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SOME REMARKS ON LINEAR SPACES OF NILPOTENT MATRICES

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Introduction.

In this paper we study linear spaces of nilpotent matrices and we are mainly concerned with linear spaces of nilpotent matrices of generic maximal rank. We attack this problem using a modern formalism of vector bundles and cohomology, and show in (2.3) and (3.3) that a pencil of nilpotent matrices of order n and constant rank $n-1$ exists if and only if n is odd. We also show in (2.3) that there is no linear space of dimension greater than two of nilpotent matrices with constant maximal rank. In the case of a pencil of nilpotent matrices with generic maximal rank, we give an upper bound for the number of points where the rank drops. The paper contains also a number of relevant examples and a list of related questions.

We have been introduced to the general problem of studying linear spaces of nilpotent matrices by professors D. Eisenbud and S. Popescu who suggested us the problem during the Pragmatic summer school 1997 [1]. They have let us know the statement of the nonexistence of pencils of odd order mentioned above, and they have also informed us that a proof of this result was already known to them. We have been let free to develop our own independent proof and publish it as a part of this paper. It is a pleasure to express our gratitude to D. Eisenbud and S.Popescu. We also heartily thank the Geometry group of Catania and EUROPROJ for the successful organization of the event of PRAGMATIC.

1. Preliminary remarks.

A matrix over an algebraically closed field of characteristics zero, denoted for convenience K, is nilpotent if and only if all its eigenvalues are zero. The set of nilpotent matrices of order n over K is an irreducible complete intersection in $gl(n, K)$ defined by the ideal generated by $\operatorname{tr}(\wedge^i A)$, $1 \leq i \leq n$. The most immediate example of a nontrivial linear space of nilpotent matrices is the set of strictly upper triangular matrices:

$$\begin{pmatrix} 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

The role of this set is manifest in the results of M. Gerstenhaber [2], [3] which we summarize in the following two statements.

Theorem 1.1. ([2]). *If V is a linear space of nilpotent matrices of order n, then* $\dim V \leq n(n-1)/2$ *and equality holds iff V is similar to the space of strictly upper triangular matrices.*

Theorem 1.2. ([3]). *If V is a linear space of nilpotent matrices such that every $A \in V$ has rank at most ρ, then*

(1) $\dim V \leq \dfrac{n(n-1)}{2} - \dfrac{(n-\rho)(n-\rho-1)}{2}$.

(2) *Moreover if equality holds then the classification of such V is known. As a consequence of this classification every such V is similar to a subspace of the strictly upper triangular matrices.*

In general it is not true that every linear space V of nilpotent matrices is similar to some subspace of upper triangular ones. This is shown by the following proposition and examples.

Proposition 1.3. *No linear space V of dimension greater than two of nilpotent matrices of order $n \geq 3$ and constant rank equal to $n-1$ is similar to a subspace of strictly upper triangular matrices.*

Proof. Suppose the contrary. By counting the dimensions, V must have nontrivial intersection with the codimension one subspace of the matrices of the form:

$$\begin{pmatrix} 0 & 0 & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

This will be a contradiction because all the matrices of the above form have rank at most $n-2$. $\square$

The following are examples of pencils of nilpotent matrices with constant maximal ranks of order 3, 5, 5 and 7 respectively, as an easy computation can show:

$$\begin{pmatrix} 0 & s & 0 \\ -t & 0 & s \\ 0 & t & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & s & 0 & 0 & 0 \\ 2t & 0 & s & 0 & 0 \\ 0 & -t & 0 & s & 0 \\ 0 & 0 & t & 0 & s \\ 0 & 0 & 0 & -2t & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & -t & 0 & 0 \\ 0 & 0 & s & 0 & -t \\ 0 & 0 & 0 & 0 & s \\ s & t & 0 & 0 & 0 \\ t & 0 & 0 & -s & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & s & 0 & 0 & 0 & 0 & 0 \\ -4t & 0 & s & 0 & 0 & 0 & 0 \\ 0 & 2t & 0 & s & 0 & 0 & 0 \\ 0 & 0 & -t & 0 & s & 0 & 0 \\ 0 & 0 & 0 & t & 0 & s & 0 \\ 0 & 0 & 0 & 0 & -2t & 0 & s \\ 0 & 0 & 0 & 0 & 0 & 4t & 0 \end{pmatrix}$$

One can easily show that if a pencil of nilpotent matrices of order $n \geq 2$ has the block form

$$N(s,t) = \begin{pmatrix} A(s,t) & C(s,t) \\ 0 & B(s,t) \end{pmatrix}$$

then there exists always an $(s_0,\ t_0) \neq (0,0)$ such that $\operatorname{rk} N(s_0,\ t_0) \leq n-2$, so it cannot have constant rank equal to $n-1$. In other words, the construction of pencils of nilpotent matrices of order n with maximal constant rank $n-1$, for arbitrarily large n, cannot be reduced to assembling blocks of nilpotent matrices of lower orders.

