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ON THE SETS OF BOUNDEDNESS OF SOLUTIONS TO DEGENERATE FOURTH-ORDER EQUATIONS WITH STRENGTHENINGLY MONOTONE PRINCIPAL PARTS, ABSORPTION AND L^1-DATA

ALEXANDER A. KOVALEVSKY - FRANCESCO NICOLOSI

We consider the Dirichlet problem for a class of degenerate nonlinear elliptic fourth-order equations with strengtheningly monotone principal parts, absorbing lower-order terms and L^1-right-hand sides. We establish existence of solutions of the given problem bounded on the sets where the behaviour of the data of the problem is regular enough.

1. Introduction

In this article we consider degenerate nonlinear elliptic fourth-order equations with strengtheningly monotone principal parts, absorbing lower-order terms and L^1-right-hand sides. A representative of such equations is the following one:

$$\sum_{|\alpha|=2} D^\alpha(\mu D^\alpha u) - \sum_{|\alpha|=1} D^\alpha(\nu |D^\alpha u|^{q-2} D^\alpha u) + |u|^{\sigma-1}u = f \text{ in } \Omega,$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n > 2$, $2 < q < n$, $\sigma > 1$, μ and ν are positive functions in Ω and $f \in L^1(\Omega)$.

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These equations belong to a class of degenerate fourth-order equations with a strengthened ellipticity and L^1-data. Existence and properties of solutions of the Dirichlet problem for equations of this class were studied in [5] and [6]. Using considerations stated in [5] and a modification of the Moser method (see for instance [9]), in [7] we proved that the given problem has solutions bounded on the sets where the behavoour of the data of the problem is regular enough. This fact has been established under the restriction

$$q < \tilde{q}(q-1)/q, \tag{1.1}$$

where q is a number connected with the rate of growth of coefficients of the equations with respect to the first-order derivatives of unknown function, and $\tilde{q}$ is an exponent characterizing an embedding of functional spaces involved.

The mentioned result of [7] can also be applied for the equations which we consider in the present work. However, due to the absorbing term $|u|^{\sigma-1}u$ in the equations and an appropriate condition on its exponent σ, really we can avoid the use of restriction (1.1), keeping at the same time the similar result on the sets of boundedness of solutions. The proof of the corresponding theorem is the main goal of the article.

We remark that only to simplify considerations concerning the pointwise convergence of the derivatives of solutions of some approximating problems, we restrict ourselves in this work with a particular case of principal parts of equations studied in [5]–[7]. The general situation can be treated following detailed constructions given in [2] for a class of nondegenerate fourth-order equations with a strengthened ellipticity and L^1-data. In its turn these constructions develop the approach proposed in [1] for nondegenerate second-order equations with L^1-data.

Finally, we note that the boundedness and Hölder continuity of solutions of nonlinear elliptic high-order equations with a strengthened ellipticity have already been investigated in [9] (nondegenerate case) and in [8] and [3] (degenerate case). However, it has been made for equations with data regular enough in all their domain of definition.

2. Preliminaries and auxiliary assertions

Let $n \in \mathbb{N}$, $n > 4$, Ω be a bounded open set of $\mathbb{R}^n$ and $q \in (4, n)$.

Let ν be a positive function on Ω such that

$$\nu \in L^1_{\mathrm{loc}}(\Omega), \quad \left(\frac{1}{\nu}\right)^{1/(q-1)} \in L^1_{\mathrm{loc}}(\Omega).$$

We denote by $W^{1,q}(\nu,\Omega)$ the set of all functions $u\in L^q(\Omega)$ having for every n-dimensional multiindex $\alpha, |\alpha|=1$, the weak derivative $D^\alpha u$ with the property $\nu^{1/q}D^\alpha u\in L^q(\Omega)$. $W^{1,q}(\nu,\Omega)$ is a Banach space with the norm

$$\|u\|_{1,q,\nu}=\left(\int_\Omega |u|^q dx+\sum_{|\alpha|=1}\int_\Omega \nu|D^\alpha u|^q dx\right)^{1/q}.$$

The closure of $C_0^\infty(\Omega)$ in $W^{1,q}(\nu,\Omega)$ is denoted by $\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$.

Let μ be a positive function on Ω such that

$$\mu\in L^1_{\text{loc}}(\Omega),\ \ \frac{1}{\mu}\in L^1_{\text{loc}}(\Omega).$$

We denote by $W^{1,q}_{2,2}(\nu,\mu,\Omega)$ the set of all functions $u\in W^{1,q}(\nu,\Omega)$ having for every n-dimensional multiindex $\alpha, |\alpha|=2$, the weak derivative $D^\alpha u$ with the property $\mu^{1/2}D^\alpha u\in L^2(\Omega)$. $W^{1,q}_{2,2}(\nu,\mu,\Omega)$ is a Banach space with the norm

$$\|u\|=\|u\|_{1,q,\nu}+\left(\sum_{|\alpha|=2}\int_\Omega \mu|D^\alpha u|^2 dx\right)^{1/2}.$$

The closure of $C_0^\infty(\Omega)$ in $W^{1,q}_{2,2}(\nu,\mu,\Omega)$ is denote by $\overset{\circ}{W}{}^{1,q}_{2,2}(\nu,\mu,\Omega)$.

Hypothesis 2.1. There exist real numbers $\tilde q>q$ and $c>0$ such that for every $u\in \overset{\circ}{W}{}^{1,q}(\nu,\Omega)$,

$$\left(\int_\Omega |u|^{\tilde q}dx\right)^{1/\tilde q}\leqslant c\left(\sum_{|\alpha|=1}\int_\Omega \nu|D^\alpha u|^q dx\right)^{1/q}.$$

Further, let $h\in C^\infty(\mathbb{R})$ be a nondecreasing functuion such that $h=0$ in $(-\infty,0]$ and $h=1$ in $[1,+\infty)$. We set

$$c'=2\max_{\mathbb{R}} h',\ \ c''=c'+\max_{\mathbb{R}}|h''|.$$

Now let for every $s\in\mathbb{N}$, $h_s:\mathbb{R}\to\mathbb{R}$ be the function such that

$$h_s(\eta)=\eta+(s+1-\eta)h(\eta-s)-(s+1+\eta)h(-\eta-s),\ \eta\in\mathbb{R}.$$

We have $\{h_s\}\subset C^\infty(\mathbb{R})$ and for every $s\in\mathbb{N}$ the following properties hold:

$$h_s(\eta)=\eta\ \text{ if }\ |\eta|\leqslant s,\quad h_s(\eta)=(s+1)\operatorname{sign}\eta\ \text{ if }\ |\eta|\geqslant s+1.$$

