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TOWARDS AN INDUCTIVE CONSTRUCTION OF SELF-ASSOCIATED SETS OF POINTS

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1. Introduction to the problem and historical background.

Castelnuovo [4] defines two sets, each of $2r+2$ points in $\mathbb{P}^r$, as *associated* if there exist two $(r+1)-gons$ (which are the configurations determined by $r+1$ linearly general points in $\mathbb{P}^r$) such that the points of one set projectively correspond to the $2r+2$ vertices of the two $(r+1)$-gons and the points of the other set projectively correspond to the $2r+2$ faces of the two given $(r+1)$-gons. This means that the two sets of points are the set of vertices and face-baricenters of the two $(r+1)$-gons, in a suitable order. In particular, when these two sets coincide, i.e., each point is homologous to itself, the $2r+2$ points are called *self-associated*.

In modern language, this is a particular case of the *Gale-Coble transform* (see [7], [8], [9] for definitions, examples and related results).

In 1889, Castelnuovo [4] showed that if $P_1, \dots, P_{2r+2}$ are $2r+2$ self-associated points in linearly general position in $\mathbb{P}^r$, not lying on a rational normal curve of degree r, then the $(r-2)$-plane Λ in $\mathbb{P}^r$, spanned by $r-1$ of them is an $(r-1)$-secant plane to the unique rational normal curve C^r of degree r through the remaining $r+3$ points. Moreover, the points of intersection of the $(r-2)$-plane Λ and C^r together with the $(r-1)$ original points on Λ form a set of $2r-2 = 2(r-2)+2$ self-associated points in $\Lambda \cong \mathbb{P}^{r-2}$. In other words, if $\Gamma = \{P_1, \dots, P_{2r+2}\} \subset \mathbb{P}^r$ is a set of self-associated points, then we may divide it in two subsets Γ_1 and Γ_2 with $|\Gamma_1| = r+3$ and $|\Gamma_2| = r-1$, respectively, such that Γ_1 is a set of points on a (well determined) rational normal curve C^r, while Γ_2 spans a $(r-1)$-secant space $\Lambda \cong \mathbb{P}^{r-2}$ to C^r, in such a way that $(\Lambda \cap C^r) \cup \Gamma_2$ is a new self-associated set in Λ.

Reversing directions, Castelnuovo's observation suggests to try to use such a description, in establishing an inductive procedure for constructing self-associated sets of points. We will discuss such an approach in small projective spaces.

Definition (Naive). Let k be a field and $\Gamma \subset \mathbb{P}^r_k = \mathbb{P}(V)$ be a set of $\gamma = r + s + 2$ labelled points, such that every subset of $\gamma - 1$ points spans $\mathbb{P}^r_k$. Choosing homogeneous coordinates for these γ points, we get a matrix $G \in M(\gamma \times (r+1); k)$ of rank $r+1$. If we transpose this matrix and take its kernel, we obtain a new matrix $G' \in M(\gamma \times (s+1); k)$. Up to an identification of $(k^\gamma)^*$ and k^γ the rows of this matrix determine a new set, Γ', of γ points in $\mathbb{P}^s_k$, which is called the *Gale-Coble transform* of Γ. If $r = s$, then both Γ and Γ' are sets of $2r+2$ points in $\mathbb{P}^r_k$. These sets are well defined up to the action of $PGL(r+1; k)$.

We refer to [7], [8] and [9] for precise definition of the Gale-Coble transform in modern terms. From now on, we consider the projective space $\mathbb{P}^r$ over the complex field $\mathbb{C}$. For all the notation used and not explained the reader is referred to [12].

2. Basic definitions and properties.

A fundamental property of a set of self-associated points, already observed by Coble [6], is the following:

Each (hyper) quadric of $\mathbb{P}^r$, which passes through $2r+1$ points of a set of $2r+2$ self-associated points, passes also through the remaining one.

This is in fact a characterization of self-associated points which are in linearly general position (see also [7], [9]).

