ON THE SINGULARITIES OF THE TRISECANT SURFACE TO A SPACE CURVE

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Let $C$ be a smooth curve in $\mathbb{P}^3$. Trisecant lines to the curve $C$ are, in general, sweeping out a (reduced) surface $\Sigma_C$ in $\mathbb{P}^3$. In this note we attempt to describe some of the singularities of $\Sigma_C$, and in particular we show that if the curve $C$ has only a finite number of quadrisection lines, then the singular locus of $\Sigma_C$ contains the quadrisection lines to $C$, and the points through which pass more trisecants than through a generic point of the trisecant surface. Several explicit examples are discussed in the last section.

1. Preliminaries.

A trisecant line to a curve $C$ of $\mathbb{P}^3$ can be understood intuitively as a limit of lines cutting the curve in three distinct points.

More precisely, a trisecant $L$ is a line such that the intersection $L \cap C$ has multiplicity at least 3, that is to say:

$$\dim_\mathbb{C}(O_L) \geq 3.$$

Higher multisections are defined in a similar way. For a space curve, one expects in general a one-dimensional family of trisecants and only finitely many higher order secants.
The following classical formula due to Cayley, see Le Barz [7] and Gruson-Peskine [2], counts (with multiplicity) the number of quadriscant lines to a smooth curve $C \subset \mathbb{P}^3$ of degree $d$ and genus $g$:

$$\frac{(d - 2)(d - 3)^2(d - 4) - (d^2 - 7d + 13 - g)g}{12}.$$

For example, in the previous formula, a quintiscant line counts in general as $(\binom{5}{4}) = 5$ quadriscants. It is important to note that the a positive result in Cayley’s formula doesn’t force $C$ to have finitely many quadriscants, although a negative result implies that $C$ has an infinite number of quadriscants.

**Example.** Let $C$ be a curve of type $(4,4)$ drawn on a smooth quadric surface $Q \subset \mathbb{P}^3$. By Bezout any quadriscant line to $C$ is contained in $Q$ and conversely both rulings of the quadric are quadriscant to $C$. $C$ has degree $8$ and genus $9$, so the Cayley’s formula yields a negative number, $-4$, agreeing with what the geometry predicts!

The family of lines in $\mathbb{P}^3$ meeting a smooth space curve $C$ has codimension 1 in the four-dimensional Grassmannian of lines in $\mathbb{P}^3$, thus by a naive dimension count we expect a one dimensional family of triscant lines to $C$, finitely many quadriscant lines to $C$ and no lines meeting $C$ five times or more. The classical triscant lemma, see [6], asserts that if the family of triscants is nonempty, then the family has dimension one, unless $C$ is a plane curve of degree $\geq 3$. Thus if the nondegenerate curve $C' \subset \mathbb{P}^3$ admits triscant lines, the triscant lines to $C$ sweep out a surface $\Sigma_C \subset \mathbb{P}^3$, called the triscant surface to $C$. A scheme structure on $\Sigma_C$ (not always necessarily reduced) may be defined using Fitting ideals, but we will not make use of it in this note.

A classical formula of Berzolari, recasted in modern terms by Le Barz [7] (see also [2]), gives the number of triscant lines to $C \subset \mathbb{P}^3$ meeting a general line in $\mathbb{P}^3$:

$$\frac{(d - 1)(d - 2)(d - 3)}{3} - (d - 2)g.$$

In cases no multiplicities are involved, Berzolari’s formula computes the degree of the triscant surface $\Sigma_C$.

**Example.** The twisted cubic $C$ has no triscant lines: $C$ is cut out by quadrics (the $2 \times 2$-minors of a $2 \times 3$ matrix with linear entries), thus it has no triscant line. Berzolari’s formula yields $0$ confirming this fact.

**Proposizione 1.** A smooth curve $C \subset \mathbb{P}^3$ has no triscant lines if only if $C$ is either a line, a conic, the twisted cubic, or an elliptic normal quartic curve.
**Proof.** If such a curve $C$ is planar, then obviously $C$ is either a line or a conic. Assume now that $C$ is nondegenerate of degree $d$. If $C$ has no trisecants the projection of $C$ from a general point of $C$ is an isomorphism onto a smooth plane curve $\tilde{C}$ of degree $d - 1$. In particular, this allows us to compute the genus of $C$:

$$g(C) = \frac{(d - 2)(d - 3)}{2}.$$ 

On the other hand, Castelnuovo’s inequality (see [3], or [4]) gives

$$g(C) \leq \begin{cases} \frac{1}{3}d^2 - d + 1 & \text{if } d \text{ is even} \\ \frac{1}{3}(d^2 - 1) - d + 1 & \text{if } d \text{ is odd.} \end{cases}$$

Combining these we deduce that $d^2 - 6d + 8 \leq 0$, which gives $2 \leq d \leq 4$. Since $C$ is not planar, the only possibilities are either $d = 3$ and $C$ is the twisted cubic, or $d = 4$ and $C$ is either rational or elliptic. In the latter case, if $C$ is elliptic then it is the complete intersection of two quadrics and thus has no trisecants. If $C$ is a rational quartic space curve, then $C$ lies on a smooth quadric surface, where it is a curve of type $(1, 3)$, thus has as trisecants all lines in one of the rulings of the quadric. This concludes the proof. \( \square \)

**Remark 1.** The proof of the preceding proposition shows in fact that a space curve has either infinitely many trisecants or none!

