover, we observe that $F(u)$ is continuously differentiable in $[0,T_1]\times[0,T_2]$ and

$$\begin{aligned}(2.9)\qquad D_tF(u)(t,s) &= \int_0^t d\tau\int_0^s D_th(\tau,\sigma)Bu(t-\tau,s-\sigma)\,d\sigma\\ &+\int_0^s h(0,\sigma)Bu(t,s-\sigma)\,d\sigma;\end{aligned}$$

$$(2.10)\qquad D_sF(u)(t,s)=\int_0^t d\tau\int_0^s D_sh(\tau,\sigma)Bu(t-\tau,s-\sigma)\,d\sigma + \int_0^t h(\tau,0)Bu(t-\tau,s)\,d\tau.$$

As is easily seen, $D_tF(u)$ is bounded in $[0,t]\times[0,s]$ and

$$(2.11)\qquad \sup_{p,r\in[0,t]\times[0,s]}\|D_tF(u)(p,r)\|\le$$

$$\le\int_0^t d\tau\int_0^s \|D_th\|_{C([0,\tau]\times[0,\sigma])}\|\Psi\|_{\mathcal{L}(X)}\|Au\|_{C([0,t-\tau]\times[0,s-\sigma];X)}\,d\sigma + \int_0^s \|h\|_{C(0\times[0,\sigma])}\|\Psi\|_{\mathcal{L}(X)}\|Au\|_{C([0,t]\times[0,s-\sigma];X)}\,d\sigma$$

$$\le (T_1T_2)^\alpha\|\Psi\|_{\mathcal{L}(X)}\|D_th\|_{C([0,T_1]\times[0,T_2])}\Big(\int_0^t d\tau\int_0^s \|Au\|^{1/(1-\alpha)}_{C([0,\tau]\times[0,\sigma];X)}\,d\sigma\Big)^{1-\alpha} + T_2^\alpha\|\Psi\|_{\mathcal{L}(X)}\|h\|_{C([0,T_1]\times[0,T_2])}\Big(\int_0^s \|Au\|^{1/(1-\alpha)}_{C([0,t]\times[0,\sigma];X)}\,d\sigma\Big)^{1-\alpha}.$$

Moreover, for any $(t_j,s_j)\in[0,t]\times[0,s]$ $(j=1,2)$, we get

$$\|D_tF(u)(t_2,s_2)-D_tF(u)(t_1,s_1)\|\le$$

$$\le\int_{t_1}^{t_2} d\tau\int_0^{s_2}\|D_th\|_{C([0,\tau]\times[0,\sigma])}\|\Psi\|_{\mathcal{L}(X)}\|Au\|_{C([0,t_2-\tau]\times[0,s_2-\sigma];X)}\,d\sigma$$

$$+\int_0^{t_1} d\tau\int_{s_1}^{s_2}\|D_th\|_{C([0,\tau]\times[0,\sigma])}\|\Psi\|_{\mathcal{L}(X)}\|Au\|_{C([0,t_2-\tau]\times[0,s_2-\sigma];X)}\,d\sigma$$

$$+\int_0^{t_1} d\tau\int_0^{s_1}\|D_th\|_{C([0,\tau]\times[0,\sigma])}\|\Psi\|_{\mathcal{L}(X)}[Au]_{C^\alpha([0,t_2-\tau]\times[0,s_2-\sigma];X)}\,d\sigma \cdot(|t_2-t_1|^2+|s_2-s_1|^2)^{\alpha/2}$$

$$+\int_{s_1}^{s_2}\|h\|_{C(0\times[0,\sigma])}\|\Psi\|_{\mathcal{L}(X)}\|Au\|_{C([0,t_2]\times[0,s_2-\sigma];X)}\,d\sigma$$

$$+\Big(\int_0^{s_1}\|h\|_{C(0\times[0,\sigma])}\|\Psi\|_{\mathcal{L}(X)}[Au]_{C^\alpha([0,t_2]\times[0,s_2-\sigma];X)}d\sigma\Big)(|t_2-t_1|^2+|s_2-s_1|^2)^{\alpha/2}$$

$$\le\|D_th\|_{C([0,T_1]\times[0,T_2])}\|\Psi\|_{\mathcal{L}(X)}\Big(\int_0^t d\tau\int_0^s\|Au\|^{1/(1-\alpha)}_{C([0,\tau]\times[0,\sigma];X)}\,d\sigma\Big)^{1-\alpha}$$

$$\cdot (T_2^\alpha |t_2 - t_1|^\alpha + T_1^\alpha |s_2 - s_1|^\alpha)$$

$$+ (T_1 T_2)^\alpha \|D_t h\|_{C([0,T_1]\times[0,T_2])} \|\Psi\|_{\mathcal{L}(X)} \Big(\int_0^t d\tau \int_0^s [Au]^{1/(1-\alpha)}_{C^\alpha([0,\tau]\times[0,\sigma];X)} \, d\sigma \Big)^{1-\alpha}$$

$$\cdot (|t_2 - t_1|^2 + |s_2 - s_1|^2)^{\alpha/2}$$

$$+ \|h\|_{C([0,T_1]\times[0,T_2])} \|\Psi\|_{\mathcal{L}(X)} \Big(\int_0^s \|Au\|^{1/(1-\alpha)}_{C([0,t]\times[0,\sigma];X)} \, d\sigma \Big)^{1-\alpha} |s_2 - s_1|^\alpha$$

