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ON THE SINGULARITIES OF THE TRISECANT SURFACE TO A SPACE CURVE

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Let C be a smooth curve in $\mathbb{P}^3$. Trisecant lines to the curve C are, in general, sweeping out a (reduced) surface Σ_C in $\mathbb{P}^3$. In this note we attempt to describe some of the singularities of Σ_C, and in particular we show that if the curve C has only a finite number of quadrisecant lines, then the singular locus of Σ_C contains the quadrisecant lines to C, and the points through which pass more trisecants than through a generic point of the trisecant surface. Several explicit examples are discussed in the last section.

1. Preliminaries.

A *trisecant line* to a curve C of $\mathbb{P}^3$ can be understood intuitively as a limit of lines cutting the curve in three distinct points.

More precisely, a trisecant L is a line such that the intersection $L \cap C$ has multiplicity at least 3, that is to say:

$$\dim_{\mathbb{C}}\left(\frac{\mathcal{O}_{\mathbb{P}^3}}{\mathcal{I}_L + \mathcal{I}_C}\right) \geq 3.$$

Higher multisecants are defined in a similar way. For a space curve, one expects in general a one-dimensional family of trisecants and only finitely many higher order secants.

The following classical formula due to Cayley, see Le Barz [7] and Gruson-Peskine [2], counts (with multiplicity) the number of quadrisecant lines to a smooth curve $C \subset \mathbb{P}^3$ of degree d and genus g:

$$\frac{(d-2)(d-3)^2(d-4)}{12} - \frac{(d^2-7d+13-g)g}{2}.$$

For example, in the previous formula, a quintisecant line counts in general as $\binom{5}{4} = 5$ quadrisecants. It is important to note that the a positive result in Cayley's formula doesn't force C to have finitely many quadrisecants, although a negative result implies that C has an infinite number of quadrisecants.

Example. Let C be a curve of type $(4, 4)$ drawn on a smooth quadric surface $Q \subset \mathbb{P}^3$. By Bezout any quadrisecant line to C is contained in Q and conversely both rulings of the quadric are quadrisecant to C. C has degree 8 and genus 9, so the Cayley's formula yields a negative number, -4, agreeing with what the geometry predicts !

The family of lines in $\mathbb{P}^3$ meeting a smooth space curve C has codimension 1 in the four-dimensional Grassmannian of lines in $\mathbb{P}^3$, thus by a naive dimension count we expect a one dimensional family of trisecant lines to C, finitely many quadrisecant lines to C and no lines meeting C five times or more. The classical trisecant lemma, see [6], asserts that if the family of trisecants is nonempty, then the family has dimension one, unless C is a plane curve of degree ≥ 3. Thus if the nondegenerate curve $C \subset \mathbb{P}^3$ admits trisecant lines, the trisecant lines to C sweep out a surface $\Sigma_C \subset \mathbb{P}^3$, called the *trisecant surface* to C. A scheme structure on Σ_C (not always necessarily reduced) may be defined using Fitting ideals, but we will not make use of it in this note.

A classical formula of Berzolari, recasted in modern terms by Le Barz [7] (see also [2]), gives the number of trisecant lines to $C \subset \mathbb{P}^3$ meeting a general line in $\mathbb{P}^3$:

$$\frac{(d-1)(d-2)(d-3)}{3} - (d-2)g.$$

In case no multiplicities are involved, Berzolari's formula computes the degree of the trisecant surface Σ_C.

Example. The twisted cubic C has no trisecant lines: C is cut out by quadrics (the 2×2-minors of a 2×3 matrix with linear entries), thus it has no trisecant line. Berzolari's formula yields 0 confirming this fact.

Proposizione 1. *A smooth curve $C \subset \mathbb{P}^3$ has no trisecant lines if only if C is either a line, a conic, the twisted cubic, or an elliptic normal quartic curve.*

Proof. If such a curve C is planar, then obviously C is either a line or a conic. Assume now that C is nondegenerate of degree d. If C has no trisecants the projection of C from a general point of C is an isomorphism onto a smooth plane curve $\bar{C}$ of degree $d - 1$. In particular, this allows us to compute the genus of C:

$$g(C) = \frac{(d-2)(d-3)}{2}.$$

On the other hand, Castelnuovo's inequality (see [3], or [4]) gives

$$g(C) \leq \begin{cases} \frac{1}{4}d^2 - d + 1 \text{ if } d \text{ is even} \\ \frac{1}{4}(d^2 - 1) - d + 1 \text{ if } d \text{ is odd.} \end{cases}$$