Moreover, for every $s \in \mathbb{N}$ and $\eta \in \mathbb{R}$ we have

$$0 \leqslant h_s'(\eta) \leqslant c', \quad |h_s''(\eta)| \leqslant c''.$$

Lemma 2.2. *Let* $q_1 \in (q, \tilde{q})$, $\tau > \tilde{q}/(\tilde{q} - q_1)$ *and* $\psi \in L^\tau(\Omega)$, $\psi \geqslant 0$ *in* Ω. *Let* Ω_0 *be a nonempty open set of* $\mathbb{R}^n$, $\Omega_0 \subset \Omega$, *and the restriction of the function* $\nu^{q_1/(q_1-q)}$ *on* Ω_0 *belongs to* $L^\tau(\Omega_0)$. *Let* $\varphi \in C_0^\infty(\Omega)$, $0 \leqslant \varphi \leqslant 1$ *in* Ω, $\operatorname{meas}\{\varphi = 1\} > 0$ *and* $\operatorname{supp}\varphi \subset \Omega_0$. *Let* $m_0, m_1, m_2 > 0, q_0 \in (q_1, \tilde{q}), u \in \overset{\circ}{W}{}^{1,q}(\nu, \Omega)$,

$$\int_\Omega |u|^{q_0} dx \leqslant m_0,$$

and let for every $s \in \mathbb{N}$, $r > 0$ *and* $t > q$ *the next inequality holds:*

$$\int_\Omega \left\{ \sum_{|\alpha|=1} \nu |D^\alpha u|^q \right\} [1 + h_s^2(u)]^r \varphi^t dx \leqslant m_1 (r+t)^{m_2} \int_\Omega [\psi + |u|^{q_1}] [1 + h_s^2(u)]^r \varphi^{t-q} dx.$$

Then

$$\operatorname{vrai}\max_{\{\varphi=1\}} |u| \leqslant M,$$

where M is a positive constant depending only on n, c, c', q, $\tilde{q}$, q_1, q_0, τ, m_0, m_1, m_2, $\operatorname{meas}\Omega$, $\max_\Omega |\nabla\varphi|$, $\|\psi\|_{L^\tau(\Omega)}$ *and the norm of the restriction of the function* $\nu^{q_1/(q_1-q)}$ *on* Ω_0 *in* $L^\tau(\Omega_0)$.

This lemma was proved in [7].

We set

$$\mu_1 = \mu^{q/(q-4)} (1/\nu)^{4/(q-4)}.$$

Lemma 2.3. *Let* $\mu_1 \in L^1(\Omega)$, $u \in \overset{\circ}{W}{}^{1,q}_{2,2}(\nu, \mu, \Omega)$ *and* $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$ *in* Ω. *Let* $q_1 \in (q, \tilde{q})$, $s \in \mathbb{N}$, $r > 0$ *and* $t > 2$. *Let* $\nu\varphi^{t-2} \in L^{q_1/(q_1-q)}(\Omega)$ *and*

$$w = u[1 + h_s^2(u)]^r \varphi^t,$$
$$z = [1 + h_s^2(u)]^r + 2r[1 + h_s^2(u)]^{r-1} h_s(u) h_s'(u) u.$$

Then $w \in \overset{\circ}{W}{}^{1,q}_{2,2}(\nu, \mu, \Omega)$ *and the following properties hold:*

(i) *for every n-dimensional multiindex* $\alpha, |\alpha| = 1$,

$$D^\alpha w = z\varphi^t D^\alpha u + tu[1 + h_s^2(u)]^r \varphi^{t-1} D^\alpha \varphi \quad \text{a.e. in } \Omega;$$

(ii) *for every n-dimensional multiindex* $\alpha, |\alpha| = 2,$

$$|D^\alpha w - z\varphi^t D^\alpha u| \leqslant 20(c'')^2(t+r)^2[1+h_s^2(u)]^r \varphi^t \left\{ \sum_{|\beta|=1} |D^\beta u|^2 \right\}$$

$$+4c'(t+r)^2(1+|u|)[1+h_s^2(u)]^r \left\{ \varphi^{t-1}|D^\alpha \varphi| + \varphi^{t-2} \sum_{|\beta|=1} |D^\beta \varphi|^2 \right\} \ \textit{a.e. in}\ \Omega.$$

This result is a particular case of Lemma 3.4 given in [7].

Along with Lemmas 2.2 and 2.3 in the proof of our main theorem the following result will be used.

Lemma 2.4. *Let* $\mu_1 \in L^1(\Omega)$, $u \in \overset{\circ}{W}{}^{1,q}_{2,2}(\nu,\mu,\Omega)$, *and let* $\chi \in C^2(\mathbb{R})$ *be the function such that* $\chi(0) = 0$ *and* χ, χ', χ'' *are bounded in* $\mathbb{R}$. *Then* $\chi(u) \in \overset{\circ}{W}{}^{1,q}_{2,2}(\nu,\mu,\Omega)$ *and the next properties hold:*

(i) *for every n-dimensional multiindex* $\alpha, |\alpha| = 1,$

$$D^\alpha \chi(u) = \chi'(u) D^\alpha u \quad \textit{a.e. in}\ \Omega;$$

(ii) *for every n-dimensional multiindex* $\alpha, |\alpha| = 2,$

$$|D^\alpha \chi(u) - \chi'(u) D^\alpha u| \leqslant |\chi''(u)| \left\{ \sum_{|\beta|=1} |D^\beta u|^2 \right\} \quad \textit{a.e. in}\ \Omega.$$

Like the previous lemma this result is established with the use of smooth approximations of functions in $\overset{\circ}{W}{}^{1,q}_{2,2}(\nu,\mu,\Omega)$.

3. Statement of the problem and some results on its solvability

We will use the following notation: Λ is the set of all n-dimensional multiindices α such that $|\alpha| = 1$ or $|\alpha| = 2$; $\mathbb{R}^{n,2}$ is the space of all sets $\xi = \{\xi_\alpha : \alpha \in \Lambda\}$ of real numbers; if a function $u \in L^1_{\text{loc}}(\Omega)$ has the weak derivatives $D^\alpha u$, $\alpha \in \Lambda$, then $\nabla_2 u = \{D^\alpha u : \alpha \in \Lambda\}$.