$$+ T_2^\alpha \|h\|_{C([0,T_1]\times[0,T_2])} \|\Psi\|_{\mathcal{L}(X)} \Big(\int_0^s [Au]^{1/(1-\alpha)}_{C^\alpha([0,t]\times[0,\sigma];X)} \, d\sigma \Big)^{1-\alpha}$$

$$\cdot (|t_2 - t_1|^2 + |s_2 - s_1|^2)^{\alpha/2}.$$

Therefore

$$[D_t F(u)]_{C^\alpha([0,t]\times[0,s];X)} \le \le (2^{1-\alpha/2} + 1) \max(1, T_1^\alpha, T_2^\alpha, (T_1 T_2)^\alpha) \|\Psi\|_{\mathcal{L}(X)} \cdot \Big[\|D_t h\|_{C([0,T_1]\times[0,T_2])} \Big(\int_0^t d\tau \int_0^s \|Au\|^{1/(1-\alpha)}_{C^\alpha([0,\tau]\times[0,\sigma];X)} \, d\sigma \Big)^{1-\alpha} + \|h\|_{C([0,T_1]\times[0,T_2])} \Big(\int_0^s \|Au\|^{1/(1-\alpha)}_{C^\alpha([0,t]\times[0,\sigma];X)} \, d\sigma \Big)^{1-\alpha} \Big]. \tag{2.12}$$

From (2.11) and (2.12) we deduce

$$\|D_t F(u)\|_{C^\alpha([0,t]\times[0,s];X)} \le \le 2(1 + 2^{-\alpha/2}) \max(1, T_1^\alpha, T_2^\alpha, (T_1 T_2)^\alpha) \|\Psi\|_{\mathcal{L}(X)} \cdot \Big[\|D_t h\|_{C([0,T_1]\times[0,T_2])} \Big(\int_0^t d\tau \int_0^s \|u\|^{1/(1-\alpha)}_{C^\alpha([0,\tau]\times[0,\sigma];\mathcal{D}(A))} \, d\sigma \Big)^{1-\alpha} + \|h\|_{C([0,T_1]\times[0,T_2])} \Big(\int_0^s \|u\|^{1/(1-\alpha)}_{C^\alpha([0,t]\times[0,\sigma];\mathcal{D}(A))} \, d\sigma \Big)^{1-\alpha} \Big]. \tag{2.13}$$

Reasoning in the same way we get

$$\|D_s F(u)\|_{C^\alpha([0,t]\times[0,s];X)} \le \le 2(1 + 2^{-\alpha/2}) \max(1, T_1^\alpha, T_2^\alpha, (T_1 T_2)^\alpha) \|\Psi\|_{\mathcal{L}(X)} \cdot \Big[\|D_s h\|_{C([0,T_1]\times[0,T_2])} \Big(\int_0^t d\tau \int_0^s \|u\|^{1/(1-\alpha)}_{C^\alpha([0,\tau]\times[0,\sigma];\mathcal{D}(A))} \, d\sigma \Big)^{1-\alpha} + \|h\|_{C([0,T_1]\times[0,T_2])} \Big(\int_0^t \|u\|^{1/(1-\alpha)}_{C^\alpha([0,\tau]\times[0,s];\mathcal{D}(A))} \, d\tau \Big)^{1-\alpha} \Big]. \tag{2.14}$$

Then, from (2.8), (2.13), (2.14), we deduce (2.7). □

2.1. *An equivalent problem.*

Theorem 2.1. *Suppose that* $u \in C^{\alpha}([0, T_1] \times [0, T_2]; \mathcal{D}(A)) \cap C^{1+\alpha}([0, T_1] \times [0, T_2]; X)$. *Then, u is a solution to the Cauchy problem* (2.1) *if and only if it is a fixed point of the operator* $\Gamma : C^{\alpha}([0, T_1] \times [0, T_2]; \mathcal{D}(A)) \to C^{\alpha}([0, T_1] \times [0, T_2]; \mathcal{D}(A))$ *defined by the following formulae, where* $G(u) = F(u) + f$*:*

$$\Gamma(u)(t, s) = \exp[b_2(s)A]u_1(b_1^{-1}(b_1(t) - b_2(s))) + \int_0^{b_2(s)} \exp[(b_2(s) - \xi)A]G(u)(b_1^{-1}(b_1(t) - b_2(s) + \xi), b_2^{-1}(\xi))\, d\xi, \tag{2.15}$$

for any $(t, s) \in E_1(T_1, T_2)$ *(cf.* (0.5)*);*

$$\Gamma(u)(t, s) = \exp[b_1(t)A]u_2(b_2^{-1}(b_2(s) - b_1(t))) + \int_0^{b_1(t)} \exp[(b_1(t) - \xi)A]G(u)(b_1^{-1}(\xi), b_2^{-1}(b_2(s) - b_1(t) + \xi))\, d\xi, \tag{2.16}$$

for any $(t, s) \in E_2(T_1, T_2)$ *(cf.* (0.6)*).*

Proof. Suppose that $u \in C^{\alpha}([0, T_1]\times[0, T_2]; \mathcal{D}(A))\cap C^{1+\alpha}([0, T_1]\times[0, T_2]; X)$ is a solution to the Cauchy problem (2.1). Then, assumptions H1–H7 of section 2 are implied by K1–K9 thanks to Lemma 2.1. Consequently, by virtue of Theorems 1.1 and 1.3, u solves equations (2.15) and (2.16).
Conversely, suppose that $u \in C^{\alpha}([0, T_1] \times [0, T_2]; \mathcal{D}(A))$ is a fixed point of operator Γ. Then, from Theorem 1.3, we deduce that $u \in C^{1+\alpha}([0, T_1] \times [0, T_2]; X) \cap C^{\alpha}([0, T_1] \times [0, T_2]; \mathcal{D}(A))$ and solves problem (2.1). □

2.2. *Basic properties of operator* Γ.