2. Nonexistence results.

We begin by restating the problem using the terminology of vector bundles. In the following any linear space of dimension $r+1$ of matrices of order n will be interpreted as a morphism of sheaves on $\mathbb{P}^r$, $\mathcal{O}^n_{\mathbb{P}^r} \xrightarrow{A} \mathcal{O}^n_{\mathbb{P}^r}(1)$. In this formalism a linear space of dimension $r+1$ of nilpotent matrices of order n of constant maximal rank is just a morphism of vector bundles $A : \mathcal{O}^n_{\mathbb{P}^r} \longrightarrow \mathcal{O}^n_{\mathbb{P}^r}(1)$ of (constant) rank $n-1$ such that $A^n = 0$, where A^n means successive composition of the corresponding twists of A. We will use this abuse of notation from now on.

Lemma 2.1. *If A is a nilpotent endomorphism of K^n of rank $n-1$ then*

(1) $\mathrm{Ker}(A^i)$ *is a subspace of dimension i, for every $1 \le i \le n$;*
(2) $\{0\} \subset \mathrm{Ker}(A) \subset \cdots \subset \mathrm{Ker}(A^n) = K^n$ *is a filtration of K^n;*
(3) *for every $1 \le i \le n-1$ we have the following exact sequences*

$$0 \longrightarrow \mathrm{Ker}(A) \longrightarrow \mathrm{Ker}(A^{i+1}) \xrightarrow{A} \mathrm{Ker}(A^i) \longrightarrow 0.$$

We have the following reformulation of the Lemma 2.1.

Lemma 2.2. *If $A : \mathcal{O}^n_{\mathbb{P}^r} \longrightarrow \mathcal{O}^n_{\mathbb{P}^r}(1)$ is a nilpotent morphism of vector bundles, then*

(1) $\mathrm{Ker}(A^i)$ *is a vector bundle of rank i, for every $1 \le i \le n$;*
(2) $\{0\} \subset \mathrm{Ker}(A) \subset \cdots \subset \mathrm{Ker}(A^n) = \mathcal{O}^n_{\mathbb{P}^r}$ *is a filtration of $\mathcal{O}^n_{\mathbb{P}^r}$;*
(3) *for every $1 \le i \le n-1$ we have the following exact sequences of vector bundles*

$$0 \longrightarrow \mathrm{Ker}(A) \longrightarrow \mathrm{Ker}(A^{i+1}) \xrightarrow{A} \mathrm{Ker}(A^i)(1) \longrightarrow 0.$$

Theorem 2.3.

(1) *If A is a linear space of nilpotent matrices of order n with maximal constant rank $n-1$, then n must be odd.*
(2) *There are no linear spaces of dimension greater or equal than three of nilpotent matrices with constant maximal rank $n-1$.*

Proof. (1) Counting the degree in the exact sequence given in the Lemma 2.2. we obtain $\deg(\mathrm{Ker}(A^{i+1})) = \deg(\mathrm{Ker}(A^i)) + \deg(\mathrm{Ker}(A)) + i$, for all $1 \leq i \leq n-1$. Summing from $i = 1$ to $n-1$, we obtain that $\deg(\mathrm{Ker}(A)) = (1-n)/2$, which implies the claim.
(2) From the first part n is odd. Using the well-known fact that $H^1(\mathcal{O}_{\mathbb{P}^r}(k)) = 0$ we obtain that all the exact sequences of the Lemma 2.2. are split and so $\mathrm{Ker}(A^i)$ is direct sum of line bundles. In particular $\mathrm{Ker}(A) = \mathcal{O}_{\mathbb{P}^r}(1/2 - n/2)$ is a direct summand of $\mathrm{Ker}(A^n) = \mathcal{O}^n_{\mathbb{P}^r}$, which is obviously a contradiction. □

The following proposition relates to the Kronecker-Weierstrass theory of pencils of matrices (see for example [4]).

Proposition 2.4. *Let Φ be a pencil of matrices with constant rank. Then*

$$A\Big(\sum_{B\in\Phi} \mathrm{Ker}\ B\Big) = \bigcap_{B\in\Phi} \mathrm{Im}\, B$$

for every $A \in \Phi$.