Combining these we deduce that $d^2 - 6d + 8 \leq 0$, which gives $2 \leq d \leq 4$. Since C is not planar, the only possibilities are either $d = 3$ and C is the twisted cubic, or $d = 4$ and C is either rational or elliptic. In the latter case, if C is elliptic then it is the complete intersection of two quadrics and thus has no trisecants. If C is a rational quartic space curve, then C lies on a smooth quadric surface, where it is a curve of type $(1, 3)$, thus has as trisecants all lines in one of the rulings of the quadric. This concludes the proof. □

Remark 1. *The proof of the preceding proposition shows in fact that a space curve has either infinitely many trisecants or none!*

The trisecant surface Σ_C of a space curve $C \subset \mathbb{P}^3$ can be described in terms of the following correspondence:
Let T be the algebraic subset of $C \times C \times C$ defined as the closure of the set of triples of points (p, q, r) of $C \times C \times C$, with $p \neq q$, $p \neq r$, $q \neq r$, such that their linear span $< p, q, r >$ is a line. Let $T^{(1)}$ denote the union of all the irreducible components of T of dimension 1, and let $\mathcal{T}$ be the image of $T^{(1)}$ via the projection π_1 to $C \times C$ defined by omitting the first factor.

The trisecant surface Σ_C coincides with $\phi(\mathcal{T} \times \mathbb{P}^1)$, where ϕ is defined by

$$\phi : \begin{array}{ccc} \mathcal{T} \times \mathbb{P}^1 \subset & C \times C \times \mathbb{P}^1 & \longrightarrow \mathbb{P}^3 \\ & (p, q; (s, t)) & \longmapsto sp + qt \end{array}$$

is the natural rational map sending the pair (p, q) to its linear span.

2. Singularities of the trisecant surface Σ_C.

We will describe simple geometric properties of Σ_C. We will assume in this section that $C \subset \mathbb{P}^3$ is a smooth irreducible nondegenerate curve. Let $n(C)$ be the biggest number $n \geq 3$ such that C has an infinite number of n-secant lines. Denote by $m(C)$ the number of $n(C)$-secant lines that pass through a generic point of Σ_C. The first number is well defined being clearly bounded by the degree d of the curve C, while the second number is clearly bounded by the number of trisecant lines, counted with multiplicity, which meet a fixed line. Observe also that, for instance, $m(C) = 2$ for curves C lying on a smooth quadric in $\mathbb{P}^3$, of type $(n(C), n(C))$.

The surface Σ_C is clearly connected being equidimensional of codimension 1. The following loci are in the singular locus of σ_C:

Theorem 1. *Let $\mathcal{S} \subset \Sigma_C$ denote the locus consisting of*
- *the $(n(C)+1)$-secant lines of C*
- *the points of intersection of at least $m(C)+1$ $n(C)$-secant lines to C.*

Then $\mathcal{S} \setminus \mathcal{B}$ is contained in $sing(\Sigma_C)$, where $\mathcal{B}$ is the set of tangency points of tangential trisecants to C.

Proof. We sketch a proof in the case where $n(C) = 3$, and thus when there are a finite number of quadrisecant lines to C. The general case is similar.

The map ϕ introduced in § 1 is generically six to one and fails to be so exactly when the image by ϕ of the point $Q \in \mathcal{T} \times \mathbb{P}^1$ is either on a quadrisecant line to C, or is the point of intersection of two trisecant lines, or is a point of tangency of a tangential trisecant to C. Notice that in the last case the fiber over $\phi(Q)$ has dimension 1.