Definition. *A linearly general set of* $2r+2$ *points* $\Gamma \subset \mathbb{P}^r$ *is called self-associated if and only if its points fail by one to impose independent conditions on the quadrics of the space.*

We can rephrase the definition in cohomological language. Let Γ be a 0-dimensional closed subscheme of $\mathbb{P}^r$, $r > 1$, and let I_Γ be its ideal sheaf. Twisting by $O_{\mathbb{P}^r}(2)$ and taking cohomology, the short exact sequence of sheaves

$$0 \to I_\Gamma \to O_{\mathbb{P}^r} \to O_\Gamma \to 0$$

yields the exact sequence

$$0 \to H^0(\mathbb{P}^r, I_\Gamma(2)) \to H^0(\mathbb{P}^r, O_{\mathbb{P}^r}(2)) \to H^0(\Gamma, O_\Gamma(2)) \to H^1(\mathbb{P}^r, I_\Gamma(2)) \to 0.$$

The kernel of the map $\varphi_2 : H^0(\mathbb{P}^r, O_{\mathbb{P}^r}(2)) \to H^0(\Gamma, O_\Gamma(2))$, consists of all the quadrics which vanish on Γ. Let

$$Coker(\varphi_2) \cong H^0(\Gamma, O_\Gamma(2))/Im(\varphi_2) \cong H^1(\mathbb{P}^r, I_\Gamma(2))$$

and put $\delta(\Gamma, 2) = \dim(Coker(\varphi_2)) = h^1(\mathbb{P}^r, I_\Gamma(2))$. From the long exact sequence, we get

$$h^0(\mathbb{P}^r, I_\Gamma(2)) = h^0(\mathbb{P}^r, O_{\mathbb{P}^r}(2)) + \delta(\Gamma, 2) - h^0(\Gamma, O_\Gamma(2)).$$

If the subscheme Γ is reduced, then $h^0(\Gamma, O_\Gamma(2)) = |supp(\Gamma)|$. Each point of Γ imposes one condition on the hypersurfaces of degree 2 in order to be contained. Therefore, $\delta(\Gamma, 2)$ is the exactly the failure of Γ to impose linearly independent conditions on hyperquadrics.

We can restate the definition of self-associated points by saying:

A set of $2r + 2$ distinct points in linearly general position $\Gamma \subset \mathbb{P}^r$, is self-associated in $\mathbb{P}^r$ if and only if $\delta(\Gamma, 2) = 1$.

In particular a set of $2r + 2$ points that lie on a rational normal curve C^r of $\mathbb{P}^r$ is self-associated, since it is contained in the $\binom{r}{2}$ quadrics that contain C^r; indeed,

$$\binom{r}{2} = \binom{r+2}{2} + \delta(\Gamma, 2) - (2r + 2) \Rightarrow \delta(\Gamma, 2) = 1.$$

Another class of self associated points are the sets of $2r + 2$ intersection points of an elliptic normal curve C of degree $r + 1$, and a general quadric in $\mathbb{P}^r$. An elliptic normal curve $C \subset \mathbb{P}^r$ is projectively normal, so Riemann-Roch gives

$$h^0(I_C(2)) = \binom{r+2}{2} - 2(r + 1) = \frac{(r + 1)(r - 2)}{2},$$

i.e. the elliptic (r+1)-curve of $\mathbb{P}^r$ lies on exactly $\frac{(r+1)(r-2)}{2}$ quadrics. Let Γ be the intersection of C with a general hyperquadric. Then Γ lies on $\frac{(r+1)(r-2)}{2} + 1 = \binom{r}{2}$ quadrics of the space, and thus $\delta(\Gamma, 2) = 1$. Therefore, Γ is a self-associated set of points in $\mathbb{P}^r$.

In low dimensions we have a complete characterization of sets of self-associated points (see [9] and the references given there, and [2], [3]).

In $\mathbb{P}^1$ any 4 points are self-associated, since two sets of 4 points are associated, in the sense of Castelnuovo, if and only if they are projectively equivalent, i.e. they have the same cross-ratio.

In $\mathbb{P}^2$ six points in linearly general position are self-associated if and only if they lie on a conic, whereas in $\mathbb{P}^3$ 8 general points are self-associated points if and only if they are the base locus of a net of 3 quadrics.