In this subsection we will show that problem (2.15)-(2.16) is uniquely solvable in $C^{\alpha}([0, T_1] \times [0, T_2]; \mathcal{D}(A))$. For this purpose, we will prove that there exists an $n \in \mathbb{N}$ such that $\Gamma^n : C^{\alpha}([0, T_1] \times [0, T_2]; \mathcal{D}(A)) \to C^{\alpha}([0, T_1] \times [0, T_2]; \mathcal{D}(A))$ is a contraction map.
We begin by proving the following generalization, to the case of two variables, of the Gronwall's inequality.

Lemma 2.3. *Suppose that* $\{\varphi_n\}_{n\in\mathbb{N}} : [0, T_1] \times [0, T_2] \to \mathbb{R}_+$ *is a sequence of real, bounded and measurable functions such that*

$$\varphi_{n+1}(t,s) \le C\Big[\Big(\int_0^t d\tau \int_0^s \varphi_n(\tau,\sigma)^{1/(1-\alpha)}\,d\sigma\Big)^{1-\alpha} + \\ + \Big(\int_0^t \varphi_n(\tau,s)^{1/(1-\alpha)}\,d\tau\Big)^{1-\alpha} + \Big(\int_0^s \varphi_n(t,\sigma)^{1/(1-\alpha)}\,d\sigma\Big)^{1-\alpha}\Big], \tag{2.17}$$

for any $(t,s) \in [0,T_1]\times[0,T_2]$*, some positive constant* C *and* $\alpha \in [0,1)$*. Then,*

$$|\varphi_{n+1}(t,s)| \le 3^{\alpha n} C^n \|\varphi_1\|_{L^\infty([0,T_1]\times[0,T_2])} \\ \cdot (T_1T_2 + T_1 + T_2 + 2)^{(n-1)(1-\alpha)} (n!)^{-1+\alpha} (ts+t+s)^{n(1-\alpha)}. \tag{2.18}$$

for any $n \in \mathbb{N}$.

Proof. We prove the lemma by induction on n. We begin by observing that

$$a^{1-\alpha} + b^{1-\alpha} + c^{1-\alpha} \le 3^\alpha (a+b+c)^{1-\alpha} \quad \forall a,b,c>0, \quad \forall \alpha \in [0,1].$$

Thanks to the previous inequality, it is easy to show that (2.18) holds for $n = 1$. Suppose now that (2.18) holds for $n = k$ and let us prove it for $n = k+1$. Then

$$\varphi_{k+1}(t,s) \le 3^\alpha C\Big[\int_0^t d\tau \int_0^s \varphi_k(\tau,\sigma)^{1/(1-\alpha)}\,d\sigma + \int_0^t \varphi_k(\tau,t)^{1/(1-\alpha)}\,d\tau + \\ + \int_0^s \varphi_k(t,\sigma)^{1/(1-\alpha)}\,d\sigma\Big]^{1-\alpha}$$

$$\le 3^{\alpha k} C^k [(k-1)!]^{-1+\alpha} \|\varphi_1\|_{L^\infty([0,T_1]\times[0,T_2])} (T_1T_2+T_1+T_2+2)^{(k-2)(1-\alpha)} \\ \cdot\Big(\int_0^t d\tau \int_0^s (\tau\sigma+\tau+\sigma)^{k-1}\,d\sigma + \int_0^t (\tau s+\tau+s)^{k-1}\,d\tau \\ + \int_0^s (t\sigma+t+\sigma)^{k-1}\,d\sigma\Big)^{1-\alpha}$$

$$\le 3^{\alpha k} C^k [(k-1)!]^{-1+\alpha} \|\varphi_1\|_{L^\infty([0,T_1]\times[0,T_2])} (T_1T_2+T_1+T_2+2)^{(k-2)(1-\alpha)} \\ \cdot\Big(\frac{1}{k(k+1)}(ts+t+s)^{k+1} + \frac{2}{k}(ts+t+s)^k\Big)^{1-\alpha} \\ \le 3^{\alpha k} C^k [(k)!]^{-1+\alpha} \|\varphi_1\|_{L^\infty([0,T_1]\times[0,T_2])} \\ \cdot (T_1T_2+T_1+T_2+2)^{(k-1)(1-\alpha)} (ts+t+s)^{k(1-\alpha)},$$

for any $(t,s) \in [0,T_1]\times[0,T_2]$. The proof is now complete. □

By virtue of Lemmata 2.2 and 2.3 we can prove the following *existence-uniqueness* result for problem (2.1).