Let $c_1, c_2, c_3 > 0$, $g \in L^1(\Omega)$, $g \geqslant 0$ in Ω, and let for every $\alpha \in \Lambda$, $A_\alpha : \Omega \times \mathbb{R}^{n,2} \to \mathbb{R}$ be a Carathéodory function. We will suppose that for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^{n,2}$,

$$\sum_{|\alpha|=1} [\nu(x)]^{-1/(q-1)} |A_\alpha(x,\xi)|^{q/(q-1)} + \sum_{|\alpha|=2} [\mu(x)]^{-1} |A_\alpha(x,\xi)|^2$$

$$\leqslant c_1 \left\{ \sum_{|\alpha|=1} \nu(x)|\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x)|\xi_\alpha|^2 \right\} + g(x). \quad (3.1)$$

Moreover, we will assume that for almost every $x \in \Omega$ and every $\xi, \xi' \in \mathbb{R}^{n,2}$ the next inequalities hold:

$$\sum_{\alpha\in\Lambda} [A_\alpha(x,\xi) - A_\alpha(x,\xi')]\,(\xi_\alpha - \xi'_\alpha)$$
$$\geqslant c_2\Big\{\sum_{|\alpha|=1} \nu(x)|\xi_\alpha - \xi'_\alpha|^q + \sum_{|\alpha|=2} \mu(x)|\xi_\alpha - \xi'_\alpha|^2\Big\}, \quad (3.2)$$

$$\sum_{|\alpha|=2} |A_\alpha(x,\xi) - A_\alpha(x,\xi')|$$
$$\leqslant c_3\Big\{\sum_{|\alpha|=1} \nu(x)|\xi_\alpha - \xi'_\alpha| + \sum_{|\alpha|=2} \mu(x)|\xi_\alpha - \xi'_\alpha|\Big\}. \quad (3.3)$$

We note that by virtue of (3.1) and (3.2) for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^{n,2}$,

$$\sum_{\alpha\in\Lambda} A_\alpha(x,\xi)\xi_\alpha$$
$$\geqslant \frac{c_2}{2}\Big\{\sum_{|\alpha|=1} \nu(x)|\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x)|\xi_\alpha|^2\Big\} - \Big(\frac{2}{c_2}+1\Big) g(x). \quad (3.4)$$

Besides, from (3.2) it follows that for almost every $x \in \Omega$ and every $\xi, \xi' \in \mathbb{R}^{n,2}, \xi \neq \xi'$,

$$\sum_{\alpha\in\Lambda} [A_\alpha(x,\xi) - A_\alpha(x,\xi')]\,(\xi_\alpha - \xi'_\alpha) > 0. \quad (3.5)$$

Let $a > 0$, $\sigma > 1$ and $f \in L^1(\Omega)$. We consider the following Dirichlet problem:

$$\sum_{\alpha\in\Lambda} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \nabla_2 u) + a|u|^{\sigma-1}u = f \ \text{ in } \ \Omega, \quad (3.6)$$

$$D^\alpha u = 0, \ \ |\alpha| = 0,1, \ \text{ on } \ \partial\Omega. \quad (3.7)$$

Definition 3.1. A W-solution of problem (3.6), (3.7) is a function $u \in \overset{\circ}{W}{}^{2,1}(\Omega) \cap L^\sigma(\Omega)$ such that:

(i) for every $\alpha \in \Lambda$, $A_\alpha(x, \nabla_2 u) \in L^1(\Omega)$;

(ii) for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \Big\{\sum_{\alpha\in\Lambda} A_\alpha(x,\nabla_2 u) D^\alpha \varphi\Big\}\, dx + a\int_\Omega |u|^{\sigma-1} u\varphi\, dx = \int_\Omega f\varphi\, dx.$$

Theorem 3.2. *Suppose that* $\mu_1 \in L^1(\Omega)$ *and the following conditions are satisfied:*

(i) *there exists a real number* $r_1 > \tilde{q}/(\tilde{q}-q)$ *such that* $\nu \in L^{r_1}(\Omega)$*;*

(ii) *there exists a real number* $r_2 > \tilde{q}(q-1)/[\tilde{q}(q-1)-q]$ *such that* $1/\mu \in L^{r_2}(\Omega)$.

Then there exists a W-solution of problem (3.6), (3.7).

We obtain this result taking into account inequalities (3.1), (3.4), (3.5) and applying Theorem 3.1 of [5].

The next result is a consequence of Theorem 5.1 of [7].

Theorem 3.3. *Suppose that all conditions of Theorem* 3.2 *are satisfied and* $q < \tilde{q}(q-1)/q$. *Let* $q_1 \in (q, \tilde{q}(q-1)/q)$, $\tau > \tilde{q}/(\tilde{q}-q_1)$ *and let* Ω_1 *be a nonempty open set of* $\mathbb{R}^n$ *such that* $\Omega_1 \subset \Omega$. *Let for every nonempty closed set G of* $\mathbb{R}^n$ *such that* $G \subset \Omega_1$ *the restrictions of the functions* $\nu^{q_1/(q_1-q)}$, μ_1, g *and* $|f|^{q_1/(q_1-1)}$ *on G belong to* $L^\tau(G)$. *Then there exists a W-solution u of problem* (3.6), (3.7) *such that for every closed set G of* $\mathbb{R}^n$ *with the properties* $G \subset \Omega_1$ *and* $\operatorname{meas} G > 0$ *we have* $\operatorname{vrai}\max_G |u| < +\infty$.

As we shall see in the next section the restricted requirement $q < \tilde{q}(q-1)/q$ in the given theorem can be omitted thanks to an appropriate condition on σ. Naturally, it will imply a change of the inclusion for q_1.

4. Main result

Theorem 4.1. *Suppose that all conditions of Theorem* 3.2 *are satisfied and* $\sigma > q$. *Let* $q < q_1 < \min(\sigma, \tilde{q})$, $\tau > \tilde{q}/(\tilde{q}-q_1)$ *and let* Ω_1 *be a nonempty open set of* $\mathbb{R}^n$ *such that* $\Omega_1 \subset \Omega$. *Let for every nonempty closed set G of* $\mathbb{R}^n$ *such that* $G \subset \Omega_1$ *the restrictions of the functions* $\nu^{q_1/(q_1-q)}$, μ_1, g *and* $|f|^{q_1/(q_1-1)}$ *on G belong to* $L^\tau(G)$. *Then there exists a W-solution u of problem* (3.6), (3.7) *such that for every closed set G of* $\mathbb{R}^n$ *with the properties* $G \subset \Omega_1$ *and* $\operatorname{meas} G > 0$ *we have* $\operatorname{vrai}\max_G |u| < +\infty$.

Proof. We make the proof in some steps.