The map ϕ is unramified at all points Q of $\mathcal{T} \times \mathbb{P}^1$, which are not of the form $(p, p; (s, t))$, with $st = 0$, in $C \times C \times \mathbb{P}^1$. We have the following diagram:

$$\begin{array}{ccc} \mathbb{C}^{10} & \rightarrow & \mathbb{C}^4 \\ \cup & & \cup \\ T_Q(C \times C \times \mathbb{P}^1) & \rightarrow & T_Q P^3 \\ \cup & & \cup \\ T_Q(\mathcal{T} \times \mathbb{P}^1) & \rightarrow & T_Q(\Sigma_C) \end{array}$$

where the differential $d\phi_Q$ is thought as a linear map from $\mathbb{C}^{10}$ to $\mathbb{C}^4$ induced by the matrix:

$$\begin{pmatrix} s & s & s & s & t & t & t & t & p_0 & q_0 \\ s & s & s & s & t & t & t & t & p_1 & q_1 \\ s & s & s & s & t & t & t & t & p_2 & q_2 \\ s & s & s & s & t & t & t & t & p_3 & q_3 \end{pmatrix}$$

with $Q = (p, q; (s, t))$. It is easy to check that $Ker(d\phi_Q) \cap T_Q(\mathcal{T} \times \mathbb{P}^1) = 0$ for points Q not of the form $(p, p; (s, t))$ $st = 0$, and the claim of the theorem follows easily from this.

Corollary 2. *If $C \subset \mathbb{P}^3$ is a smooth, nodegenerate curve of degree $d \geq 7$ and genus g, having only finitely many quadrisecant lines and tangential trisecant lines, then $C \subset sing(\Sigma_C)$.*

Proof. By the Plücker formulas (see [3], p.291) it follows that the number of trisecants through a general point p of C is

$$\delta(C) = \frac{(d-2)(d-3)}{2} - g.$$

We look now to the projection from a general point q of Σ_C. Any triple point of the projection drops the genus by 3, while nodes and tacnodes drop the genus by 1 or 2, respectively. We deduce that

$$g \leq \frac{(d-1)(d-2)}{2} - 3m(C),$$

where $m(C)$ denotes as above the number of trisecant lines through a general point of Σ_C. Combining these two relations with Castelnuovo's inequality, we deduce, for $d \geq 7$, that $\delta(C) > m(C)$, which implies that Σ_C is singular along the curve C. □

3. Some examples of trisecant surfaces Σ_C.

(1) If C is a curve of bidegree (a, b) drawn on a smooth quadric Q with $b \geq 3$ or $a \geq 3$, then Σ_C is just the quadric surface. Note also that $m(C) = 1$ or 2 depending on wether one or both rulings of Q are trisecant to C. In this case Σ_C has no singularities, but Berzolari's formula counts (usually) the degree of a multiplicity structure on this surface.

(2) If C is the intersection of a quadric cone Q and a smooth cubic surface V in $\mathbb{P}^3$, then again $\Sigma_C = Q$, the vertex of the cone being the only singularity of the trisecant surface.

(3) Assume now C is a smooth curve of genus 2 and degree 5, traced on a quadric cone. Then C is linked to a ruling of the cone in the complete intersection of the cone with a cubic surface passing through the vertex. In this case there is only one trisecant line passing through the general point of C.

In fact, conversely, assume now $C \subset \mathbb{P}^3$ is such that there is a unique (genuine) trisecant line through the general point p of C. Then, on one hand $(d-2)(d-3)/2 - g(C) = 1$ or 2 (depending on whether the projection of C from p acquires an ordinary node or a tacnode), on the other hand the genus must satisfy Castelnuovo's inequality. Hence $d \leq 6$, and the possibilities are $d = 4, g = 0, d = 5, g = 1, 2$ or $d = 6, g = 4, 5$. Obviously, the last case doesn't exist! If C is rational quartic curve, then C lies on a smooth quadric as a divisor of type $(1, 3)$, and so there is a unique trisecant line through the general point of the curve. If C is a quintic of genus 2, then again C lies on a quadric surface Q (by Riemann-Roch). Either Q is smooth and C is a divisor of type $(2, 3)$, so there is a unique trisecant line through the general point of the curve, or Q is a quadric cone and we are in the previously described case. If C has degree 6 and genus 4, then either C is either the complete intersection of a quadric cone and a cubic or is a curve of type $(3, 3)$ on a smooth quadric, in which case there are two trisecant lines through the general point of the curve.

In conclusion, curves of degree 5 and genus 2, and curves of degree 6 and genus 4 traced on a quadric cone, are the only space curves with a single (genuine) trisecant through the general point of the curve, such that Σ_C has a finite non empty singular locus (1 point).

