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GALE TRANSFORM FOR SETS OF POINTS ON THE VERONESE SURFACE IN $\mathbb{P}^5$

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Introduction.

The present work stems from an attempt to finding a general result concerning the Gale transform of sets of points contained in a smooth projective surface.
In the classical literature there are many sparse results concerning the Gale transform (classically called "association") of special configurations of points (see [1], [2], [3]) but no general geometric description of the Gale transform of a collection of points belonging to a given projective variety is known. A first fundamental result in this direction in the case of projective curves is the Theorem of Goppa [6] (see also the first section of this paper) which asserts that if a set of points is contained in some projective curve, then its Gale transform is contained in a different embedding of the same curve.
Very recently, D. Eisenbud and S. Popescu gave a generalization of the Goppa's theorem to rational scrolls and Veronese embeddings (see [5]). For instance, they proved that if a set is contained in a rational scroll then its Gale transform is contained in another scroll which is obtained via an elementary transformation of the first.

No general geometric result about the Gale transform of sets of points contained in a general projective surface seems presently to be known.
In this paper we will study the Gale transform of sets of point contained in a Veronese quartic surface in $\mathbb{P}^5$. We will also give partial results concerning

other Veronese surfaces.
The following statement collects our main result:

Theorem. *Let* Γ *be a general finite set of points contained in the Veronese quartic surface* $\mathcal{V} \subset \mathbb{P}^5$. *Then its Gale transform* Γ' *can be characterized as follows:*

- *a) if* $\deg(\Gamma) = 11$, *then the Gale transform* $\Gamma' \subset \mathbb{P}^4$ *is contained in two del Pezzo quartic surfaces whose intersection is the union of the set* Γ' *and a line* L. *In this case* $\Gamma' \cup L$ *is the scheme theoretic intersection of 4 quadrics;*
- *b) if* $\deg(\Gamma) = 12$, *then the Gale transform* $\Gamma' \subset \mathbb{P}^5$ *is contained in a Segre threefold* $W \simeq \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$, *on which it is the base locus of a net in* $H^0(O_W(1,2))$. *In this case, there exists a plane* Π *such that the union* $\Gamma' \cup \Pi$ *is the scheme theoretic intersection of* 6 *quadrics;*
- *c) if* $\deg(\Gamma) > 12$, *then* Γ' *is contained in a cone* $\mathcal{K}$ *over the Veronese surface* $\mathcal{V}$ *and the Gale transformation of* Γ' *is the projection from the vertex of* $\mathcal{K}$.

The outline of the paper is the following:
In the first section we will collect general facts concerning the Gale transform. In the second we will recall well known properties of the del Pezzo's surfaces. The last three sections of the paper will be devoted to the proof of the Theorem above. We will also give in the last section a partial result concerning other Veronese surfaces.

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1. Definitions and preliminary results.

We begin with the definition of the Gale transform as given in the paper [4], [5] (our ground field will always be $\mathbb{C}$).

Definition. *Let* Γ *be a Gorenstein finite scheme. The Gale transform of a linear series* (V, L) *on* Γ *is the linear series* $(V^\perp, K_\Gamma \otimes L^{-1})$ *where* K_Γ *denotes the canonical line bundle of* Γ *and* $V^\perp \subset H^0(K_\Gamma \otimes L^{-1})$ *denotes the annihilator of* $V \subset H^0(L)$ *with respect to the Serre duality (after the choice of a trace map).*

The following proposition is proved in [4].