Step 1. Let us show that for every function $\psi \in L^\infty(\Omega)$ there exists a function $\bar{\psi} \in \overset{\circ}{W}{}^{1,q}_{2,2}(\nu,\mu,\Omega) \cap L^\sigma(\Omega)$ such that for every $\varphi \in \overset{\circ}{W}{}^{1,q}_{2,2}(\nu,\mu,\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 \bar{\psi}) D^\alpha \varphi \Big\} dx + a \int_\Omega |\bar{\psi}|^{\sigma-1} \bar{\psi} \varphi \, dx = \int_\Omega \psi \varphi \, dx. \qquad (4.1)$$

Let $\{\chi_k\} \subset C^2(\mathbb{R})$ be a sequence of functions such that for every $k \in \mathbb{N}$,

$$\chi_k(s) = s \text{ if } |s| \leqslant k, \tag{4.2}$$

$$|\chi_k| \leqslant 3k \text{ in } \mathbb{R}, \tag{4.3}$$

$$0 < \chi_k' \leqslant 1 \text{ in } \mathbb{R}, \tag{4.4}$$

$$|\chi_k''| \leqslant \frac{8}{k}\, \chi_k' \text{ in } \mathbb{R}. \tag{4.5}$$

As far as the construction of such a sequence of functions is concerned see [2].

Let $\psi \in L^\infty(\Omega)$. From the proof of Theorem 3.4 of [6] it follows that there exists a function $\bar{\psi} \in \overset{\circ}{W}{}^{2,1}(\Omega) \cap L^\sigma(\Omega)$ such that the following properties hold:

($*_1$) for every $\alpha \in \Lambda$, $A_\alpha(x, \nabla_2 \bar{\psi}) \in L^1(\Omega)$;

($*_2$) for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \Big\{ \sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 \bar{\psi}) D^\alpha \varphi \Big\} dx + a \int_\Omega |\bar{\psi}|^{\sigma - 1} \bar{\psi} \varphi \, dx = \int_\Omega \psi \varphi \, dx;$$

($*_3$) for every $k \in \mathbb{N}$, $\chi_k(\bar{\psi}) \in \overset{\circ}{W}{}^{1,q}_{2,2}(\nu, \mu, \Omega)$ and

$$\int_\Omega \Big\{ \sum_{|\alpha|=1} \nu |D^\alpha \chi_k(\bar{\psi})|^q + \sum_{|\alpha|=2} \mu |D^\alpha \chi_k(\bar{\psi})|^2 \Big\} dx \leqslant c_4 \int_\Omega \psi \chi_k(\bar{\psi}) dx + c_5,$$

where c_4 and c_5 are positive constants depending only on n, q, c_1, c_2 and the norms in $L^1(\Omega)$ of the functions g and μ_1.

Since $\psi \in L^\infty(\Omega)$, $\bar{\psi} \in L^\sigma(\Omega)$ and $\sigma > q$, from property ($*_3$) we deduce that the sequence $\{\chi_k(\bar{\psi})\}$ is bounded in $W^{1,q}_{2,2}(\nu, \mu, \Omega)$. Hence taking into account that $\chi_k(\bar{\psi}) \to \bar{\psi}$ strongly in $L^q(\Omega)$, we obtain $\bar{\psi} \in \overset{\circ}{W}{}^{1,q}_{2,2}(\nu, \mu, \Omega)$.

Let $\varphi \in \overset{\circ}{W}{}^{1,q}_{2,2}(\nu, \mu, \Omega) \cap L^\infty(\Omega)$. We take a sequence $\{\varphi_j\} \subset C_0^\infty(\Omega)$ bounded in $L^\infty(\Omega)$ and such that $\|\varphi_j - \varphi\| \to 0$ and $\varphi_j \to \varphi$ a.e. in Ω. Then by property ($*_2$) for every $j \in \mathbb{N}$ we have

$$\int_\Omega \Big\{ \sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 \bar{\psi}) D^\alpha \varphi_j \Big\} dx + a \int_\Omega |\bar{\psi}|^{\sigma - 1} \bar{\psi} \varphi_j \, dx = \int_\Omega \psi \varphi_j \, dx.$$

Hence passing to the limit as $j \to \infty$, we get (4.1). This completes the first step.

Step 2. For every $l \in \mathbb{N}$ we define the function $f_l : \Omega \to \mathbb{R}$ by

$$f_l(x) = \begin{cases} f(x) & \text{if } |f(x)| \leqslant l, \\ 0 & \text{if } |f(x)| > l. \end{cases}$$

By the assertion established at the first step we have: if $l \in \mathbb{N}$, there exists a function $u_l \in \overset{\circ}{W}{}^{1,q}_{2,2}(\nu,\mu,\Omega) \cap L^{\sigma}(\Omega)$ such that for every $\varphi \in \overset{\circ}{W}{}^{1,q}_{2,2}(\nu,\mu,\Omega) \cap L^{\infty}(\Omega)$,

$$\int_{\Omega} \Big\{ \sum_{\alpha\in\Lambda} A_{\alpha}(x,\nabla_2 u_l) D^{\alpha}\varphi \Big\} dx + a\int_{\Omega} |u_l|^{\sigma-1} u_l \varphi\, dx = \int_{\Omega} f_l \varphi\, dx. \tag{4.6}$$

We fix $k, l \in \mathbb{N}$. Due to (4.2)–(4.5) and Lemma 2.4 $\chi_k(u_l) \in \overset{\circ}{W}{}^{1,q}_{2,2}(\nu,\mu,\Omega) \cap L^{\infty}(\Omega)$. Then by (4.6)

$$\int_{\Omega} \Big\{ \sum_{\alpha\in\Lambda} A_{\alpha}(x,\nabla_2 u_l) D^{\alpha}\chi_k(u_l) \Big\} dx + a\int_{\Omega} |u_l|^{\sigma-1} u_l \chi_k(u_l)\, dx = \int_{\Omega} f_l \chi_k(u_l)\, dx.$$

Hence taking into account property (i) of Lemma 2.4, we get

$$\int_{\Omega} \Big\{ \sum_{\alpha\in\Lambda} A_{\alpha}(x,\nabla_2 u_l) D^{\alpha}u_l \Big\} \chi_k'(u_l)\, dx + a\int_{\Omega} |u_l|^{\sigma-1} u_l \chi_k(u_l)\, dx \leqslant \int_{\Omega} f_l \chi_k(u_l)\, dx + J_l, \tag{4.7}$$

where

$$J_l = \int_{\Omega} \Big\{ \sum_{|\alpha|=2} |A_{\alpha}(x,\nabla_2 u_l)|\, |D^{\alpha}\chi_k(u_l) - \chi_k'(u_l) D^{\alpha}u_l| \Big\} dx.$$