(4) Let $C \subset \mathbb{P}^3$ be a smooth rational sextic curve, thus a smooth projection to $\mathbb{P}^3$ of the rational normal sextic in $\mathbb{P}^6$. We will describe in the sequel the trisecant surface for various such projections. If C is a curve of type $(1, 5)$ on a smooth quadric, then as seen above the quadric is the trisecant surface to C. We will therefore assume henceforth that C is not contained in any quadric surface.

Lemma 3. *The curve C has only a finite number of 4-secant lines.*

Proof. Riemann-Roch gives $\chi(\mathcal{O}_C(3)) = 19$, while $h^0(\mathcal{O}_{\mathbb{P}^3}(3)) = 20$, so C is contained in a cubic surface V, which cannot be a cone over a plane cubic curve since C is rational. Thus either V is smooth or has at most 4 nodes, or V is non-normal in which case it is ruled and has a double line as singular locus. In this last case, the cubic is the projection of a smooth cubic scroll $\tilde{V}$ in $\mathbb{P}^4$ from a point outside it. $\tilde{V}$ is the blow-up of $\mathbb{P}^2$ in a point, embedded in $\mathbb{P}^4$ via the linear system $|H| = |2l - E|$, where l is the class of a line in $\mathbb{P}^2$ and E is the exceptional divisor. Adjunction on $\tilde{V}$ shows that the (pullback of the) curve C belongs to the linear system $|5l - 4E|$. In particular the double line of V is 5-secant to C, but C has no further 4-secant lines on V. By Bezout all 4-secant lines to C lie on the cubic so we are done in this case. If the cubic surface V is normal, then it contains only finitely many lines, and Bezout's theorem allows to conclude the proof. □

As we have seen in the proof of the previous lemma, the sextic C lies on a cubic surface $V \subset \mathbb{P}^3$. By Bezout all 4-secant lines to C lie also on V. For simplicity we will assume in the sequel that V is smooth, and thus that it is the embedding in $\mathbb{P}^3$ of $\mathbb{P}^2$ blown up in 6 points, via the linear system $|H| = |3l - \sum_{i=1}^{6} E_i|$, where l is the class of a line in $\mathbb{P}^2$ and E_i, $i = 1, \dots, 6$, are the exceptional divisors.

There are several choices for C on the cubic, and we will discuss here only two of them.

If $C \in |4l - \sum_{i=1}^{3} 2E_i|$ is a smooth rational sextic curve, then C has exactly 6 disjoint 4-secant lines on V. (Observe that in this case V is the unique cubic surface containing C.) In terms of the basis of the Picard lattice of V the 4-secant lines are

$$2l - E_i - E_j - \sum_{k=4}^{6} E_k, \qquad \{i, j\} \subset \{1, 2, 3\}, \quad i \neq j,$$

(3 such lines) and

$$l - E_i - E_j, \qquad \{i, j\} \subset \{4, 5, 6\}, \quad i \neq j,$$

(again 3 such lines). All the other lines on V have secancy ≤ 2 with C. This agrees with Cayley's formula which gives 6 quadrisecant lines to C. These 6 lines are in the singular locus of the trisecant surface Σ_C, which is also singular along C.

A more special case, corresponds to C a smooth rational sextic curve in the linear system $|3l - 2E_1 - E_2|$. Observe that in this case C is in fact contained in a pencil of cubic surfaces. An analysis as above shows that

$$2l - \sum_{i=2}^{6} E_i,$$

is a 5-secant line to C, while

$$2l - E_1 - E_3 - E_4 - E_5 - E_6$$

is a genuine quadrisecant line to C. There are no further quadrisecant lines since the union of C with the 5-secant line and "twice" the quadrisecant line is the complete intersection of the pencil of cubics containing C (since $(3l - 2E_1 - E_2) + (2l - E_1 - E_3 - E_4 - E_5 - E_6) + 2(2l - \sum_{i=2}^{6} E_i) = 3H$). In this case the 5-secant line counts as 5 quadrisecant lines in Cayley's formula.

Note also that in this case several trisecant lines to C lie on the cubic V, for instance $2l - E_1 - E_2 - E_j - E_k - E_l$, where the triplet $\{i, j, k\} \subset \{3, 4, 5, 6\}$, accounts three such trisecant lines.

Finally note that Plücker's formula gives 6 trisecant lines passing through a generic point of C, thus the curve C is in the singular locus of Σ_C in both studied cases. In particular, the singular locus of Σ_C is not discrete and not irreducible.

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