Proposition. *Let Γ be a finite reduced scheme. If the linear series (V, L) defines an embedding of Γ in $\mathbb{P}^r := \mathbb{P}(V)$, $r \geq 1$, with ideal sheaf $\mathcal{I}_\Gamma$, then Serre duality induces a natural identification $V^\perp \simeq H^1(\mathcal{I}_\Gamma(1))^*$.*

We will denote in the sequel by $\Gamma' \subset \mathbb{P}(V^\perp) := \mathbb{P}^s$ the embedding given by the Gale transform linear series (where always $r + s + 2 = \deg(\Gamma)$).
The following fundamental theorem characterize the Gale transform of schemes contained in projective curves:

Goppa's Theorem. ([6], [4], [5]). *If $\Gamma \subset \mathbb{P}^r$ is as above and such that $\Gamma \subset C \subset \mathbb{P}(H^0(O_C(H))) = \mathbb{P}^r$, where C is a smooth linearly normal curve, then the Gale transform Γ' is the image (in its linear span) of $\Gamma \subset C$ under the linear system $\mid K_C + \Gamma - H \mid$.*

The following lemma will be useful in the sequel:

Lemma. *If Γ is a reduced finite scheme contained in a Veronese surface $\mathcal{V}_d$, where $\mathbb{P}^2 \simeq \mathcal{V}_d \subset \mathbb{P}^{\frac{(d+2)(d+1)}{2}-1} := \mathbb{P}^N$, then $H^1(\mathcal{I}_{\Gamma,\mathbb{P}^N}(1))^* \simeq H^1(\mathcal{I}_{\Gamma,\mathbb{P}^2}(d))^*$.*

Proof. This follows taking cohomology in the sequence

$$0 \longrightarrow \mathcal{I}_{\mathcal{V}_d}(1) \longrightarrow \mathcal{I}_\Gamma(1) \longrightarrow \mathcal{I}_{\Gamma,\mathcal{V}_d}(1) \longrightarrow 0$$

since $H^1(\mathcal{I}_{\mathcal{V}_d}(1)) = H^2(\mathcal{I}_{\mathcal{V}_d}(1)) = 0$. $\square$

Combining the proposition above with the previous lemma we find that the Gale transform of a set $\Gamma \subset \mathcal{V}_d$ is induced by the linear series $H^1(\mathcal{I}_{\Gamma,\mathbb{P}^2}(d))^*$.

2. Some further useful results.

In this section we will begin the study of the Gale transform of a set of points $\Gamma \subset \mathbb{P}^5$ contained in a Veronese quartic surface $\mathcal{V}$, when $\deg(\Gamma) = 11, 12$ (the case of 10 points, the first interesting one, is treated in [5]). We essentially show that the Gale transform $\Gamma' \subset \mathbb{P}^h$ $h = 4, 5$ lies on "many" del Pezzo's surfaces.
To this end, start with $\Gamma \subset \mathbb{P}^2 \cong \mathcal{V}_2 \subset \mathbb{P}^5$, $\deg(\Gamma) = 11$ or 12, consider a general pencil $\mathcal{L} \in \mathbf{G}r(2, H^0(\mathcal{I}_{\Gamma,\mathbb{P}^2}(4)))$, and let C denote a smooth quartic curve in the pencil $\mathcal{L}$. The base locus of the pencil $\mathcal{L}$ is the set $\Gamma \cup \Gamma_\mathcal{L}$, where $\Gamma_\mathcal{L}$ is a residual set such that $\deg(\Gamma_\mathcal{L}) = 4$, or 5.
The following lemma relates part of the cohomology of $\mathcal{I}_\Gamma$ in terms of that of the ideal sheaf of $\Gamma_\mathcal{L}$.

Lemma. *For* $i = 2, 3$ *we have isomorphisms:*

$$H^1(\mathcal{I}_{\Gamma,\mathbb{P}^2}(i))^* \simeq H^0(\mathcal{I}_{\Gamma_{\mathcal{L}},\mathbb{P}^2}(5-i)).$$

