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A SUMMER SCHOOL IN CATANIA

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In the summer of 1997 we ran the PRAGMATIC summer school in Catania. After many different thoughts about the format, we decided to try an experiment: We made a list of 18 research problems (reproduced below) and handed it out on the first day of class. Early in the first week we went through the list briefly explaining the background of the problems, and asked each student to choose a problem to work on, either singly or with one or two others. We lectured on subjects necessary for understanding and attacking the problems: our lectures occupied each morning for four weeks. (There was a break of a fifth week in the middle in which the students worked on the problems without us.) In the afternoons in the first two weeks the students presented seminar talks on relevant papers, while in the last two weeks each student spoke on his/her own researches. In addition we met individually two or three times with the students working on each problem. (The local organizers had found us a lovely place to live by the Mediterranean sea, and most of the students were also able to join us for a mathematical lunch and a swim sometime during the program!)

The choice of problems seemed to us very important. Our criteria were: that the answer was genuinely unknown and of interest to us; that *some* progress on the problem could be made by an industrious student using one of the computer algebra systems adapted to Algebraic Geometry, namely CoCoA [13], Macaulay [7], Macaulay2 [35], and Singular [36]; and that a strong student could, with luck and hard work, make a theoretical advance. We ran extensive tutorials and computer sessions to teach the students how to use Macaulay2.

The students worked hard on their problems, and (it seemed to us) learned a good deal. In some cases they discovered results that were new and surprising to us. During the last week, the students were asked to write up what they had done. Most of these writeups appear in this volume. One student, whose project led to a correct proof for a well-known but unproved "theorem" of Severi, will appear elsewhere (see Russo (1998)). (Violo (1998) found a proof independently around the same time.)

In the meantime, some other mathematicians have contributed to the solutions of some of the problems we posed; see the list of problems below for references of which we are aware.

The Summer School culminated in a workshop run by the "Syzygies" node of Europroj. Several of the students had the excitement of presenting talks in the workshop on their progress with the research problems, and everyone enjoyed hearing and meeting the many participants who came from the rest of Europe.

At every stage of planning and running the summer school we were helped enormously by the local organizers, Alfio Ragusa and Giuseppe Paxia. They did everything from juggling financial support from different agencies to assisting with good cheer at the final parties (one in our yard for the summer school and one in a local restaurant for the workshop). In between they did all the many things necessary to make the school run smoothly and to make the participants, from many faraway lands, feel at home. They and Rosario Strano also contributed greatly to the mathematical pleasures of the school by speaking in the seminars and by discussing mathematics with us and with the students. We thank them all for this Sicilian adventure.

Research Problems: PRAGMATIC 1997.

In the following list "unknown" means unknown to the authors. Listed with each problem is the suggestion we gave on how to begin, and some relevant references (often coming from work done during this summer school!) We also do not try to give exhaustive references, but merely mention in most of the cases only those relevant to the suggested approach.

1. L'vovsky's [44] "periodicity conjecture": Let $C \subset \mathbb{P}^n$ be a curve, and let Y be an arbitrary subscheme in $\mathbb{P}^n$. Examine L'vovski's conjecture that the resolution of $Y \cup \Gamma$ is eventually periodic in $\gamma = \deg(\Gamma)$ of period $d = \deg(C)$, where Γ is a "sufficiently general" set of points of C.
(Suggestion: Show first the conjecture for all large enough sets of points $\Gamma \subset C$, in the case where C is a rational normal curve.)
A complete proof to L'vovsky's conjecture was obtained during the workshop by

Mustaţă [48]. He also states and discusses a version of the Minimal Resolution Conjecture in this context.

2. Castelnuovo-Mumford regularity of square free monomial ideals: Examine the conjecture that if I is an ideal generated by square free monomials (that is I is the Stanley-Reisner ideal associated to a simplicial complex), then the Castelnuovo-Mumford regularity $\operatorname{reg}(I)$ is bounded by the number of facets d in the corresponding simplicial complex (and thus by the arithmetic degree). In the case where I is of pure codimension c, and the simplicial complex is connected in codimension one, examine the conjecture that the regularity is bounded by $d - c + 1$. These are versions of conjectures by Bayer, see Bayer-Mumford [6] and Eisenbud-Goto [20], respectively. See also Sturmfels [54] for a weaker result.
(Suggestion: Look first at simplicial complexes with few facets.)
Both conjectures have been proved, by Terai and Frühbis-Terai, respectively: the first turned out to be an easy application of the theory developed in Terai [57], the second and a version of it for simplicial complexes which are not necessarily connected in codimension one, are proved by Terai [58], and Frühbis-Terai [28].

3. Estimate the annihilator of $\wedge^{c+1} I$, where I is an ideal of codimension c. See Eisenbud-Green [21] for specific conjectures.

4. Torsion in the symmetric algebra: Let A be a graded domain generated in degree one, and let $I \subset A$ be a homogeneous ideal. The symmetric algebra of I is then a bigraded algebra. In which bidegrees is there A-torsion?
(Suggestion: Consider first primary monomial ideals with few generators. A reference for the codimension two case is Polini-Ulrich [50].)