Using (3.4) and (4.4), we obtain the inequality

$$\frac{c_2}{2}\int_{\Omega} \Big\{ \sum_{|\alpha|=1} \nu|D^{\alpha}u_l|^q + \sum_{|\alpha|=2} \mu|D^{\alpha}u_l|^2 \Big\} \chi_k'(u_l)\, dx \leqslant \int_{\Omega} \Big\{ \sum_{\alpha\in\Lambda} A_{\alpha}(x,\nabla_2 u_l) D^{\alpha}u_l \Big\} \chi_k'(u_l)\, dx + \Big(\frac{2}{c_2}+1\Big)\int_{\Omega} g\, dx, \tag{4.8}$$

and using property (ii) of Lemma 2.4, (3.1), (4.4) and (4.5), we establish that

$$J_l \leqslant \frac{c_2}{4}\int_{\Omega} \Big\{ \sum_{|\alpha|=1} \nu|D^{\alpha}u_l|^q + \sum_{|\alpha|=2} \mu|D^{\alpha}u_l|^2 \Big\} \chi_k'(u_l)\, dx + \frac{c_2}{4}\int_{\Omega} g\, dx + 8n^2\,[32n^q(c_1+1)/c_2]^{(q+4)/(q-4)} \int_{\Omega} \mu_1\, dx. \tag{4.9}$$

We note that the latter estimate is derived in the same way as an analogous estimate in the proof of Theorem 3.4 of [6].

Setting

$$c_6 = c_2 \int_\Omega g\,dx + 8n^2\,[32n^q(c_1+1)/c_2]^{(q+4)/(q-4)} \int_\Omega \mu_1\,dx,$$

from (4.7)–(4.9) we deduce that

$$\frac{c_2}{4}\int_\Omega \Big\{ \sum_{|\alpha|=1} \nu|D^\alpha u_l|^q + \sum_{|\alpha|=2} \mu|D^\alpha u_l|^2 \Big\} \chi_k'(u_l)\,dx + a\int_\Omega |u_l|^{\sigma-1}u_l\chi_k(u_l)\,dx$$
$$\leqslant \int_\Omega f_l\chi_k(u_l)\,dx + c_6.$$

Hence taking into account (4.2)–(4.4) and the definition of the function f_l, we get

$$a\int_\Omega |u_l|^\sigma|\chi_k(u_l)|\,dx \leqslant 3k\int_\Omega |f|\,dx + c_6, \tag{4.10}$$

$$\frac{c_2}{4}\int_\Omega \Big\{ \sum_{|\alpha|=1} \nu|D^\alpha u_l|^q + \sum_{|\alpha|=2} \mu|D^\alpha u_l|^2 \Big\} \chi_k'(u_l)\,dx \leqslant 3k\int_\Omega |f|\,dx + c_6. \tag{4.11}$$

Since by (4.2) and (4.4) for every $s \in \mathbb{R}, |s| \geqslant k$, the inequality $|\chi_k(s)| \geqslant k$ holds, from (4.10) we obtain

$$a\int_{\{|u_l|\geqslant k\}} |u_l|^\sigma dx \leqslant 3\int_\Omega |f|\,dx + c_6. \tag{4.12}$$

This completes the second step.

Step 3. From (4.12) we obtain that there exists $c_7 > 0$ such that for every $l \in \mathbb{N}$,

$$\int_\Omega |u_l|^\sigma dx \leqslant c_7. \tag{4.13}$$

Now we fix an arbitrary multiindex α, $|\alpha| = 1$, and $l \in \mathbb{N}$. Let $k \in \mathbb{N}$, $k_1 \in \mathbb{N}$ and

$$k^{q/(1+\sigma)} < k_1 \leqslant k^{q/(1+\sigma)} + 1. \tag{4.14}$$

We have

$$\operatorname{meas}\{\nu^{1/q}|D^\alpha u_l| \geqslant k\} \leqslant \operatorname{meas}\{|u_l| \geqslant k_1\} + \operatorname{meas} G, \tag{4.15}$$

where $G = \{|u_l| < k_1,\ \nu^{1/q}|D^\alpha u_l| \geqslant k\}$. By (4.13)

$$\operatorname{meas}\{|u_l| \geqslant k_1\} \leqslant c_7 k_1^{-\sigma}, \tag{4.16}$$

and using (4.11), (4.2) and (4.4), for the measure of the set G we obtain the following estimate:

$$\operatorname{meas} G \leqslant \frac{4}{c_2}\left\{3\int_\Omega |f|dx + c_6\right\}k_1 k^{-q}. \tag{4.17}$$

From (4.14)–(4.17) we get that for every multiindex α, $|\alpha| = 1$, and $k, l \in \mathbb{N}$,

$$\operatorname{meas}\{\nu^{1/q}|D^\alpha u_l| \geqslant k\} \leqslant c_8 k^{-q\sigma/(1+\sigma)}, \tag{4.18}$$

where c_8 is a positive constant depending only on c_2, c_6, c_7 and the norm of the function f in $L^1(\Omega)$.

Due to (4.18) and Lemma 2.6 of [2] we have

$$\begin{aligned}&\text{if } \alpha \text{ is a multiindex and } |\alpha| = 1, \text{for every } \lambda \in (0, q\sigma/(1+\sigma))\\ &\text{the sequence} \{\nu^{1/q}D^\alpha u_l\} \text{ is bounded in } L^\lambda(\Omega).\end{aligned} \tag{4.19}$$

Next, it is clear that there exists a number $t_1 > 1$ such that

$$\frac{1}{2} - \frac{1}{q} - \frac{q}{4\tilde{q}(q-1)} < \frac{1}{t_1} < \frac{q-1}{q} - \frac{1}{\sigma q}. \tag{4.20}$$

Define

$$t_2 = \frac{qt_1}{4q-(q-4)t_1}.$$

From (4.20) it follows that

$$t_2 < \frac{\tilde{q}(q-1)}{\tilde{q}(q-1)-q}, \qquad \frac{t_1}{t_1-1} < \frac{q\sigma}{\sigma+1}. \tag{4.21}$$

The first of this inequalities and condition (ii) of Theorem 3.2 imply that

$$\frac{1}{\mu} \in L^{t_2}(\Omega). \tag{4.22}$$

Using the definition of the function μ_1 and Young's inequality, we obtain

$$\left(\frac{1}{\nu}\right)^{t_1/q} = \mu_1^{(q-4)t_1/4q} \cdot \left(\frac{1}{\mu}\right)^{t_1/4} \leqslant \mu_1 + \left(\frac{1}{\mu}\right)^{t_2}.$$