Proof. Consider the sequences

$$0 \longrightarrow O_{\mathbb{P}^2}(i-4) \longrightarrow \mathcal{I}_\Gamma(i) \longrightarrow O_C(il-\Gamma) \longrightarrow 0$$

and

$$0 \longrightarrow O_{\mathbb{P}^2}(j-4) \longrightarrow \mathcal{I}_{\Gamma_{\mathcal{L}}}(j) \longrightarrow O_C(jl-\Gamma_{\mathcal{L}}) \longrightarrow 0,$$

where l denotes the divisor class of the embedding of C in the plane. The first sequence and Serre duality give $H^1(\mathcal{I}_\Gamma(i))^* \simeq H^1(O_C(il-\Gamma))^* \simeq H^0(O_C(\Gamma+(1-i)l))$, while the second yields $H^0(\mathcal{I}_{\Gamma_{\mathcal{L}}}(j)) \simeq H^0(O_C(jl-\Gamma_{\mathcal{L}}))$, when $j < 4$. The claim follows since $\Gamma \cup \Gamma_{\mathcal{L}} \simeq 4l$ on C. □

Let $S_{\Gamma_{\mathcal{L}}} := \mathbb{P}^2(\Gamma_{\mathcal{L}})$ be the blow up of $\mathbb{P}^2$ along $\Gamma_{\mathcal{L}}$, and denote by $\pi_{\mathcal{L}}$ the natural map to the plane. The linear system $H^0(\mathcal{I}_{\Gamma_{\mathcal{L}}}(3))$ provides an embedding $i_{\Gamma_{\mathcal{L}}} : S_{\Gamma_{\mathcal{L}}} \longrightarrow \mathbb{P}^h$ ($h = 4, 5$) as a del Pezzo surface. To avoid cumbersome notations, we will denote by $\mathcal{L}$ both the linear pencil of plane quartics and its proper transform in $S_{\Gamma_{\mathcal{L}}}$. Notice that the base locus of $\mathcal{L}$ in $S_{\Gamma_{\mathcal{L}}}$ is $\pi_{\mathcal{L}}^{-1}(\Gamma)$.

Proposition. *For any pencil* $\mathcal{L}$*, the Gale transform of* $\Gamma \subset \mathcal{V}_2 \subset \mathbb{P}^5$ *is* $\Gamma' := i_{\Gamma_{\mathcal{L}}}(\Gamma) \subset S_{\Gamma_{\mathcal{L}}}$*. Conversely, if a set* $\Gamma' \subset S_{\Gamma_{\mathcal{L}}} \subset \mathbb{P}^h$ *is the base locus of the pencil* $\mathcal{L}$ *in* $S_{\Gamma_{\mathcal{L}}}$*, then its Gale transform* $\Gamma \subset \mathbb{P}^5$ *is contained in a Veronese surface* $\mathcal{V}_2 \subset \mathbb{P}^5$.

Proof. The proposition is a consequence of the Goppa's theorem applied to a general (smooth) curve C belonging to the pencil $\mathcal{L}$.
Indeed, if $\Gamma \subset C \subset \mathcal{V}_2$, then Goppa's theorem says that Γ' is the image of Γ under linear system $|K_C + \Gamma - 2l|$. But $\Gamma \cup \Gamma_{\mathcal{L}} \simeq 4l$ on C, whence $|K_C + \Gamma - 2l| = |\Gamma - l| = |3l - \Gamma_{\mathcal{L}}|$, which is the linear series giving the embedding of the corresponding del Pezzo's surface.
Conversely, suppose now that Γ' is the base locus of $\mathcal{L}$ in $S_{\Gamma_{\mathcal{L}}}$. Then the union $\pi_{\mathcal{L}}(\Gamma') \cup \Gamma_{\mathcal{L}}$ is the base locus of a pencil of plane quartics. Let C be a general quartic curve in this pencil. Again, Goppa's theorem tells us that the Gale transform Γ of Γ' is the image of $\pi_{\mathcal{L}}(\Gamma')$ in the linear system $|K_C + \pi_{\mathcal{L}}(\Gamma') - (3l - \Gamma_{\mathcal{L}})|$ on C. We may conclude since the previous linear series is just $|2l|$, which defines the Veronese surface in $\mathbb{P}^5$. □

3. The Gale transform of a general set of 11 points on the Veronese surface.

This section is devoted to the proof of the following:

Theorem. *Suppose Γ is a general set of 11 points on the Veronese surface $\mathcal{V}_2 \subset \mathbb{P}^5$. Then, the Gale transform Γ' of Γ is contained in two del Pezzo quartic surfaces whose intersection is the union of Γ' and a line L.*