Theorem 2.2. *There exists* $n_0 \in \mathbb{N}$ *such that* $\Gamma^n : C^\alpha([0, T_1]\times[0, T_2]; \mathcal{D}(A)) \to C^\alpha([0, T_1] \times [0, T_2]; \mathcal{D}(A))$ *is a contraction map for* $n \geq n_0$. *Consequently, problem* (2.1) *admits an unique solution* $u \in C^\alpha([0, T_1] \times [0, T_2]; \mathcal{D}(A)) \cap C^{1+\alpha}([0, T_1] \times [0, T_2]; X)$. *Moreover, u satisfies the following estimates:*

$$(2.19)\quad \|u\|_{C^\alpha([0,T_1]\times[0,T_2];\mathcal{D}(A))} \leq D_1\big(\|u_1\|_{C^\alpha([0,T_1];\mathcal{D}(A))} + \|u_2\|_{C^\alpha([0,T_2];\mathcal{D}(A))} + \|f\|_{C^\alpha([0,T_1]\times[0,T_2];X)} + \|D_t f\|_{C^\alpha(E_1(T_1,T_2);X)} + \|D_s f\|_{C^\alpha(E_2(T_1,T_2);X)} + \|Au_1 + f(\cdot, 0)\|_{B([0,T_1];\mathcal{D}(A))} + \|Au_2 + f(0, \cdot)\|_{B([0,T_2];\mathcal{D}_A(\alpha,\infty))}\big);$$

$$(2.20)\quad \|u\|_{C^{1+\alpha}([0,T_1]\times[0,T_2];X)} \leq D_2\big(\|u_1\|_{C^\alpha([0,T_1];\mathcal{D}(A))} + \|u_2\|_{C^\alpha([0,T_2];\mathcal{D}(A))} + \|u_1'\|_{C([0,T_1];X)} + \|u_2'\|_{C([0,T_2];X)} + \|f\|_{C^\alpha([0,T_1]\times[0,T_2];X)} + \|D_t f\|_{C^\alpha(E_1(T_1,T_2);X)} + \|D_s f\|_{C^\alpha(E_2(T_1,T_2);X)} + \|Au_1 + f(\cdot, 0)\|_{B([0,T_1];\mathcal{D}_A(\alpha,\infty))} + \|Au_2 + f(0, \cdot)\|_{B([0,T_2];\mathcal{D}_A(\alpha,\infty))}\big),$$

where

$$D_1 = C_1 g(C_2\|h\|_{C^1([0,T_1]\times[0,T_2])}),$$
$$D_2 = C_3\|h\|_{C^1([0,T_1]\times[0,T_2])} g(C_2\|h\|_{C^1([0,T_1]\times[0,T_2])}) + C_4,$$

C_j $(j = 1, \ldots, 4)$ *are positive constants depending on* α, M, T_k, $\|a_k\|_{C^\alpha([0,T_k])} + \|1/a_k\|_{C^\alpha([0,T_k])}$ $(k = 1, 2)$ *and g is an increasing and analytic function in* $\mathbb{R}_+$ *such that* $g(0) = 1$.

Proof. Let us split Γ as the sum of two terms: $\Gamma u = k + Tu$. Here k is the solution to problem (2.1) and T is the linear operator defined on $C^\alpha([0, T_1] \times [0, T_2]; \mathcal{D}(A))$ by

$$(2.21)\quad Tu(t, s) = \int_0^{\min(b_1(t),b_2(s))} \exp(\xi A)Fu(b_1^{-1}(b_1(t) - \xi), b_2^{-1}(b_2(s) - \xi))\, d\xi,$$

for any $(t, s) \in [0, T_1] \times [0, T_2]$.
By Theorem 1.3 and Lemma 2.2, we deduce that Γ maps $C^\alpha([0, T_1] \times [0, T_2]; \mathcal{D}(A))$ into itself. Moreover, for any $v_j \in C^\alpha([0, T_1] \times [0, T_2]; \mathcal{D}(A))$ $(j = 1, 2)$ and any $n \in \mathbb{N}$, we have $\Gamma^n(v_2) - \Gamma^n(v_1) = T^n v_2 - T^n v_1$. Therefore we can limit ourselves to proving that, for n sufficiently large, T^n is a linear contraction in $C^\alpha([0, T_1] \times [0, T_2]; \mathcal{D}(A))$. Applying estimate (1.44) to the interval

$[0,t]\times[0,s]$ ($(t,s)\in[0,T_1]\times[0,T_2]$) and taking Lemma 2.2 into account, we deduce that there exists a positive constant $\widetilde{C}$ such that

$$\begin{aligned}(2.22)\qquad &\|Tv\|_{C^\alpha([0,t]\times[0,s];\mathcal{D}(A))}\le\\ &\le\widetilde{C}\|h\|_{C^1([0,T_1]\times[0,T_2])}\Big[\Big(\int_0^t d\tau\int_0^s\|v\|^{1/(1-\alpha)}_{C^\alpha([0,\tau]\times[0,\sigma];\mathcal{D}(A))}d\sigma\Big)^{1-\alpha}\\ &+\Big(\int_0^t\|v\|^{1/(1-\alpha)}_{C^\alpha([0,\tau]\times[0,s];\mathcal{D}(A))}d\tau\Big)^{1-\alpha}+\Big(\int_0^t\|v\|^{1/(1-\alpha)}_{C^\alpha([0,t]\times[0,\sigma];\mathcal{D}(A))}d\sigma\Big)^{1-\alpha}\Big],\end{aligned}$$