In $\mathbb{P}^4$, apart from 10 points on a rational normal quartic, Bath [3] has claimed that the general 10 self-associated points are a hyperquadric section of a normal elliptic quintic curve. In this general case, the rational normal quartic through any 7 of the 10 points meets the 2-plane, spanned by the remaining 3, in further 3 points such that the 6 points on the plane lie on a conic; this is a special case of Castelnuovo's remark [4]. For a proof of Bath' assertion and for other results on self-associated sets in small projective spaces we refer the reader to [9].

3. Self-associated sets in $\mathbb{P}^2$ and $\mathbb{P}^3$.

We try now to investigate under which extra conditions, one may hope for an inductive description of self-associated sets of points, as suggested by Castelnuovo's remark above.

The case of 6 self-associated points in $\mathbb{P}^2$ is trivial. Namely start with a point (which one may see as a self-associated set of $\mathbb{P}^0$) of the projective plane $\mathbb{P}^2$, and choose 4 general additional points, say $\{p_1, \dots, p_4\}$. Let C^2 be the unique conic through the 5 points $\{p, p_1, \dots, p_4\}$. Any further point on C^2, together with the previous 5 points, determines a set of 6 self-associated points of the plane.

We will describe now a similar construction in $\mathbb{P}^3$. Start with a set of 4 distinct points in $\mathbb{P}^1$, which form always a self-associated set. Consider this $\mathbb{P}^1$ embedded as a line L in $\mathbb{P}^3$ and denote the 4 points on L as $\{s_1, s_2, p_7, p_8\}$. Choose 4 further general points in $\mathbb{P}^3$, none belonging to the line L, say $\{p_1, p_2, p_3, p_4\}$. The unique twisted cubic C^3, passing through $\{p_1, p_2, p_3, p_4, s_1, s_2\}$ has the line L as its chord through s_1 and s_2. Thes configuration $C^3 \cup L$ is the complete intersection of 2 quadrics of the space, say Q_1 and Q_2. This intersection is the union of a divisor of type (1,2) and one of type (1,0) on a smooth quadric. Let now Q_3 be a general quadric through the 6 points $\Gamma'' = \{p_1, p_2, p_3, p_4, p_7, p_8\}$, thus which does not contain the line L or the twisted cubic. This is possible since $h^0(I_{\gamma''}(2)) = 4$. Therefore,

$$Q_1 \cap Q_2 \cap Q_3 = (C^3 \cup L) \cap Q_3 = \Gamma'' \cup \Gamma'''$$

is a set of 8 points:

a) 4 points on C^3, $\{p_1, p_2, p_3, p_4\}$;

b) 2 points on L, $\{p_7, p_8\}$;

c) 2 new points which must lie on the twisted cubic. Denote these points by $\{p_5, p_6\}$.
The set $\Gamma_1 = \{p_1, \ldots, p_6\}$ lies on C^3, whereas $\Gamma_2 = \{p_7, p_8\}$ lies on L. $\Gamma_2 \cup \{s_1, s_2\}$ is self-associated in L and $L \cap C^3 = \{s_1, s_2\}$. Finally, $\Gamma = \Gamma_1 \cup \Gamma_2$ lies on a net of 3 quadrics, and in fact is the base locus of the net

$$\{\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3 \mid \lambda_i \in k\}.$$

By our definition, for general choices, this means that Γ is a self-associated set of 8 points in $\mathbb{P}^3$.

4. Self-associated sets of points in $\mathbb{P}^4$.

We discuss now an "inductive" construction of self-associated sets of 10 points in $\mathbb{P}^4$.

Recall from our discussion in section 2, that simple examples of such configurations are given by the intersection of a normal elliptic quintic with a general hyperquadric, or by any set of 10 points on a rational normal quartic of $\mathbb{P}^4$ (which lies on $\binom{4}{2} = 6$ quadrics).

Bath and Babbage claimed that the first of these examples is the general case of $\mathbb{P}^4$ (see [9] for a modern proof), and that always in the general case, the unique rational normal quartic through any 7 points of that set meets the 2-plane spanned by the remaining 3 in 3 further points (so the 2-plane is a 3-secant plane to the rational quartic). Moreover, the set of 6 points on the 2-plane is in linearly general position and lies on a conic, thus it is self-associated in $\mathbb{P}^2$.

We also remark here that hyperquadric sections of elliptic normal curves do not account for the general self-associated set in $\mathbb{P}^r$, for $r \geq 5$ (see [2] and [9]). See also [9] for a conjecture concerning general self-associated sets in $\mathbb{P}^5$.