The trisecant surface $\Sigma_C$ of a space curve $C \subset \mathbb{P}^3$ can be described in terms of the following correspondence:

Let $T$ be the algebraic subset of $C \times C \times C$ defined as the closure of the set of triples of points $(p, q, r)$ of $C \times C \times C$, with $p \neq q, p \neq r, q \neq r$, such that their linear span $<p, q, r>$ is a line. Let $T^{(1)}$ denote the union of all the irreducible components of $T$ of dimension $1$, and let $\mathcal{T}$ be the image of $T^{(1)}$ via the projection $\pi_1$ to $C \times C$ defined by omitting the first factor.

The trisecant surface $\Sigma_C$ coincides with $\phi(\mathcal{T} \times \mathbb{P}^1)$, where $\phi$ is defined by

$$\phi : \mathcal{T} \times \mathbb{P}^1 \subset C \times C \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

$$(p, q; (s, t)) \longmapsto sp + qt$$

is the natural rational map sending the pair $(p, q)$ to its linear span.
2. Singularities of the trisecant surface $\Sigma_C$.

We will describe simple geometric properties of $\Sigma_C$. We will assume in this section that $C \subseteq \mathbb{P}^3$ is a smooth irreducible nondegenerate curve. Let $n(C)$ be the biggest number $n \geq 3$ such that $C$ has an infinite number of $n$-secant lines. Denote by $m(C)$ the number of $n(C)$-secant lines that pass through a generic point of $\Sigma_C$. The first number is well defined being clearly bounded by the degree $d$ of the curve $C$, while the second number is clearly bounded by the number of trisecant lines, counted with multiplicity, which meet a fixed line. Observe also that, for instance, $m(C) = 2$ for curves $C$ lying on a smooth quadric in $\mathbb{P}^3$, of type $(n(C), n(C))$.

The surface $\Sigma_C$ is clearly connected being equidimensional of codimension 1. The following loci are in the singular locus of $\sigma_C$:

**Theorem 1.** Let $S \subseteq \Sigma_C$ denote the locus consisting of

- the $(n(C) + 1)$-secant lines of $C$
- the points of intersection of at least $m(C) + 1$ $n(C)$-secant lines to $C$.

Then $S \setminus B$ is contained in $\text{sing}(\Sigma_C)$, where $B$ is the set of tangency points of tangential trisecants to $C$.

**Proof.** We sketch a proof in the case where $n(C) = 3$, and thus when there are a finite number of quadrirsectant lines to $C$. The general case is similar.

The map $\phi$ introduced in § 1 is generically six to one and fails to be so exactly when the image by $\phi$ of the point $Q \in T \times \mathbb{P}^1$ is either on a quadrirsectant line to $C$, or is the point of intersection of two trisecant lines, or is a point of tangency of a tangential trisecant to $C$. Notice that in the last case the fiber over $\phi(Q)$ has dimension 1.

The map $\phi$ is unramified at all points $Q$ of $T \times \mathbb{P}^1$, which are not of the form $(p, p; (s, t))$, with $st = 0$, in $C \times C \times \mathbb{P}^1$. We have the following diagram:

$$
\begin{array}{ccc}
\mathbb{C}^{10} & \rightarrow & \mathbb{C}^4 \\
\cup & & \cup \\
T_Q(C \times C \times \mathbb{P}^1) & \rightarrow & T_Q \mathbb{P}^3 \\
\cup & & \cup \\
T_Q(T \times \mathbb{P}^1) & \rightarrow & T_Q(\Sigma_C)
\end{array}
$$

where the differential $d\phi_Q$ is thought as a linear map from $\mathbb{C}^{10}$ to $\mathbb{C}^4$ induced by the matrix:

$$
\begin{pmatrix}
s & s & s & s & t & t & t & t & p_0 & q_0 \\
s & s & s & s & t & t & t & t & p_1 & q_1 \\
s & s & s & s & t & t & t & t & p_2 & q_2 \\
s & s & s & s & t & t & t & t & p_3 & q_3
\end{pmatrix}
$$
with $Q = (p, q; (s, t))$. It is easy to check that $\text{Ker}(d\phi_Q) \cap T_Q(T \times \mathbb{P}^1) = 0$ for points $Q$ not of the form $(p, p; (s, t)) st = 0$, and the claim of the theorem follows easily from this.