In this case, $h^0(\mathcal{I}_{\Gamma',\mathbb{P}^2}(2)) = 4$ and $(I_{\Gamma'})_2$ generates the homogeneous ideal of $\Gamma' \cup L$.
Conversely, if Γ' is a general set of 11 points in $\mathbb{P}^4$ such that $h^0(\mathcal{I}_{\Gamma'}(2)) = 4$, and such that the generators of $H^0(\mathcal{I}_{\Gamma'}(2))$ define scheme theoretically the union of Γ' with a line, then the Gale transform $\Gamma \subset \mathbb{P}^5$ of Γ' is contained in a Veronese surface.

The proof of the theorem is a consequence of the following three lemmas. We keep in force the notations introduced in the last section, and so $\mathcal{L} \in \mathbf{G}r(2, H^0(\mathcal{I}_{\Gamma,\mathbb{P}^2}(4)))$ denotes a general pencil and C is a general quartic quartic belonging to $\mathcal{L}$. The base locus of $\mathcal{L}$ is a set $\Gamma \cup \Gamma_{\mathcal{L}}$, where $\Gamma_{\mathcal{L}}$ is a residual set of 5 points.
Furthermore, $S_{\Gamma_{\mathcal{L}}} := \mathbb{P}^2(\Gamma_{\mathcal{L}}) \subset \mathbb{P}^4$ will denote the del Pezzo's embedding via $i_{\mathcal{L}}$, and $\pi_{\mathcal{L}}$ its natural projection onto the plane. Finally, let $D_{\mathcal{L}}$ be the unique conic containing the set $\Gamma_{\mathcal{L}}$, and by $L_{\mathcal{L}}$ its image with respect to $i_{\mathcal{L}}$. Notice that $L_{\mathcal{L}}$ is a line in $\mathbb{P}^4$.

Lemma 1. *The line $L_{\mathcal{L}}$ does not depend on the choice of the pencil $\mathcal{L}$. (Thus we will denote it in the sequel by L).*

Proof. It suffices to notice that $L_{\mathcal{L}}$ is the base component of the subseries given by the image of the natural map $H^0(\mathcal{I}_{\Gamma_{\mathcal{L}}}(2)) \times H^0(O_{\mathbb{P}^2}(1)) \longrightarrow H^0(\mathcal{I}_{\Gamma_{\mathcal{L}}}(3))$, which is independent of the choice of the pencil $\mathcal{L}$. This is a consequence of the isomorphisms established in the lemma proved in Section 2. $\square$

Combining Lemma 1 above with the proposition proved in section II, we deduce that $S_{\Gamma_{\mathcal{L}}} \cap S_{\Gamma_{\overline{\mathcal{L}}}} \supset \Gamma \cup L$ for any pair of pencils $\mathcal{L}$ and $\overline{\mathcal{L}}$ as above. Moreover, as the following lemma shows, we have equality as soon as the two pencils are general:

Lemma 2. *For a general choice of pencils $\mathcal{L}$ and $\overline{\mathcal{L}}$, $S_{\Gamma_{\mathcal{L}}} \cap S_{\Gamma_{\overline{\mathcal{L}}}} = \Gamma' \cup L$.*

Proof. A del Pezzo surface $S_{\Gamma_{\mathcal{L}}}$ is a complete intersection of two quadrics in $\mathbb{P}^4$. The quadrics defining $S_{\Gamma_{\overline{\mathcal{L}}}}$ restrict to a pencil $|Q| \subset \mathbb{P}(H^0(O_{S_{\Gamma_{\mathcal{L}}}}(2)))$, and the intersection $S_{\Gamma_{\mathcal{L}}} \cap S_{\Gamma_{\overline{\mathcal{L}}}}$ is just the base locus of $|Q|$. Clearly, a divisor in