Hence, taking into account the inclusion $\mu_1 \in L^1(\Omega)$ and (4.22), we deduce that

$$\frac{1}{\nu} \in L^{t_1/q}(\Omega). \tag{4.23}$$

This with (4.19) and the second inequality in (4.21) implies that for every multiindex α, $|\alpha| = 1$, the sequence $\{D^\alpha u_l\}$ is bounded in $L^1(\Omega)$. In its turn, the latter fact and (4.13) allow us to conclude that the sequence $\{u_l\}$ is bounded in $\overset{\circ}{W}{}^{1,1}(\Omega)$. Therefore, there exist an increasing sequence $\{l_i\} \subset \mathbb{N}$ and a functuion $u \in L^1(\Omega)$ such that

$$\|u_{l_i} - u\|_{L^1(\Omega)} \to 0, \tag{4.24}$$

$$u_{l_i} \to u \text{ a.e. in } \Omega. \tag{4.25}$$

Step 4. Let $l, m \in \mathbb{N}$. By Lemma 2.4 we have $\chi_1(u_l - u_m) \in \overset{\circ}{W}{}^{1,q}_{2,2}(\nu, \mu, \Omega)$ and the following properties hold:

($*_4$) if $|\alpha| = 1$, $D^\alpha \chi_1(u_l - u_m) = \chi_1'(u_l - u_m) D^\alpha (u_l - u_m)$ a.e. in Ω;

($*_5$) if $|\alpha| = 2$, $|D^\alpha \chi_1(u_l - u_m) - \chi_1'(u_l - u_m) D^\alpha (u_l - u_m)|$

$$\leqslant |\chi_1''(u_l - u_m)| \sum_{|\beta|=1} |D^\beta (u_l - u_m)|^2 \text{ a.e. in } \Omega.$$

From (4.6) it follows that

$$\int_\Omega \left\{ \sum_{\alpha \in \Lambda} \left[A_\alpha(x, \nabla_2 u_l) - A_\alpha(x, \nabla_2 u_m)\right] D^\alpha \chi_1(u_l - u_m) \right\} dx \leqslant \int_\Omega |f_l - f_m| \, |\chi_1(u_l - u_m)| \, dx.$$

Hence, using inequalities (3.2), (3.3), properties (4.2)–(4.5), ($*_4$), ($*_5$) and making considerations analogous to those given in [4], we get

$$\int_{\{|u_l - u_m| \leqslant 1\}} \left\{ \sum_{|\alpha|=1} \nu |D^\alpha u_l - D^\alpha u_m|^q + \sum_{|\alpha|=2} \mu |D^\alpha u_l - D^\alpha u_m|^2 \right\} dx \leqslant \frac{6}{c_2} \int_\Omega |f_l - f_m| \, dx + c_9 \int_{\{|u_l - u_m| > 1\}} (\nu + \mu_1) \, dx, \tag{4.26}$$

where c_9 is a positive constant depending only on n, q, c_2 and c_3.

From (4.25), (4.26) and the strong convergence of $\{f_l\}$ to f in $L^1(\Omega)$ we derive that: if $|\alpha| = 1$, there exists a measurable function $v_\alpha : \Omega \to \mathbb{R}$ such that $\nu^{1/q} D^\alpha u_{l_i} \to v_\alpha$ in measure; if $|\alpha| = 2$, there exists a measurable function $w_\alpha : \Omega \to \mathbb{R}$ such that $\mu^{1/2} D^\alpha u_{l_i} \to w_\alpha$ in measure.

Clearly, without loss of generality we may assume that

$$\text{if } |\alpha| = 1, \ \nu^{1/q} D^\alpha u_{l_i} \to v_\alpha \ \text{ a.e. in } \Omega, \tag{4.27}$$

$$\text{if } |\alpha| = 2, \ \mu^{1/2} D^\alpha u_{l_i} \to w_\alpha \ \text{ a.e. in } \Omega. \tag{4.28}$$

Step 5. We set $\sigma_1 = \tilde{q}(q-1)/q$ and fix $k, l \in \mathbb{N}$. Since $\chi_k(u_l) \in \overset{\circ}{W}{}^{1,q}(\nu, \Omega)$, by virtue of Hypothesis 2.1, assertion (i) of Lemma 2.4, (4.4) and (4.11) we have

$$\int_\Omega |\chi_k(u_l)|^{\tilde{q}} dx \leqslant c_{10} k^{\tilde{q}/q},$$

where c_{10} is a positive constant depending only on q, $\tilde{q}$, c, c_2, c_6 and the norm of the function f in $L^1(\Omega)$. Then taking into account that $|\chi_k(s)| \geqslant k$ if $|s| \geqslant k$, we obtain

$$\operatorname{meas}\{|u_l| \geqslant k\} \leqslant c_{10} k^{-\sigma_1}.$$

Using this estimate, (4.11) and Lemma 2.6 of [2], by analogy with (4.19) we get

$$\begin{aligned} &\text{if } \alpha \text{ is a multiindex and } |\alpha| = 1, \text{for every } \lambda \in (0, q\sigma_1/(1+\sigma_1)) \\ &\text{the sequence } \{\nu^{1/q} D^\alpha u_l\} \text{ is bounded in } L^\lambda(\Omega); \end{aligned} \tag{4.29}$$

$$\begin{aligned} &\text{if } \alpha \text{ is a multiindex and } |\alpha| = 2, \text{for every } \lambda \in (0, 2\sigma_1/(1+\sigma_1)) \\ &\text{the sequence } \{\mu^{1/2} D^\alpha u_l\} \text{ is bounded in } L^\lambda(\Omega). \end{aligned} \tag{4.30}$$

Owing to (4.19), the second inequality in (4.21), (4.23), (4.24), (4.27), condition (ii) of Theorem 3.2, (4.28) and (4.30) for every $\alpha \in \Lambda$ there exists the weak derivative $D^\alpha u$, $D^\alpha u \in L^1(\Omega)$ and

$$D^\alpha u_{l_i} \to D^\alpha u \ \ \text{a.e. in } \Omega, \tag{4.31}$$

$$D^\alpha u_{l_i} \to D^\alpha u \ \ \text{strongly in } L^1(\Omega). \tag{4.32}$$

From (4.24) and (4.32) it follows that $u_{l_i} \to u$ strongly in $W^{2,1}(\Omega)$ and therefore, $u \in \overset{\circ}{W}{}^{2,1}(\Omega)$.