**Corollary 2.** If $C \subset \mathbb{P}^3$ is a smooth, nodegenerate curve of degree $d \geq 7$ and genus $g$, having only finitely many quadririscant lines and tangential triscant lines, then $C \subset \text{sing}(\Sigma C)$.

**Proof.** By the Plücker formulas (see [3], p.291) it follows that the number of triscants through a general point $p$ of $C$ is

$$\delta(C) = \frac{(d-2)(d-3)}{2} - g.$$ 

We look now to the projection from a general point $q$ of $\Sigma C$. Any triple point of the projection drops the genus by 3, while nodes and tacnodes drop the genus by 1 or 2, respectively. We deduce that

$$g \geq \frac{(d-1)(d-2)}{2} - 3m(C),$$

where $m(C)$ denotes as above the number of triscant lines through a general point of $\Sigma C$. Combining these two relations with Castelnuovo’s inequality, we deduce, for $d \geq 7$, that $\delta(C) > m(C)$, which implies that $\Sigma C$ is singular along the curve $C$. $\square$

3. **Some examples of triscant surfaces $\Sigma C$.**

(1) If $C$ is a curve of bidegree $(a, b)$ drawn on a smooth quadric $Q$ with $b \geq 3$ or $a \geq 3$, then $\Sigma C$ is just the quadric surface. Note also that $m(C) = 1$ or 2 depending on whether one or both rulings of $Q$ are triscant to $C$. In this case $\Sigma C$ has no singularities, but Berzolari’s formula counts (usually) the degree of a multiplicity structure on this surface.

(2) If $C$ is the intersection of a quadric cone $Q$ and a smooth cubic surface $V$ in $\mathbb{P}^3$, then again $\Sigma C = Q$, the vertex of the cone being the only singularity of the triscant surface.

(3) Assume now $C$ is a smooth curve of genus 2 and degree 5, traced on a quadric cone. Then $C$ is linked to a ruling of the cone in the complete intersection of the cone with a cubic surface passing through the vertex. In this case there is only one triscant line passing through the general point of $C$. 
In fact, conversely, assume now \( C \subset \mathbb{P}^3 \) is such that there is a unique (genuine) trisecant line through the general point \( p \) of \( C \). Then, on one hand \((d - 2)(d - 3)/2 - g(C) = 1 \) or \(2 \) (depending on whether the projection of \( C \) from \( p \) acquires an ordinary node or a tacnode), on the other hand the genus must satisfy Castelnuovo’s inequality. Hence \( d \leq 6 \), and the possibilities are \( d = 4, g = 0, d = 5, g = 1, 2 \) or \( d = 6, g = 4, 5 \). Obviously, the last case doesn’t exist! If \( C \) is rational quartic curve, then \( C \) lies on a smooth quadric as a divisor of type \((1, 3)\), and so there is a unique trisecant line through the general point of the curve. If \( C \) is a quintic of genus 2, then again \( C \) lies on a quadric surface \( Q \) (by Riemann-Roch). Either \( Q \) is smooth and \( C \) is a divisor of type \((2, 3)\), so there is a unique trisecant line through the general point of the curve, or \( Q \) is a quadric cone and we are in the previously described case. If \( C \) has degree 6 and genus 4, then either \( C \) is either the complete intersection of a quadric cone and a cubic or is a curve of type \((3, 3)\) on a smooth quadric, in which case there are two trisecant lines through the general point of the curve.

In conclusion, curves of degree 5 and genus 2, and curves of degree 6 and genus 4 traced on a quadric cone, are the only space curves with a single (genuine) trisecant through the general point of the curve, such that \( \Sigma_C \) has a finite non empty singular locus (1 point).

(4) Let \( C \subset \mathbb{P}^3 \) be a smooth rational sextic curve, thus a smooth projection to \( \mathbb{P}^3 \) of the rational normal sextic in \( \mathbb{P}^6 \). We will describe in the sequel the trisecant surface for various such projections. If \( C \) is a curve of type \((1, 5)\) on a smooth quadric, then as seen above the quadric is the trisecant surface to \( C \). We will therefore assume henceforth that \( C \) is not contained in any quadric surface.