We will describe in the sequel a construction of sets of self-associated points $\mathbb{P}^4$, starting from self-associated sets in $\mathbb{P}^2$ (which as we have seen are well understood).

Start with a $\mathbb{P}^2$ linearly embedded in $\mathbb{P}^4$, and let $E \cong \mathbb{P}^1$ be a line in $\mathbb{P}^4$ disjoint from the chosen plane. One may link the chosen plane in the complete intersection of two hyperquadrics containing it and the line E. For general choices, the residual surface is a smooth cubic rational scroll $S = S_{1,2} \subset \mathbb{P}^4$, which contains E as directrix and meets the initial plane along a (smooth) conic G. Counting parameters, it is easy to see that any smooth conic in the fixed $\mathbb{P}^2$ may be obtained as intersection, for appropriate choices of E and the linking hypersurfaces. In terms of the representation of S as $\mathbb{P}^2(x_0)$, the blowing-up of the projective plane at a point, $S \subset \mathbb{P}^4$ is embedded by the linear system

$|H| = |2l - E|$, where l is the class of a line in $\mathbb{P}^2$ and E is the exceptional divisor. Recall that $l^2 = 1$, $E_0^2 = -1$, $E_0 \cdot l = 0$. In terms of this basis $G \in |l|$.

We need to consider the following linear systems on S:

a) A general member $F \in |3l - 2E|$ is embedded via $|H|$ as a rational normal quartic curve in $\mathbb{P}^4$. Indeed, $deg(\Phi_H(F)) = (3l - 2E) \cdot (2l - E) = 4$, and F is not contained in any hyperplane, since $| H - F |= \emptyset$; therefore F is a rational normal quartic. Notice also that dim $| 3l - 2E_0 |= 6$.

b) A general divisor D in the linear system $| 3l - E_0 |$ is mapped via $|H|$ to an elliptic normal quintic in $\mathbb{P}^4$. Notice that dim $| 3l - E_0 |= 8$.

An easy calculation gives: $F \cdot D = (3l - 2E_0) \cdot (3l - E_0) = 9 - 2 = 7$, $G \cdot F = (3l - 2E_0) \cdot (l) = 3$, $G \cdot D = (l) \cdot (3l - E_0)$.

Choose now 6 general points on the conic G, say $\{s_1, s_2, s_3, p_8, p_9, p_{10}\}$. They form a self-associated set in the projective plane spanned by G.

From the above observations it follows that we may choose a (smooth) rational normal quartic $F \in |3l - 2E|$ on the scroll S, subject to pass through the points $\{s_1, s_2, s_3\}$. Similarly, we may choose an elliptic normal quintic $D \in |3l - E|$ on S containing the other 3 points on the conic, $\{p_8, p_9, p_{10}\}$. The 0-dimensional scheme Γ, defined as the intersection $D \cdot (F + G)$ is a set of 10 points on the elliptic quintic D. The subscheme $\Gamma \subset \mathbb{P}^4$ is self-associated since Γ as a divisor on D is equivalent to the divisor cut by a hyperquadric of the space. Namely

$$F + G \sim 3l - 2E + l = 4l - 2E = 2H,$$

i.e. the divisor $F + G$ belongs to the linear series $| \ 2H \ |$ on the scroll, and since both S and the elliptic quintic D are arithmetically Cohen-Macaulay the claim follows (since each normal elliptic quintic is contained in 5 quadrics of the space, the 10 points lie on 6 linearly independent quadrics, that is $\delta(\Gamma, 2) = 1$).