for any $v\in C^\alpha([0,T_1]\times[0,T_2];\mathcal{D}(A))$. By iteration we deduce that

$$\begin{aligned}(2.23)\qquad &\|T^nv\|_{C^\alpha([0,t]\times[0,s];\mathcal{D}(A))}\le\\ &\le\widetilde{C}\|h\|_{C^1([0,T_1]\times[0,T_2])}\Big[\Big(\int_0^t d\tau\int_0^s\|T^{n-1}v\|^{1/(1-\alpha)}_{C^\alpha([0,\tau]\times[0,\sigma];\mathcal{D}(A))}d\sigma\Big)^{1-\alpha}\\ &+\Big(\int_0^t\|T^{n-1}v\|^{1/(1-\alpha)}_{C^\alpha([0,\tau]\times[0,s];\mathcal{D}(A))}d\tau\Big)^{1-\alpha}\\ &+\Big(\int_0^s\|T^{n-1}v\|^{1/(1-\alpha)}_{C^\alpha([0,t]\times[0,\sigma];\mathcal{D}(A))}d\sigma\Big)^{1-\alpha}\Big],\qquad\forall n\in\mathbb{N}.\end{aligned}$$

Applying Lemma 2.3 to the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ defined by $\varphi_k=T^{k-1}u$ for any $k\in\mathbb{N}$ we deduce that

$$\begin{aligned}(2.24)\quad \|T^nv\|_{C^\alpha([0,t]\times[0,s];\mathcal{D}(A))}&\le 3^{\alpha n}\widetilde{C}^n(T_1T_2+T_1+T_2+2)^{(2n-1)(1-\alpha)}\\ &\cdot(n!)^{-1+\alpha}\|h\|^n_{C^1([0,T_1]\times[0,T_2])}\|v\|_{C^\alpha([0,T_1]\times[0,T_2];\mathcal{D}(A))}.\end{aligned}$$

Consequently, according to a well-known result (cf. [11]), the equation $\Gamma(u)=u$ admits an unique solution $u\in C^\alpha([0,T_1]\times[0,T_2];\mathcal{D}(A))\cap C^{1+\alpha}([0,T_1]\times[0,T_2];X)$ given by $u=\sum_{n=0}^\infty T^nk=(I-T)^{-1}k$. By means of (2.24) we easily derive (2.19).

Then, from Theorem 1.3 we deduce that Γ maps $C^\alpha([0,T_1]\times[0,T_2];\mathcal{D}(A))$ into $C^{1+\alpha}([0,T_1]\times[0,T_2];X)$. Moreover, from (1.45) and Lemma 2.2, we get the following estimate:

$$\begin{aligned}(2.25)\quad \|u\|_{C^{1+\alpha}([0,T_1]\times[0,T_2];X)}&\le\|k\|_{C^{1+\alpha}([0,T_1]\times[0,T_2];X)}+\|Tu\|_{C^{1+\alpha}([0,T_1]\times[0,T_2];X)}\\ &\le\|k\|_{C^{1+\alpha}([0,T_1]\times[0,T_2];X)}\\ &+\overline{C}\|h\|_{C^1([0,T_1]\times[0,T_2])}\Big[\Big(\int_0^t d\tau\int_0^s\|u\|^{1/(1-\alpha)}_{C^\alpha([0,\tau]\times[0,\sigma];\mathcal{D}(A))}d\sigma\Big)^{1-\alpha}\\ &+\Big(\int_0^t\|u\|^{1/(1-\alpha)}_{C^\alpha([0,\tau]\times[0,s];\mathcal{D}(A))}d\tau\Big)^{1-\alpha}+\Big(\int_0^t\|u\|^{1/(1-\alpha)}_{C^\alpha([0,t]\times[0,\sigma];\mathcal{D}(A))}d\sigma\Big)^{1-\alpha}\Big],\end{aligned}$$

for some positive constant $\overline{C}$. Hence, (2.20) follows from (2.19) and (2.25).

□

3. Some applications.

In this section we are concerned with the following problem: *determine a function* $u : [0, T_1] \times [0, T_2] \times \overline{\Omega} \to \mathbb{R}$ *solution to the following problem*:

$$(3.1)\quad \begin{cases} a_1(t)D_t u(t,s,x) + a_2(s)D_s u(t,s,x) - \mathcal{A}u(t,s,x) = \\ \quad = f(t,s,x) + \int_0^t d\tau \int_0^s h(t-\tau, s-\sigma)\mathcal{B}u(\tau,\sigma,x)\, d\sigma, \\ \qquad\qquad (t,s,x) \in [0,T_1] \times [0,T_2] \times \Omega, \\ u(t,0,x) = u_1(t,x), \qquad (t,x) \in [0,T_1] \times \overline{\Omega}, \\ u(0,s,x) = u_2(s,x), \qquad (s,x) \in [0,T_2] \times \overline{\Omega}, \\ u(t,s,x) = 0, \qquad (t,s,x) \in [0,T_1] \times [0,T_2] \times \partial\Omega. \end{cases}$$