Thus, taking into account (4.13) and (4.25), we have

$$u \in \overset{\circ}{W}{}^{2,1}(\Omega) \cap L^\sigma(\Omega). \tag{4.33}$$

Moreover, using (3.1), (4.29)–(4.31) and condition (i) of Theorem 3.2, we establish that for every $\alpha \in \Lambda$,

$$A_\alpha(x, \nabla_2 u) \in L^1(\Omega), \tag{4.34}$$

$$A_\alpha(x, \nabla_2 u_{l_i}) \to A_\alpha(x, \nabla_2 u) \ \text{ strongly in } L^1(\Omega). \tag{4.35}$$

Step 6. Let $\varphi \in C_0^\infty(\Omega)$. By virtue of (4.6) for every $i \in \mathbb{N}$ we have

$$\int_\Omega \left\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u_{l_i}) D^\alpha \varphi \right\} dx + a \int_\Omega |u_{l_i}|^{\sigma-1} u_{l_i} \varphi \, dx = \int_\Omega f_{l_i} \varphi \, dx. \tag{4.36}$$

Due to (4.35)

$$\int_\Omega \left\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u_{l_i}) D^\alpha \varphi \right\} dx \to \int_\Omega \left\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u) D^\alpha \varphi \right\} dx, \tag{4.37}$$

and owing to the strong convergence of $\{f_l\}$ to f in $L^1(\Omega)$ we get

$$\int_\Omega f_{l_i} \varphi \, dx \to \int_\Omega f \varphi \, dx. \tag{4.38}$$

Let us show that

$$\int_\Omega |u_{l_i}|^{\sigma-1} u_{l_i} \varphi \, dx \to \int_\Omega |u|^{\sigma-1} u \varphi \, dx. \tag{4.39}$$

First of all, by analogy with Lemma 7.3 of [2] we establish that there exists s positive constant c_{11} such that for every $k, l \in \mathbb{N}$,

$$a \int_{\{|u_l| \geqslant 2k\}} |u_l|^\sigma dx \leqslant 2 \int_{\{|u_l|>k\}} |f| dx + c_{11} k^{-1}. \tag{4.40}$$

Now let us fix an arbitrary $\varepsilon > 0$. Clearly, there exists $\varepsilon_1 > 0$ such that for every measurable set G with $\operatorname{meas} G \leqslant \varepsilon_1$ we have

$$\int_G |f| dx \leqslant \varepsilon a/2. \tag{4.41}$$

We fix $k \in \mathbb{N}$ such that $k \geqslant \max\{(c_7/\varepsilon_1)^{1/\sigma}, c_{11}/\varepsilon a\}$. Then using (4.13), (4.40) and (4,41), we obtain that for every $l \in \mathbb{N}$,

$$\int_{\{|u_l| \geqslant 2k\}} |u_l|^\sigma dx \leqslant 2\varepsilon. \tag{4.42}$$

Owing to (4.25) and D. Egorov's theorem there exists a measurable set $\Omega' \subset \Omega$ such that

$$\int_{\Omega\setminus\Omega'} |u|^\sigma dx \leqslant \varepsilon, \tag{4.43}$$

$$\operatorname{meas}(\Omega \setminus \Omega') \leqslant \varepsilon/(2k)^\sigma, \tag{4.44}$$

$$|u_{l_i}|^{\sigma-1} u_{l_i} \to |u|^{\sigma-1} u \ \text{ uniformly in } \ \Omega'. \tag{4.45}$$

From (4.45) it follows that there exists $i_0 \in \mathbb{N}$ such that for every $i \in \mathbb{N}$, $i \geqslant i_0$,

$$\int_{\Omega'} \big||u_{l_i}|^{\sigma-1}u_{l_i} - |u|^{\sigma-1}u\big|dx \leqslant \varepsilon. \tag{4.46}$$

Let $i \in \mathbb{N}$, $i \geqslant i_0$. By (4.43) and (4.46)

$$\int_{\Omega} \big||u_{l_i}|^{\sigma-1}u_{l_i} - |u|^{\sigma-1}u\big|dx \leqslant \int_{\Omega\setminus\Omega'} |u_{l_i}|^{\sigma}dx + 2\varepsilon,$$

and using (4.42) and (4.44) we get

$$\begin{aligned}\int_{\Omega\setminus\Omega'} |u_{l_i}|^{\sigma}dx &= \int_{(\Omega\setminus\Omega')\cap\{|u_{l_i}|<2k\}} |u_{l_i}|^{\sigma}dx + \int_{(\Omega\setminus\Omega')\cap\{|u_{l_i}|\geqslant 2k\}} |u_{l_i}|^{\sigma}dx \\ &\leqslant (2k)^{\sigma}\operatorname{meas}(\Omega\setminus\Omega') + \int_{\{|u_{l_i}|\geqslant 2k\}} |u_{l_i}|^{\sigma}dx \leqslant 3\varepsilon.\end{aligned}$$

Therefore,

$$\int_{\Omega} \big||u_{l_i}|^{\sigma-1}u_{l_i} - |u|^{\sigma-1}u\big|dx \leqslant 5\varepsilon.$$

This allows us to conclude that assertion (4.39) holds.

Now from (4.36)–(4.39) we get

$$\int_{\Omega}\Big\{\sum_{\alpha\in\Lambda} A_{\alpha}(x,\nabla_2 u)D^{\alpha}\varphi\Big\}dx + a\int_{\Omega}|u|^{\sigma-1}u\varphi\,dx = \int_{\Omega} f\varphi\,dx.$$

This with (4.33) and (4.34) implies that the function u is a W-solution of problem (3.6), (3.7).

Step 7. It remains to show that for every closed set G of $\mathbb{R}^n$ with the properties $G \subset \Omega_1$ and $\operatorname{meas} G > 0$ we have $\operatorname{vrai}\max_G |u| < +\infty$.

Let G be a closed set of $\mathbb{R}^n$, $G \subset \Omega_1$ and $\operatorname{meas} G > 0$. We set $\rho = \operatorname{dist}(G, \partial\Omega_1)$ and $\Omega_0 = \{x \in \mathbb{R}^n : d(x,G) < \rho/2\}$. Evidently, $\overline{\Omega_0} \subset \Omega_1$. Let $\chi : \Omega \to \mathbb{R}$ be the characteristic function of the set Ω_0. We define

$$\psi = \big(\nu^{q_1/(q_1-q)} + \mu_1 + g + |f|^{q_1/(q_1-1)}\big)\chi\,.$$

By conditions of the theorem we have $\psi \in L^{\tau}(\Omega)$ and the restriction of the function $\nu^{q_1/(q_1-q)}$ on Ω_0 belongs to $L^{\tau}(\Omega_0)$.