Conversely, let Γ be a general hyperquadric section of a quintic elliptic normal curve $D \subset \mathbb{P}^4$, and let F be the (unique) rational normal quartic passing through a subset $\Gamma_1 \subset \Gamma$ of 7 points. There are 5 hyperquadrics containing D and F imposes only two extra conditions in order to be contained in one of them. It follows that there are 3 hyperquadrics containing both the elliptic normal curve D and the rational quartic F. Moreover, these 3 hyperquadrics cut out a smooth rational cubic scroll in $\mathbb{P}^4$: Indeed, the secant variety to the elliptic normal quintic D is a quintic hypersurface V $in\mathbb{P}^4$. There are already $\binom{7}{2} = 21$ chords of D that meet the rational quartic F, so Bezout's theorem implies that F must be contained in the secant variety V. Let now S be the union of all secant lines to D that meet the quartic curve F. No two secant lines of D meet outside the elliptic curve, thus S is a ruled surface, rational since the rulings are parametrized by the rational quartic F. It is easily seen that S is indeed a smooth rational cubic scroll in $\mathbb{P}^4$, such that F is a section of the scroll and D is a bisection. In the basis l and E of $S = \mathbb{P}^2(x_0)$, one sees immediately that $F \in |3l - 2E|$ and $D \in |3l - E|$, as above. In other words, the above given description of 10 self-associated points in $\mathbb{P}^4$ is the most general one.

5. A possible approach in higher dimensional projective spaces.

As mentioned above, in $\mathbb{P}^r$ for $r > 4$ self-associated sets are much more complicated, because the intersection of a normal elliptic curve of degree $r+1$ with a general quadric is not anymore the general case. We may start as above, with the projective space $\mathbb{P}^{r-2}$, viewed as an $(r-2)$-plane Λ in $\mathbb{P}^r$, with a set of $2r-2$ self-associated points. We may divide this set of points into two subsets, say Δ_1 and Δ_2, each of cardinality $r-1$, and then consider 4 further general points in $\mathbb{P}^r$ and the unique rational normal curve C^r of degree r passing through these 4 points and those of one of these sets, say Δ_1.

The set Δ_1 should play the role of the $r-1$ point set Γ_2 of § 1. We would like to find $r-1$ further points on the rational normal curve in such a way that they form, together with the 4 general chosen points, the set Γ_1 of § 1.

A useful observation is the fact that in $\mathbb{P}^r$ there are $r-1$ linearly independent hyperquadrics containing the rational normal curve C^r and the $(r-2)$-plane. Suppose, in fact, that coordinates are chosen in the projective space such that the $(r-2)$-plane has equations

$$x_0 = x_1 = 0,$$

then the quadrics of the space containing this $(r-2)$-plane are of the form

$$F_r := \{x_0 l_0 + x_1 l_1 \mid l_o,\ l_1 \in (\mathbb{C}[x_0, \dots, x_r])_1\},$$

and they form a space of dimension $2r+1$. In order to contain the rational normal curve, we have to impose further $r-2$ conditions, since Γ_2 already lies on the $r-2$-plane. Thus, we get $2r+1-(r-2) = r-1$ linearly independent quadrics containing both the rational normal curve and the $(r-2)$-plane.

We think that, by using the fact that the $2r-2$ points in $\mathbb{P}^{r-2}$ lie on exactly $\binom{r-2}{2}$ quadrics, the fundamental step would be to find a suitable rational normal scroll of degree $r-1$ in $\mathbb{P}^r$ passing through both the 4 points on C^r and the points on the $(r-2)$-plane of Γ_2, and which meets on C^r further $r-1$ points. Moreover, since each scroll is the intersection of $\binom{r-1}{2}$ quadrics, we have to find a scroll for which the quadrics defining it are linearly independent from those containing $C^r \cup \Lambda$. This would imply that these $2r + 2$ points form a set Γ which lies on $(r-1) + \binom{r-1}{2} = \binom{r}{2}$ quadrics. This means that $\delta(\Gamma, 2) = 1$.

A result of Fano [10] concerning rational normal scrolls in $\mathbb{P}^r$ says that there are ∞^{r-1} scrolls of degree $r-1$ containing a fixed rational normal curve of degree r. This suggests to consider also the rational normal curve D^r containing the set Γ_1 and the chosen 4 general points. Thus the rational normal curves D^r and C^r share 4 general points and are such that one passes through Γ_1 and the other one through Γ_2. We know that in the ideal I_{C^r} we may find $r-1$ hyperquadrics containing Λ, and Fano's result says that there are ∞^{r-1} scrolls of degree $r-1$ containing D^r. We would like to be able to find a suitable such scroll containing both C^r and D^r such that we may select on it the desired numbers of points failing to impose independent conditions on quadrics, as explained before.

Further investigations in dimension $r > 4$ might lead to a general construction of self-associated points in $\mathbb{P}^r$.

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