Here Ω denotes any open set in $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class C^2. Moreover, $\mathcal{A}$ and $\mathcal{B}$ denote the second order linear operators, formally defined by

$$(3.2)\ (\mathcal{A}u)(x) = \sum_{i,j=1}^{n} c_{i,j}(x) D_i D_j u(x) + \sum_{j=1}^{n} c_j(x) D_j u(x) + c(x)u(x),\ x \in \Omega;$$

$$(3.3)\ (\mathcal{B}u)(x) = \sum_{i,j=1}^{n} d_{i,j}(x) D_i D_j u(x) + \sum_{j=1}^{n} d_j(x) D_j u(x) + d(x)u(x),\ x \in \Omega.$$

We assume that $c_{i,j}, c_j, c, d_{i,j}, d_j, d$ are continuous functions in $\overline{\Omega}$ $(i, j = 1, \dots, n)$ and

$$(3.4)\qquad \sum_{i,j=1}^{n} c_{i,j}(x)\xi_i\xi_j \ge \nu|\xi^2|, \quad \forall x \in \overline{\Omega}, \quad \forall \xi \in \mathbb{R}^n,$$

for some positive constant ν.

3.1. *The case* $X = L^p(\Omega)$ *with* $p \in (1, +\infty)$.

The realization in $L^p(\Omega)$ of the linear operator $\mathcal{A}$ generates an analytic semigroup provided we choose $\mathcal{D}(\mathcal{A}) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [7], Theorems 3.1.2, 3.1.3).
By the Agmon-Douglis-Nirenberg *a priori* estimates for regular domains (see [7], Theorem 3.1.1), we deduce that the graph-norm of $\mathcal{D}(\mathcal{A})$ is equivalent to the $W^{2,p}(\Omega)$-norm. From Theorem A we obtain the following existence and uniqueness theorem.

Theorem 3.1.1. *Suppose that*

K10 $a_1 \in C^\alpha([0,T_1])$, $(\alpha \in (0,1))$, $m_1 \leq a_1(t) \leq M_1$, $t \in [0,T_1]$ $(0 < m_1 < M_1)$;

K11 $a_2 \in C^\alpha([0,T_2])$, $(\alpha \in (0,1))$ (cf. (0.7)), $m_1 \leq a_2(s) \leq M_1$, $s \in [0,T_2]$;

K12 $f \in C^\alpha([0,T_1]\times[0,T_2]; L^p(\Omega))$, $D_t f \in C^\alpha(E_1(T_1,T_2); L^p(\Omega))$, $D_s f \in C^\alpha(E_2(T_1,T_2); L^p(\Omega))$;

K13 $h \in C^1([0,T_1]\times[0,T_2])$;

K14 $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{B})$;

K15 $u_j \in C^\alpha([0,T_j]; W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)) \cap C^{1+\alpha}([0,T_j]; L^p(\Omega))$ $(j=1,2)$;

K16 $\mathcal{A}u_1 + f(\cdot,0) \in B([0,T_1]; \mathcal{D}_{\mathcal{A}}(\alpha,\infty))$, $\mathcal{A}u_2 + f(0,\cdot) \in B([0,T_2]; \mathcal{D}_{\mathcal{A}}(\alpha,\infty))$;

K17 $u_1(0) = u_2(0)$, $\quad a_1(0)u_1'(0) + a_2(0)u_2'(0) - \mathcal{A}u_1(0) = f(0,0)$.

Then, problem (3.1) *admits an unique solution* $u \in C^\alpha([0,T_1]\times[0,T_2]; W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)) \cap C^{1+\alpha}([0,T_1]\times[0,T_2]; L^p(\Omega))$. *In particular, if* $p > n/2$, $u \in C([0,T_1]\times[0,T_2]\times\overline{\Omega})$.

3.2. *The case* $X = L^\infty(\Omega)$.

The realization in $L^\infty(\Omega)$ of the linear operator defined (3.2) is a generator of an analytic semigroup provided we choose $\mathcal{D}(\mathcal{A}) = \{u \in \cap_{p\geq 1} W^{2,p}_{\mathrm{loc}}(\Omega) : u, \mathcal{A}u \in L^\infty(\Omega),\ u(x)_{|\partial\Omega} \equiv 0\}$ (see [7], Corollary 3.1.21). Moreover, $\mathcal{D}(\mathcal{A})$ is continuously embedded in $C^1(\overline{\Omega})$ (see [7], Theorem 3.1.19). From Theorem A, we immediately deduce the following theorem.

Theorem 3.2.1. *Suppose that*

K18 $a_1 \in C^\alpha([0,T_1])$, $(a \in (0,1))$, $m_1 \leq a_1(t) \leq M_1$, $t \in [0,T_1]$ $(0 < m_1 < M_1)$;

K19 $a_2 \in C^\alpha([0,T_2])$, $(\alpha \in (0,1))$ (cf. (0.7)), $m_1 \leq a_2(s) \leq M_1$, $s \in [0,T_2]$;

K20 $f \in C^\alpha([0,T_1]\times[0,T_2]; L^\infty(\Omega))$, $D_t f \in C^\alpha(E_1(T_1,T_2); L^\infty(\Omega))$, $D_s f \in C^\alpha(E_2(T_1,T_2); L^\infty(\Omega))$;

K21 $h \in C^1([0,T_1]\times[0,T_2])$;

K22 $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{B})$;

K23 $u_j \in C^\alpha([0,T_j]; \mathcal{D}(\mathcal{A})) \cap C^{1+\alpha}([0,T_j]; L^\infty(\Omega))$ $(j=1,2)$;

K24 $\mathcal{A}u_1 + f(\cdot,0) \in B([0,T_1]; \mathcal{D}_{\mathcal{A}}(\alpha,\infty))$, $\mathcal{A}u_2 + f(0,\cdot) \in B([0,T_2]; \mathcal{D}_{\mathcal{A}}(\alpha,\infty))$;

K25 $u_1(0) = u_2(0)$, $\quad a_1(0)u_1'(0) + a_2(0)u_2'(0) - \mathcal{A}u_1(0) = f(0,0)$.