$|Q|$ is the union of L with a curve which varies in a subseries $|\overline{Q}|$ contained in $\mathcal{L}$. The base locus of Q is the union of L and the base locus of $|\overline{Q}|$.
There are two possibilities: either $|\overline{Q}|$ fills $\mathcal{L}$ or not. In the first case we are done because the base locus of $\mathcal{L}$ on $S_{\Gamma_{\mathcal{L}}}$ is Γ'. In the second case $|Q|$ is represented by a fixed curve $L \cup X$, where X belongs to both pencils $\mathcal{L}$ and $\overline{\mathcal{L}}$. So, the two pencils contain isomorphic curves. Now, recall that two plane quartics (canonical curves) are isomorphic iff there is a linear automorphism of the plane sending one curve to the other. But the only automorphism of the plane fixing Γ is the identity, so two general pencils $\mathcal{L}$ and $\overline{\mathcal{L}}$ have no curve in common, and this concludes the proof. □

Lemma 3. *Let $\Gamma' \subset \mathbb{P}^4$ be a set of 11 points such that $h^0(\mathcal{I}_{\Gamma'}(2)) = 4$, and such that these quadrics define scheme theoretically the union of Γ' and a disjoint line L. Then the Gale transform $\Gamma \subset \mathbb{P}^5$ of Γ' is contained in a Veronese surface.*

Proof. The main point to prove is that there is a smooth del Pezzo quartic S containing both L and Γ'.
We begin by showing that the general quadric Y in $H^0(\mathcal{I}_{\Gamma' \cup L}(2))$ is smooth. By Bertini, Y is smooth outside $\Gamma' \cup L$, since $H^0(\mathcal{I}_{\Gamma' \cup L}(2))$ generates $\mathcal{I}_L(2)$. Since Γ' is reduced, for each point $\gamma \in \Gamma'$, the maximal ideal m_γ in the local ring at γ is generated by a basis in $H^0(\mathcal{I}_{\Gamma' \cup L}(2))$, whence Y must be smooth at γ.
We are left to check the smoothness along L. Since $H^0(\mathcal{I}_{\Gamma' \cup L}(2))$ generates $\mathcal{I}_L(2)$, the natural evaluation $H^0(\mathcal{I}_{\Gamma' \cup L}(2)) \otimes \mathcal{O}_L \longrightarrow \mathcal{N}_L^*(2)$ is surjective (where $\mathcal{N}_L^*$ denotes the conormal bundle of L). The quadric Y is singular at some point $p \in L$ iff the corresponding section of $\mathcal{N}_L^*(2)$ vanishes at p. Thus it suffices to prove that the general quadric Y vanishes nowhere in $\mathcal{N}_L^*(2)$, and this is a consequence of a well known result of Serre which is stated at the end of the section for the reader's convenience.
To prove now that $\Gamma' \cup L$ is contained in a smooth quartic del Pezzo surface, it suffices to repeat the argument above for a general member in the linear system $H^0(\mathcal{I}_{\Gamma' \cup L, Y}(2))$. Again, smoothness at each point of Γ' follows because $H^0(\mathcal{I}_{\Gamma' \cup L, Y}(2))$ generates the maximal ideal at such points. Finally, smoothness along L is again a consequence of the fact that $H^0(\mathcal{I}_{\Gamma' \cup L, Y}(2))$ generates the conormal bundle of L in Y.
In conclusion, there exists a smooth del Pezzo quartic S containing both L and Γ', and moreover S can be identified to $S_{\Gamma_{\mathcal{L}}}$ for some pencil $\mathcal{L}$, whose elements contain both L and Γ'. The remaining quadrics in $H^0(\mathcal{I}_{\Gamma' \cup L, Y}(2))$ cut on $S_{\Gamma_{\mathcal{L}}}$ a linear series whose moving part is $\mathcal{L}$, and whose base locus is Γ'. The lemma follows now from the proposition proved in Section 2. □

Proposition. (Serre). *Let E be a vector bundle over a smooth variety M such that* $\operatorname{rank}(E) > \dim(M)$. *If E is generated by global sections, then the general* *element of a vector space $W \subset H^0(E)$ (of dimension $>$ rank(E)) generating E vanishes nowhere.*