We fix a function $\varphi \in C_0^{\infty}(\Omega)$ such that $0 \leqslant \varphi \leqslant 1$ in Ω, $\varphi = 1$ in G, $\operatorname{supp}\varphi \subset \Omega_0$ and set

$$m(\varphi) = 1 + \max_{\Omega}\Big\{\sum_{|\beta|=1}|D^{\beta}\varphi|^2 + \sum_{|\beta|=2}|D^{\beta}\varphi|\Big\}.$$

We also fix $l \in \mathbb{N}$ and set

$$\Phi_l = \sum_{|\alpha|=1} \nu |D^\alpha u_l|^q + \sum_{|\alpha|=2} \mu |D^\alpha u_l|^2.$$

Let $s \in \mathbb{N}$, $r > 0$ and $t > q$. Define

$$w_l = u_l[1 + h_s^2(u_l)]^r \varphi^t,$$
$$z_l = [1 + h_s^2(u_l)]^r + 2r[1 + h_s^2(u_l)]^{r-1} h_s(u_l) h_s'(u_l) u_l.$$

By virtue of Lemma 2.3 $w_l \in \overset{\circ}{W}{}^{1,q}_{2,2}(\nu, \mu, \Omega)$ and the following properties hold:

($*_6$) for every n-dimensional multiindex α, $|\alpha| = 1$,

$$D^\alpha w_l = z_l \varphi^t D^\alpha u_l + t u_l [1 + h_s^2(u_l)]^r \varphi^{t-1} D^\alpha \varphi \quad \text{a.e. in } \Omega;$$

($*_7$) for every n-dimensional multiindex α, $|\alpha| = 2$,

$$|D^\alpha w_l - z_l \varphi^t D^\alpha u_l| \leqslant 20(c'')^2 (t+r)^2 [1 + h_s^2(u_l)]^r \varphi^t \Big\{ \sum_{|\beta|=1} |D^\beta u_l|^2 \Big\}$$
$$+ 4c' m(\varphi)(t+r)^2 (1 + |u_l|)[1 + h_s^2(u_l)]^r \varphi^{t-2} \quad \text{a.e. in } \Omega.$$

Besides, due to Lemma 2.4 and (4.3) $\{\chi_k(w_l)\} \subset \overset{\circ}{W}{}^{1,q}_{2,2}(\nu, \mu, \Omega) \cap L^\infty(\Omega)$. Then by (4.6) for every $k \in \mathbb{N}$ we have

$$\int_\Omega \Big\{ \sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 u_l) D^\alpha \chi_k(w_l) \Big\} dx + a \int_\Omega |u_l|^{\sigma-1} u_l \chi_k(w_l)\, dx = \int_\Omega f_l \chi_k(w_l)\, dx.$$

From this, taking into account (4.2), (4.4) and the definition of the function w_l, we infer that for every $k \in \mathbb{N}$,

$$\int_\Omega \Big\{ \sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 u_l) D^\alpha \chi_k(w_l) \Big\} dx \leqslant \int_\Omega f_l \chi_k(w_l)\, dx.$$

Hence, taking into consideration Lemma 2.4 and properties of the functions χ_k, $k \in \mathbb{N}$, and passing to the limit as $k \to \infty$, we get

$$\int_\Omega \Big\{ \sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 u_l) D^\alpha w_l \Big\} dx \leqslant \int_\Omega f_l w_l\, dx. \tag{4.47}$$

We set

$$I_l = \int_\Omega \Big\{ \sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 u_l) [D^\alpha w_l - z_l \varphi^t D^\alpha u_l] \Big\} dx.$$

From (3.4) and (4.47) it follows that

$$\frac{c_2}{2}\int_\Omega \Phi_l[1+h_s^2(u_l)]^r\varphi^t\,dx \leqslant \int_\Omega f_l w_l\,dx - I_l + \Big(\frac{2}{c_2}+1\Big)(1+4c'r)\int_\Omega g\,[1+h_s^2(u_l)]^r\varphi^t\,dx. \quad (4.48)$$

Due to the definitions of the functions f_l, w_l and ψ we have

$$\int_\Omega f_l w_l\,dx \leqslant \int_\Omega [\psi+|u_l|^{q_1}][1+h_s^2(u_l)]^r\varphi^t\,dx, \quad (4.49)$$

$$\int_\Omega g\,[1+h_s^2(u_l)]^r\varphi^t\,dx \leqslant \int_\Omega \psi\,[1+h_s^2(u_l)]^r\varphi^t\,dx. \quad (4.50)$$

In order to estimate the integral I_l we use (3.1) and properties $(*_6)$ and $(*_7)$ and argue as in analogous situation in the proof of Theorem 5.1 of [7]. Then we obtain

$$|I_l| \leqslant \frac{c_2}{4}\int_\Omega \Phi_l\,[1+h_s^2(u_l)]^r\varphi^t\,dx + c_{12}(r+t)^{q^2/(q-4)}\int_\Omega [\psi+1+|u_l|^{q_1}]\,[1+h_s^2(u_l)]^r\varphi^{t-q}dx,$$

where c_{12} is a positive constant depending only on n, q, $\tilde{q}$, c'', c_1, c_2 and $m(\varphi)$.

The latter estimate and (4.48)–(4.50) imply that

$$\int_\Omega \Big\{\sum_{|\alpha|=1} \nu|D^\alpha u_l|^q\Big\}\,[1+h_s^2(u_l)]^r\varphi^t\,dx \leqslant c_{13}(r+t)^{q^2/(q-4)}\int_\Omega [\psi+1+|u_l|^{q_1}]\,[1+h_s^2(u_l)]^r\varphi^{t-q}\,dx,$$

where c_{13} is a positive constant depending only on the same parameters as the constant c_{12}.

Hence, taking into account (4.13) and applying Lemma 2.2, we deduce that there exists a positive constant $M>0$ such that for every $l\in\mathbb{N}$, $\operatorname{vrai}\max_G |u_l| \leqslant M$. This and (4.25) imply that $\operatorname{vrai}\max_G |u| \leqslant M$.

The theorem is proved.

From Theorem 4.1 we obtain the following consequence.

Corollary 4.2. *Suppose that all conditions of Theorem* 3.2 *are satisfied and* $\sigma > q$. *Let the functions* ν, μ_1, g *and* f *belong to* $L^\infty_{\rm loc}(\Omega)$. *Then there exists a* W*-solution* u *of problem* (3.6), (3.7) *such that* $u\in L^\infty_{\rm loc}(\Omega)$.

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ALEXANDER A. KOVALEVSKY
Department of Nonlinear Analysis
Institute of Applied Mathematics and Mechanics

Rosa Luxemburg St. 74
83114 Donetsk, Ukraine
e-mail: alexkvl@iamm.ac.donetsk.ua

FRANCESCO NICOLOSI
Dipartimento di Matematica e Informatica
Universita' degli Studi di Catania
Viale A.Doria 6
95125 Catania, Italia
e-mail: fnicolosi@dmi.unict.it