First a lemma:

Lemma 2.5. *For any pencil A of matrices of order n of constant rank ρ, there is a diagram with exact rows and columns.*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}(A) & \longrightarrow & \mathcal{O}^m_{\mathbb{P}^1} & \longrightarrow & \mathcal{O}^{m-n+\rho}_{\mathbb{P}^1}(1) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}(A) & \longrightarrow & \mathcal{O}^n_{\mathbb{P}^1} & \xrightarrow{A} & \mathrm{Im}(A) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}^{n-m}_{\mathbb{P}^1} & = & \mathcal{O}^{n-m}_{\mathbb{P}^1} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $m = h^0(\mathrm{Ker}(A)^*)$, $Im(A) \subset \mathcal{O}^n_{\mathbb{P}^1}(1)$ is the image of A.

Proof. By a celebrated theorem, proved in various versions over the years by Hilbert, Birkhoff, Grothendieck, any holomorphic vector bundle over $\mathbb{P}^1$ is a direct sum of line bundles. This implies that $\mathcal{I} = Im(A)$ and $\mathcal{K} = \mathrm{Ker}(A)$ are direct sums of bundles over $\mathbb{P}^1$.
From the exact sequence $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}^n_{\mathbb{P}^1} \longrightarrow \mathcal{I} \longrightarrow 0$ we get that any direct summand of $\mathcal{K}$ has non-positive degree and that any direct summand of $\mathcal{I}$ has degree zero or one. Taking cohomology in the dualized exact sequence $0 \longrightarrow \mathcal{I}^* \longrightarrow \mathcal{O}^n_{\mathbb{P}^1} \longrightarrow \mathcal{K}^* \longrightarrow 0$ and observing that $H^1(\mathcal{I}^*) = 0$ we get a surjection $H^0(\mathcal{O}^n_{\mathbb{P}^1}) \longrightarrow H^0(\mathcal{K}^*) \longrightarrow 0$. The above observation about the direct summands of $\mathcal{K}$ will imply the exactness of the evaluation sequence for $\mathcal{K}^*$.

$$0 \longrightarrow \mathrm{Ker}(ev_{\mathcal{K}^*}) \longrightarrow H^0(\mathcal{K}^*) \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\mathcal{K}^*} \mathcal{K}^* \longrightarrow 0.$$

Putting the two exact sequences together we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}^* & \longrightarrow & \mathcal{O}^n_{\mathbb{P}^1} & \longrightarrow & \mathcal{K}^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Ker}(ev_{\mathcal{K}^*}) & \longrightarrow & H^0(\mathcal{K}^*) \otimes \mathcal{O}_{\mathbb{P}^1} & \xrightarrow{ev_{\mathcal{K}^*}} & \mathcal{K}^* & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

An easy diagram chase shows that the left vertical morphism is surjective. Since $H^0(\mathrm{Ker}(ev_{\mathcal{K}^*})) = 0$, $\mathrm{Ker}(ev_{\mathcal{K}^*})$ must be a direct sum of line bundles of degree -1. Completing the diagram with kernels, dualizing and computing the ranks, we get the claim of the proposition. □

Proof of Proposition 2.4. It is easy to see that the first row in the diagram of Lemma 2.5 can be written as

$$0 \longrightarrow \mathcal{K} \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{A} U \otimes \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0,$$

where $W = \sum_{B\in\Phi} \mathrm{Ker}\ B$ while $U \cong H^0\mathcal{I}(-1) \subseteq H^0(\mathcal{O}^n_{\mathbb{P}^1})$ is $\bigcap_{B\in\Phi} \mathrm{Im}\, B$. □

In case $r = 1$ and $n = 2k + 1$, it is interesting to find explicitly the filtration of $\mathcal{O}^n_{\mathbb{P}^1}$ corresponding to a pencil of constant maximal rank.

Proposition 2.6.
(1) $\mathrm{Ker}(A) = \mathcal{O}_{\mathbb{P}^1}(-k)$.
(2) $\mathrm{Ker}(A^{2k+1}) = \mathcal{O}^{2k+1}_{\mathbb{P}^1}$.
(3) $\mathrm{Ker}(A^2) = \mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(1-k)$.
(4) $\mathrm{Ker}(A^{2k}) = \mathcal{O}^k_{\mathbb{P}^1}(-1) \oplus \mathcal{O}^k_{\mathbb{P}^1}$.