5. Investigate the symbolic powers of the homogeneous ideals of rational normal curves, that is, the ideals of homogeneous forms vanishing to given order along a rational normal curve. Try other curves too!
(Suggestion: The symbolic powers of the ideal of the twisted cubic coincide with the usual powers, but this is already not anymore true for the ideal of a rational normal quartic curve, since its second symbolic power contains a cubic generator (the equation of the secant variety of the rational normal quartic). On the other hand, for the scroll $\mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$, whose ideal is defined by the minors of the generic $2 \times (n + 1)$-matrix, the symbolic powers are known to coincide with the ordinary powers by DeConcini-Eisenbud-Procesi [18]. Consider successive hyperplane sections, interpolating between this variety and the rational normal curve.)
Conca [15] has recently described symbolic powers of ideals of rational normal

curves. Essentially, his result shows that, as predicted, the equations of the various higher secant varieties are the only new generators for the symbolic powers of the rational normal curves.

6. Let $S \subset \mathbb{P}^r$ be a smooth rational normal scroll of codimension c, and let $X \subset S$ be a divisor of class $aH - bF$, where H is the hyperplane section and F the class of a ruling. It is known that the homogeneous coordinate ring of X is Cohen-Macaulay if and only if $-1 \leq b \leq c$. The minimal free resolution is known when $-1 < b$. (see Eisenbud [19], Appendix 2.6.) Investigate the minimal free resolution in the case $b = -1$!
(Suggestion: Start with the 3 different scrolls in $\mathbb{P}^5$.)
Note: This problem has reportedly been solved by Uwe Nagel (not yet available in preprint).

7. Let $X \subset \mathbb{P}^{34}$ be the 3-uple embedding of $\mathbb{P}^4$, and let $Y \subset \mathbb{P}^{69}$ be the 4-uple embedding of $\mathbb{P}^4$. The union of the 7-secant $\mathbb{P}^6$ to X is a hypersurface $SX \subset \mathbb{P}^{34}$, although one would expect from a dimension count that $SX = \mathbb{P}^{34}$ (Palatini [1902]). Similarly, the union $SY \subset \mathbb{P}^{69}$ of the 14-secant $\mathbb{P}^{13}$ to Y is a hypersurface. Investigate the degrees and the equations of these two hypersurfaces! These are two of the exceptional cases in the interpolation theorem of Alexander-Hirschowitz [2]–[4], which relate to the classical Waring problem: determine the minimal s such that a general degree d polynomial f can be expressed as a sum of d^{th} powers of s linear forms. See Geramita's notes [29] on the Waring problem, Iarrobino [41] and Ciliberto-Hirschowitz [12].
(Suggestion: Study (higher) secant varieties of interesting subvarieties of X and Y.)

8. Investigate, at least by rank distribution, the linear spaces of nilpotent matrices in the ring of $n \times n$-matrices (as subvarieties of the Grassmannian.) Some interesting and slightly relevant references: Motzkin-Taussky [45]–[47], and Gerstenhaber [31]–[34]. See perhaps also Eisenbud-Saltman [26].
(Suggestion: Do the cases $n \leq 4$.)
Some progress in this direction has been made by Causa-Re-Teodorescu [10].

9. Toric varieties and Minkowski sums: Let $\Gamma, \Gamma' \subset \mathbb{N}^r \setminus \{0\}$ be finite sets. Let I be the (binomial) ideal of relations on the monomials corresponding to the points of $\Gamma + \Gamma'$. The relations $(a + b) + (c + d) = (a + d) + (b + c)$ for $a, c \in \Gamma$, and $b, d \in \Gamma'$ correspond to 2×2 determinants in I. Investigate when these 2×2-minors generate the binomial ideal I!
(Suggestion: Investigate the condition that the minors form a Gröbner basis with respect to some term order. Investigate these conditions when Γ and Γ' are replaced by $m\Gamma$ and $n\Gamma'$ for $m, n >> 0$. Start with $r = 1$, look also at Segre

and Veronese embeddings, and the toric del Pezzo surfaces. What about secant varieties? See Eisenbud-Koh-Stillman [23] for the related problem in the case of curves.)

10. Investigate the regularity of powers, radical, products and sums of homogeneous ideals. For instance, is it true that $\mathrm{reg}(I^n) \leq n \cdot \mathrm{reg}(I)$, or even better that $\mathrm{reg}(I^n) \leq (n-1)d + \mathrm{reg}(I)$, when I is generated in degrees at most d? Chandler [11] has shown this for artinian ideals, and the weaker version for ideals of dimension 1; see also Geramita-Gimigliano-Pitteloud [30]. What about the inequality $\mathrm{reg}(I \cdot J) \leq \mathrm{reg}(I) + \mathrm{reg}(J)$? Also is it always true that $\mathrm{reg}(\sqrt{I}) \leq \mathrm{reg}(I)$? Ravi [51] has checked this in a number of cases, for instance when I is a monomial ideal.
(Suggestion: Consider first squarefree monomial ideals)
A counterexample to the first question (namely, the Stanley-Reisner ideal of the "usual" triangulation of $\mathbb{P}^2(\mathbb{R})$, for $n = 2$) was found by Terai (unpublished) during the summer school. See also Sturmfels [55] for a discussion of this example. A linear asymptotic bound for $\mathrm{reg}(I^n)$ has been recently obtained by Swanson [56] and Cutkosky-Herzog-Trung [16].