Then, problem (3.1) *admits an unique solution* $u \in C^\alpha([0,T_1]\times[0,T_2]; \mathcal{D}(\mathcal{A})) \cap C^{1+\alpha}([0,T_1]\times[0,T_2]; L^\infty(\Omega))$. *In particular,* $u \in C([0,T_1]\times[0,T_2]\times\overline{\Omega})$.

3.3. *The case* $X = C(\overline{\Omega})$.

The realization in $C(\overline{\Omega})$ of the linear operator in (3.2) defines a generator of an analytic semigroup provided we choose $\mathcal{D}(\mathcal{A}) = \{u \in \cap_{p\geq 1} W^{2,p}_{\text{loc}}(\Omega) : u, \mathcal{A}u \in C(\overline{\Omega}),\ u(x)_{|\partial\Omega} \equiv 0\}$ (see [7], Corollary 3.1.21). Moreover, $\mathcal{D}(\mathcal{A})$ is continuously embedded into $C^1(\overline{\Omega})$ (see [7], Theorem 3.1.19). From Theorem A, we immediately derive the following theorem.

Theorem 3.3.1. *Suppose that*

K26 $a_1 \in C^\alpha([0, T_1])$, $(\alpha \in (0, 1))$, $m_1 \leq a_1(t) \leq M_1$, $t \in [0, T_1]$ $(0 < m_1 < M_1)$;

K27 $a_2 \in C^\alpha([0, T_2])$, $(\alpha \in (0, 1))$ *(cf.* (0.7)*)*, $m_1 \leq a_2(s) \leq M_1$, $s \in [0, T_2]$;

K28 $f \in C^\alpha([0, T_1] \times [0, T_2];\ C(\overline{\Omega}))$, $D_t f \in C^\alpha(E_1(T_1, T_2);\ C(\overline{\Omega}))$, $D_s f \in C^\alpha(E_2(T_1, T_2);\ C(\overline{\Omega}))$;

K29 $h \in C^1([0, T_1] \times [0, T_2])$;

K30 $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{B})$;

K31 $u_j \in C^\alpha([0, T_j];\ \mathcal{D}(\mathcal{A})) \cap C^{1+\alpha}([0, T_j];\ C(\overline{\Omega}))$ $(j = 1, 2)$;

K32 $\mathcal{A}u_1 + f(\cdot, 0) \in B([0, T_1];\ \mathcal{D}_{\mathcal{A}}(\alpha, \infty))$, $\mathcal{A}u_2 + f(0, \cdot) \in B([0, T_2];\ \mathcal{D}_{\mathcal{A}}(\alpha, \infty))$;

K33 $u_1(0) = u_2(0), \quad a_1(0)u_1'(0) + a_2(0)u_2'(0) - \mathcal{A}u_1(0) = f(0, 0)$.

Then problem (3.1) *admits an unique solution* $u \in C^\alpha([0, T_1] \times [0, T_2];\ \mathcal{D}(\mathcal{A})) \cap C^{1+\alpha}([0, T_1] \times [0, T_2];\ C(\overline{\Omega}))$. *In particular,* $u \in C^1([0, T_1] \times [0, T_2] \times \overline{\Omega})$.

4. Appendix.

This section is devoted to show a physical motivation for the integrodifferential equation in (0.1), which is related to two different times.
Let S be a smooth enough surface containing a spatial domain V. We consider a physical system F, where some diffusion process with memory take place. We assume that the evolution of a part of the system can be accessible only subsequently. We describe the evolution of the system by using two different times t and s, by two conservation laws concerning $\rho_1(t, x)$ and $\rho_2(s, x)$. These quantities account for, e.g., the density, the temperature, or more generally, some measurable physical quantities related to two parts F_1 and F_2 that constitute F. We introduce now the following notation:

- ν is the outward normal unit vector to the surface S,
- q_i $(i = 1, 2)$ are the generation rates per volume unit and per time unit related to the physical quantity of F_i,

- J_i $(i = 1, 2)$ are the flux densities per surface unit and per *proper* time unit related to F_i.

We now make our first basic assumption: *the variations of functions ρ_1 and ρ_2 are negligible with respect to s and t, respectively.* Consequently, from the conservation principle we get the following equations:

$$(4.1)\quad D_t \int_V \alpha_1 \rho_1(t, s, x)\, dx = -\int_S J_1(t, s, x) \cdot \nu\, d\eta + \int_V q_1(t, s, x)\, dx;$$

$$(4.2)\quad D_s \int_V \alpha_2 \rho_2(t, s, x)\, dx = -\int_S J_2(t, s, x) \cdot \nu\, d\eta + \int_V q_2(t, s, x)\, dx,$$

where α_j $(j = 1, 2)$ are proportionality constants.
By means of the divergency theorem and by the arbitrariness of the volume V, the evolution equations of the system turn out to be

$$\alpha_1 D_t \rho_1(t, s, x) = -\mathrm{div} J_1(t, s, x) + f_1(t, s, x); \tag{4.3}$$

$$\alpha_2 D_s \rho_1(t, s, x) = -\mathrm{div} J_2(t, s, x) + f_2(t, s, x). \tag{4.4}$$