Proof. (1) and (2) are already proved. (3) follows, since for $i = 1$ the exact sequence in Lemma 2.2 splits.
(4) This is a corollary of the Lemma 2.5, since the elements of the pencil are nilpotent matrices of maximal rank, $Im(A) = \mathrm{Ker}(A^{n-1})(1)$ and $\rho = n - 1$, $m = h^0(\mathcal{K}^*) = (n+1)/2$. The simple observation that the right vertical exact sequence is split completes the proof. □

An analysis of extensions over $\mathbb{P}^1$ of vector bundles by lines bundles shows the following results.

Proposition.
(1) $n = 3$, $\mathrm{Ker}(A) = \mathcal{O}_{\mathbb{P}^1}(-1)$, $\mathrm{Ker}(A^2) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\mathrm{Ker}(A^3) = \mathcal{O}_{\mathbb{P}^1}{}^3$.
(2) $n = 5$, $\mathrm{Ker}(A) = \mathcal{O}_{\mathbb{P}^1}(-2)$, $\mathrm{Ker}(A^2) = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\mathrm{Ker}(A^3) = \mathcal{O}_{\mathbb{P}^1}(-1)^3$ as in the second example of Section 1 or $\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ as in the third example of Section 1, $\mathrm{Ker}(A^4) = \mathcal{O}_{\mathbb{P}^1}{}^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^2$, $\mathrm{Ker}(A^5) = \mathcal{O}_{\mathbb{P}^1}{}^5$

To find an upper bound for the number of points where the rank drops, in the case of a pencil of nilpotent matrices with generic maximal rank we will follow more carefully the previous computation of deg(Ker(A)). Jordan block decomposition gives the following lemma.

Lemma 2.7. *If A is a pencil of nilpotent matrices with generic maximal rank, then*
(1) $\mathrm{Ker}(A^i)$ *is a torsion free sheaf of rank i, for every $1 \leq i \leq n$;*
(2) $\{0\} \subseteq \mathrm{Ker}(A) \subseteq \cdots \subseteq \mathrm{Ker}(A^n) = \mathcal{O}^n_{\mathbb{P}^1}$ *is a filtration of* $\mathcal{O}^n_{\mathbb{P}^1}$*;*
(3) *for every $1 \leq i \leq n-1$ we have the exact sequences of locally free sheaves*

$$0 \longrightarrow \mathrm{Ker}(A) \longrightarrow \mathrm{Ker}(A^{i+1}) \xrightarrow{A} \mathrm{Ker}(A^i)(1) \longrightarrow \mathcal{T}_i \longrightarrow 0$$

where $\mathcal{T}_i$ is a torsion sheaf on $\mathbb{P}^1$, supported in $z_1, \cdots, z_p$ such that $\dim_K \mathcal{T}_{i,z_j} = \mathrm{Card}\{k | n^j_k = i+1\} - \mathrm{Card}\{k \mid n^j_k \geq 1\}$*, and where $n^j_1, \cdots, n^j_m$ is the partition of n corresponding to A_{z_j}.*

Theorem 2.8. *The number of points where the rank drops, in the case of a pencil of nilpotent matrices with generic maximal rank is $n(n-1)/2$.*

Proof. Computing inductively, from the above sequences, the Chern classes of $\mathrm{Ker}(A^i)$ and summing from $i = 1$ to $n - 1$ we obtain that $n \cdot \deg(\mathrm{Ker}(A)) \geq p - n(n-1)/2$. Since $\mathrm{Ker}(A) \subseteq \mathcal{O}^n_{\mathbb{P}^1}$, it must have negative degree and this completes the proof. □

3. Existence results.

In this section we construct a pencil of nilpotent constant maximal rank matrices in each odd dimension n. Let us denote with $A_n(s,t) = A_n(\alpha_1, \cdots, \alpha_{n-1})(s,t)$ a pencil of $n \times n$, matrices of the following form:

$$A_n(s,t) = \begin{pmatrix} 0 & s & 0 & \cdots & 0 \\ \alpha_1 t & 0 & s & \ddots & \vdots \\ 0 & \alpha_2 t & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & s \\ 0 & \cdots & 0 & \alpha_{n-1} t & 0 \end{pmatrix}$$

Our aim is to find pencils of nilpotent matrices of constant rank $n-1$ among the pencils given above.

Lemma 3.1. *The determinant of A_n is* $\begin{cases} (-st)^{\frac{n}{2}} \prod_{i=1}^{i=\frac{n}{2}} \alpha_{2i-1}, & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd.} \end{cases}$

The following lemma takes care of the constancy of the rank.