**Lemma 3.** The curve \( C \) has only a finite number of 4-secant lines.

**Proof.** Riemann-Roch gives \( \chi(\mathcal{O}_C(3)) = 19 \), while \( h^0(\mathcal{O}_{\mathbb{P}^3}(3)) = 20 \), so \( C \) is contained in a cubic surface \( V \), which cannot be a cone over a plane cubic curve since \( C \) is rational. Thus either \( V \) is smooth or has at most 4 nodes, or \( V \) is non-normal in which case it is ruled and has a double line as singular locus. In this last case, the cubic is the projection of a smooth cubic scroll \( \bar{V} \) in \( \mathbb{P}^4 \) from a point outside it. \( \bar{V} \) is the blow-up of \( \mathbb{P}^2 \) in a point, embedded in \( \mathbb{P}^4 \) via the linear system \( |H| = |2l - E| \), where \( l \) is the class of a line in \( \mathbb{P}^2 \) and \( E \) is the exceptional divisor. Adjunction on \( \bar{V} \) shows that the pullback of the curve \( C \) belongs to the linear system \( |5l - 4E| \). In particular the double line of \( V \) is 5-secant to \( C \), but \( C \) has no further 4-secant lines on \( V \). By Bezout all 4-secant lines to \( C \) lie on the cubic so we are done in this case. If the cubic surface \( V \) is normal, then it contains only finitely many lines, and Bezout’s theorem allows to conclude the proof. \( \square \)
As we have seen in the proof of the previous lemma, the sextic \( C \) lies on a cubic surface \( V \subset \mathbb{P}^3 \). By Bezout all 4-secant lines to \( C \) lie also on \( V \). For simplicity we will assume in the sequel that \( V \) is smooth, and thus that it is the embedding in \( \mathbb{P}^3 \) of \( \mathbb{P}^2 \) blown up in 6 points, via the linear system \( |H| = |3l - \sum_{i=1}^{6} E_i| \), where \( l \) is the class of a line in \( \mathbb{P}^2 \) and \( E_i, i = 1, \ldots, 6 \), are the exceptional divisors.

There are several choices for \( C \) on the cubic, and we will discuss here only two of them.

If \( C \in |4l - \sum_{i=1}^{3} 2E_i| \) is a smooth rational sextic curve, then \( C \) has exactly 6 disjoint 4-secant lines on \( V \). (Observe that in this case \( V \) is the unique cubic surface containing \( C \).) In terms of the basis of the Picard lattice of \( V \) the 4-secant lines are

\[
2l - E_i - E_j - \sum_{k=4}^{6} E_k, \quad \{i, j\} \subset \{1, 2, 3\}, \quad i \neq j.
\]

(3 such lines) and

\[
l - E_i - E_j, \quad \{i, j\} \subset \{4, 5, 6\}, \quad i \neq j,
\]

(again 3 such lines). All the other lines on \( V \) have secancy \( \leq 2 \) with \( C \). This agrees with Cayley’s formula which gives 6 quadriscant lines to \( C \). These 6 lines are in the singular locus of the trisecant surface \( \Sigma_C \), which is also singular along \( C \).

A more special case, corresponds to \( C \) a smooth rational sextic curve in the linear system \( |3l - 2E_1 - E_2| \). Observe that in this case \( C \) is in fact contained in a pencil of cubic surfaces. An analysis as above shows that

\[
2l - \sum_{i=2}^{6} E_i,
\]

is a 5-secant line to \( C \), while

\[
2l - E_1 - E_3 - E_4 - E_5 - E_6
\]

is a genuine quadriscant line to \( C \). There are no further quadriscant lines since the union of \( C \) with the 5-secant line and “twice” the quadriscant line is the complete intersection of the pencil of cubics containing \( C \) (since \((3l - 2E_1 - E_2) + (2l - E_1 - E_3 - E_4 - E_5 - E_6) + 2(2l - \sum_{i=2}^{6} E_i) = 3H\)). In this case the 5-secant line counts as 5 quadriscant lines in Cayley’s formula.
Note also that in this case several trisecant lines to \( C \) lie on the cubic \( V \), for instance \( 2l - E_1 - E_2 - E_3 - E_l \), where the triplet \( \{i, j, k\} \subset \{3, 4, 5, 6\} \), accounts three such trisecant lines.

Finally note that Plücker’s formula gives 6 trisecant lines passing through a generic point of \( C \), thus the curve \( C \) is in the singular locus of \( \Sigma_C \) in both studied cases. In particular, the singular locus of \( \Sigma_C \) is not discrete and not irreducible.

I would like to thank Professor Ch. Peskine for suggesting me to study this last example.

I was introduced to this problem, during the PRAGMATIC’97 summer school that took place in Catania. I would like to thank Professors D. Eisenbud and S. Popescu for guidance and for introducing me to the use of the computer algebra softwares Macaulay/Macaulay2. I am also grateful to the organizers of PRAGMATIC’97, and the faculty of the University of Catania for the warm and pleasant atmosphere they provided during the summer school.

REFERENCES

[1] A. Geramita, Inverse systems of fat points, the curve seminar at Queens, vol. X, Queen’s university publications, Kingston, ON.


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