Let us now suppose that the thermic memory is accounted by a convolution kernel $h(t, s)$ depending on the two times, which in turn, accounts for the two process F_i.
Consequently, we obtain, as a generalization of the thermic diffusion process, the following laws linking J_i, ρ_i and h, $(i = 1, 2)$:

$$\begin{aligned} J_1(t, s, x) = {}& -D_1 \nabla \rho_1(t, s, x) - \\ & - k_1 \int_0^t \int_0^s h(t - \tau, s - \sigma) \nabla \rho_1(\tau, \sigma, x)\, d\tau d\sigma; \end{aligned} \tag{4.5}$$

$$\begin{aligned} J_2(t, s, x) = {}& -D_2 \nabla \rho_2(t, s, x) - \\ & - k_2 \int_0^t \int_0^s h(t - \tau, s - \sigma) \nabla \rho_2(\tau, \sigma, x)\, d\tau d\sigma. \end{aligned} \tag{4.6}$$

Hence, we get a system of two integrodifferential equations for ρ_i, $(i = 1, 2)$

$$\begin{aligned} \alpha_1 D_t \rho_1(t, s, x) = {}& \mathrm{div} \big(D_1 \nabla \rho_1(t, s, x) \\ & + k_1 \int_0^t \int_0^s h(t - \tau, s - \sigma) \nabla \rho_1(\tau, \sigma, x)\, d\tau d\sigma \big) + f_1(t, s, x); \end{aligned} \tag{4.7}$$

$$\alpha_2 D_s \rho_2(t,s,x) = \operatorname{div}\Big(D_2 \nabla \rho_2(t,s,x) + k_2 \int_0^t \int_0^s h(t-\tau, s-\sigma) \nabla \rho_2(\tau,\sigma,x)\, d\tau d\sigma\Big) + f_2(t,s,x). \tag{4.8}$$

We now make our second basic assumption: *the vectors* (α_1, D_1, k_1) *and* (α_2, D_2, k_2) *are proportional with a proportionality constant* λ. It is now more convenient to sum or subtract memberwise equations (4.7) and (4.8) and to study the evolution of the quantities $\rho_1 + \rho_2$ and $\rho_1 - \rho_2$. In the case when ρ_1 and ρ_2 denote some densities, the linear combination $\rho_1 + \lambda\rho_2$ has an evident physical meaning.
Setting

$$\rho(t,s,x) := \rho_1(t,s,x) + \lambda\rho_2(t,s,x); \tag{4.9}$$

$$f(t,s,x) := f_1(t,s,x) + f_2(t,s,x). \tag{4.10}$$

from (4.7) and (4.8) and our basic assumption on the behaviour of ρ_1 and ρ_2, we deduce

$$D_t\rho(t,s,x) + D_s\rho(t,s,x) = D_t\rho_1(t,s,x) + \lambda D_s\rho_2(t,s,x) + + D_s\rho_1(t,s,x) + \lambda D_t\rho_2(t,s,x) \simeq D_t\rho_1(t,s,x) + \lambda D_s\rho_2(t,s,x). \tag{4.11}$$

Consequently, in a first approximation, we can assume that ρ satisfies the integrodifferential equation

$$\alpha_1(D_t\rho(t,s,x) + D_s\rho(t,s,x)) = \operatorname{div}\Big(D_1\nabla\rho(t,s,x) + k_1 \int_0^t \int_0^s h(t-\tau, s-\sigma)\nabla\rho(\tau,\sigma,x)\, d\tau d\sigma\Big) + f(t,s,x). \tag{4.12}$$

Assuming that our kernels do not depend on the spatial variables, we derive the equation

$$D_t\rho(t,s,x) + D_s\rho(t,s,x) = \operatorname{div}(D\nabla\rho(t,s,x)) + \int_0^t \int_0^s h(t-\tau, s-\sigma)\operatorname{div}(k\nabla\rho(\tau,\sigma,x))\, d\tau d\sigma + f(t,s,x), \tag{4.13}$$

where

$$D = D_1/\alpha_1, \quad k = k_1/\alpha_1. \tag{4.14}$$

Finally, setting

$$A(x)\rho(t,s,x) := \mathrm{div}(D\nabla\rho(t,s,x)); \tag{4.15}$$

$$B(x)\rho(t,s,x) := \mathrm{div}(k\nabla\rho(t,s,x)), \tag{4.16}$$

we obtain the equation

$$\begin{aligned} D_t\rho(t,s,x) + D_s\rho(t,s,x) &= A(x)\rho(t,s,x) \\ &+ \int_0^t \int_0^s h(t-\tau, s-\sigma)\mathrm{div}\, B(x)\rho(\tau,\sigma,x)\, d\tau d\sigma + f(t,s,x). \end{aligned} \tag{4.17}$$

We conclude by nothing that equation (4.13) can be assumed as a model of diffusion processes with a memory depending on both usual time t and some additional physical quantity s.
In this sense equation (4.13) is related to the two well-known papers by Friedman [1] and Lions [6] dealing with the differential case with more time variable (cf. also [2], [3], [5], [8], [9], [10], [13]).

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