Lemma 3.2. *For n odd A_n represents a pencil of constant rank $n-1$ if and only if $\alpha_1, \cdots, \alpha_n$ are all non zero.*

In order to impose the nilpotency condition to the matrix A_n it is natural to introduce the following projective variety. We set X_n to be the variety in the projective space $\mathbb{P}^{n-2}$ with homogeneous coordinates $a = (\alpha_1, \cdots, \alpha_{n-1})$ defined by the ideal

$$I = (\mathrm{tr} \wedge^2 A_n(1,1), \cdots, \mathrm{tr} \wedge^{2i} A_n(1,1), \cdots, \mathrm{tr} \wedge^{n-1} A_n(1,1)).$$

The points $a \in X_n$ give all the pencils of nilpotent matrices of the form A_n since, by lemma, $\mathrm{tr} \wedge^i A_n = \mathrm{tr} \wedge^i A_n(1,1)(st)^i$ for i even, and $\mathrm{tr} \wedge^i A_n = 0$ for i odd. We want to show the existence of points $a \in X_n$ such that $a = (\alpha_1, \ldots, \alpha_n)$ has every entry α_i different from zero, and this will complete our existence result about pencils of nilpotent matrices with constant rank. This fact is contained in the following result.

Theorem 3.3.
(1) *If Y is an irreducible component of X_n then Y is not contained in any coordinate hyperplane $(\alpha_i = 0)$.*
(2) *X_n is a complete intersection of dimension $(n-3)/2$.*

Proof. By induction on the odd nonnegative integers n. For $n = 3$ the variety X_3 is defined by the single equation $\alpha_1 + \alpha_2 = 0$ and the claim follows in this case. In general let H_i be the hyperplane $\alpha_i = 0$ and let us consider the intersection $X_n \cap H_i$. The matrix A_n, for $a \in X_n \cap H_i$, takes the block form:

$$A_n = \begin{pmatrix} B_i & S \\ 0 & C_{n-i} \end{pmatrix}$$

where $B_i = A_i(\alpha_1, \cdots, \alpha_{i-1})$, $C_{n-i} = A_{n-1}(\alpha_{i+1}, \cdots, \alpha_{n-1})$ and S has s in the position (i,1) and zero elsewhere. A simple induction shows that

$$A^k = \begin{pmatrix} B^k & \sum_{j=0}^{k-1} A^{k-j-1} S B^j \\ 0 & C^k \end{pmatrix}$$

and so A_n is nilpotent if and only if the matrices B_i and C_{n-i} are nilpotent. Because n is odd, either i or $n - i$ is even, let's say i. From Lemma 3.1 we know that $\det B_i = \pm\alpha_1\alpha_3 \cdots \alpha_{2i-1}$, so at least one α_j must be zero for some odd j. Then the matrix B_i itself takes a block form:

$$B_i = \begin{pmatrix} E_j & S \\ 0 & D_{i-j} \end{pmatrix}$$

where E_j and D_{i-j} are nilpotent matrices of the usual form, this time both of odd order. This shows that the original matrix A_n is of the block form:

$$A_n = \begin{pmatrix} E & S & 0 \\ 0 & D & S \\ 0 & 0 & C \end{pmatrix}$$

where E, D, C represent nilpotent pencils of matrices, of the same type as above, of odd orders a, b, c respectively. (a, b, c counting the numbers of α's appearing in each block, and satisfying the relation $a + b + c = n$.) Let us call $\tilde{X}_a$, $\tilde{X}_b$, $\tilde{X}_c$ the affine cones respectively associated to the varieties X_a, X_b, X_c. Then we have actually shown that $X_n \cap H_i$ is a union of the projectivizations of some affine cones of the form $\tilde{X}_a \times \tilde{X}_b \times \tilde{X}_c$. By the inductive hypothesis, every such piece has projective dimension $(a - 1)/2 + (b - 1)/2 + (c - 1)/2$, that is, $(n-3)/2$. If Y is any irreducible component of X_n we already know that $\dim Y \geq (n-3)/2$, since X_n is defined by $(n-1)/2$ equations. This implies that every component is properly intersected by each of the coordinate hyperplanes $(\alpha_i = 0)$, and furthermore it has dimension exactly $(n - 1)/2$, which proves both statements of the theorem. □

The pencils of type A_n do not exhaust all the possible similarity classes of pencils of constant rank $n-1$, for orders $n \geq 5$. Indeed we know examples of pencils of nilpotent matrices of constant rank $n-1$ not similar to any pencil of the form A_n. There are three questions which deserve further study:

(1) To find the similarity classes among the pencils of the form A_n.
(2) To find all the filtration of $\mathcal{O}^n_{\mathbb{P}^1}$ that corresponds to a pencil of nilpotent matrices of constant maximal rank.
(3) To complete the classification of pencils of constant rank.

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