11. Investigate the free resolution of the ideal of forms of degree d in 3 variables vanishing at a given set of general points with prescribed multiplicities.
(Suggestion: The Hilbert function of this ideal is known; see Alexander-Hirschowitz [4]. This suggests a conjecture for the shape of the minimal resolution. See Harbourne [38],[39] for related information.)

12. Let $V \subset \mathbb{P}^5$ be the Veronese surface and let $\Gamma \subset V$ be a set of γ distinct points. Does the Gale transform of Γ lie on some interesting rational surface? What about other surfaces? In the case of rational normal scrolls the answer is yes; in the analogous question for curves the answer is the content of Goppa's classical result in coding theory. See Eisenbud-Popescu [24],[25] for definitions, details and further references.
Some progress in the case of points on Veronese surfaces has been made by Davide [17].

13. Let $\Gamma \subset \mathbb{P}^r$ be a set of $2r + 2$ points in linearly general position such that the homogeneous coordinate ring of Γ is Gorenstein (that is Γ is "self-associated"). Let Γ_1 be a subset of $r + 3$ points, and let Γ_2 be the remaining points. Let C be the unique rational normal curve through Γ_1, and let L be the $(r-2)$-plane spanned by the residual set Γ_2. Castelnuovo showed that $L \cap C$ consists of $r - 1$ further points, and $(L \cap C) \cup \Gamma_2$ is a self-associated set of points in L. Investigate conditions leading to a converse, that is, to an inductive construction of self-associated sets. See Castelnuovo [9] for the

original treatment, and Eisenbud-Popescu (1998) for a modern treatment and further references to classical literature.
A discussion of self-associated sets of points in $\mathbb{P}^3$ and $\mathbb{P}^4$ from this perspective is the content of Flamini [27].

14. Let C be a curve of genus 2. Construct rank 2 vector bundles E on C such that $\mathcal{O}_{\mathbb{P}(E)}(1)$ has degree 8 and 6 sections. Investigate the condition that this linear series is very ample. (It was shown by Ionescu [42] that this sometimes happens, but no explicit example is known.)

15. Let $C \subset \mathbb{P}^3$ be a smooth, nondegenerate curve. Find the surface swept out by the 3-secant lines to C, and investigate its singularities. Are they "usually" just the 4-secant lines? See Le Barz (1982), and Gruson-Peskine [37], for modern proofs of Berzolari's formula for the degree of this surface, and Cayley's formula for the number of 4-secant lines.
(Suggestion: Work out some simple cases: quintic, and sextic rational curves, or the complete intersection of two cubic surfaces.)
See Bertin [8] for a discussion of a number of examples.

16. Compute the scheme of k-secant m-planes to a smooth curve C in $\mathbb{P}^r$. A formula for the degree of this scheme is given in Arbarello-Cornalba-Griffiths-Harris [5]. There are some cases when this formula gives a negative number. Compute the k-secant scheme in such a case, and guess the meaning (if any) of this number.

17. Severi [53] gives a classification of surfaces in $\mathbb{P}^5$ having only one "apparent node" — that is, having only one secant line through a general point of $\mathbb{P}^5$ - thus their generic projection to $\mathbb{P}^4$ has just one node. Unfortunately, his argument has a gap: he does not consider the case of surfaces containing infinitely many planar curves. Construct some surfaces of this type, and see how small a number of apparent nodes you can get! Some surfaces with infinitely many planar curves can be constructed as hypersurface sections of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. (Severi's examples of such surfaces are the rational normal scrolls, and the quintic del Pezzo surface.)
The gaps in Severi's proof have been recently filled (independently) by Russo (1998) and Violo (1998).

18. Consider a $p \times q$ matrix with entries forms of given degrees d_{ij} in a sufficiently general way that its ideal of $k \times k$ minors has "expected" codimension $(p - k + 1)(q - k + 1)$. Assume the matrix is homogeneous in the sense that $d_{ij} + d_{kl} = d_{il} + d_{jk}$. The degree of the corresponding determinantal variety is determined by these conditions; it is a function of the d_{ij}. In case all $d_{ij} = 1$, the formula is classical, see Room [52]. A beautiful explicit formula was found by Geramita in case $p = k, q = k + 1$ and generalized to all cases $p = k$ by Herzog and Trung [40]. Find a formula in some cases where $k < p, q$; even the case $k = 2$, $p = q = 3$ is unknown!

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