4. The Gale transform of a general set of 12 points on the Veronese surface.

In this section we prove the following

Theorem. *Suppose Γ is a general set of 12 points on the Veronese surface in $\mathbb{P}^5$. Then, the Gale transform Γ' of Γ is contained in a Segre threefold $V \simeq \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$, where it is the base locus of a net in $H^0(O_V(1,2))$. In this case, there exists a plane Π such that the union $\Gamma' \cup \Pi$ is the scheme theoretic intersection of 6 quadrics.*

Conversely, if $\Gamma' \subset \mathbb{P}^5$ is a general set of 12 points such that there exists a plane Π whose union with Γ' is the scheme theoretic intersection of 6 quadrics, then the Gale transform Γ of Γ' is contained in a Veronese surface in $\mathbb{P}^5$.

As above, each pencil $\mathcal{L} \subset H^0(\mathcal{I}_{\Gamma,\mathbb{P}^2}(4))$ determines a del Pezzo quintic surface $S_{\Gamma_{\mathcal{L}}} \subset \mathbb{P}^5$ containing the Gale transform Γ'. The surface $S_{\Gamma_{\mathcal{L}}}$ has a pencil of conics (which are the images of conics containing $\Gamma_{\mathcal{L}}$). The pencil of planes spanned by these conics build up a Segre threefold $V_{\mathcal{L}}$ containing $S_{\Gamma_{\mathcal{L}}}$.

The Segre threefold $V_{\mathcal{L}}$ does not depend on the pencil $\mathcal{L}$ (the proof is the same as for Lemma 1 of the third section). Therefore we have a family of del Pezzo quintic surfaces in $\mathbb{P}^5$, each containing Γ' and they all are contained in a fixed Segre threefold $V \simeq \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. Recall that any del Pezzo surface on V is a divisor of type $(1,2)$ whose union with a plane of V is the complete intersection of V with a quadric of $\mathbb{P}^5$.

Lemma. *If $\mathcal{L}$ and $\overline{\mathcal{L}}$ are general, then $\pi_{\mathcal{L}}(S_{\Gamma_{\mathcal{L}}} \cap S_{\Gamma_{\overline{\mathcal{L}}}})$ is a quartic of $\mathcal{L}$.*

Proof. The union of $S_{\Gamma_{\overline{\mathcal{L}}}}$ with any plane of V is cut out by some quadric of $\mathbb{P}^5$, hence $\pi_{\mathcal{L}}(S_{\Gamma_{\mathcal{L}}} \cup S_{\Gamma_{\overline{\mathcal{L}}}})$ is contained in a plane sextic which splits in a conic Δ, containing $\Gamma_{\mathcal{L}}$, and a quartic of $\mathcal{L}$. Hence, to complete the proof, it suffices to note that if $\mathcal{L}$ and $\overline{\mathcal{L}}$ are general, then $S_{\Gamma_{\mathcal{L}}}$ does not contain the conic Δ. $\square$

We are ready to prove the theorem.

Proof. The first statement is a direct consequence of the last lemma because if we choose three general pencils $\mathcal{L}$, $\mathcal{L}_1$ and $\mathcal{L}_2$, then the set $S_{\Gamma_{\mathcal{L}}} \cap S_{\Gamma_{\mathcal{L}_1}} \cap S_{\Gamma_{\mathcal{L}_2}}$ is the base locus of $\mathcal{L}$ in $S_{\Gamma_{\mathcal{L}}}$ which is Γ'. To conclude the first part of the statement it suffices to recall that both $S_{\Gamma_{\mathcal{L}_1}}$ and $S_{\Gamma_{\mathcal{L}_2}}$ are residual to a fixed plane $\Pi \subset V$ in the intersection of V with suitable quadrics.

To prove the converse, we observe that the union of 12 general points with a plane of $\mathbb{P}^5$ is always contained in a Segre threefold (which is cut out by three quadrics). The remaining three quadrics containing $\Pi \cup \Gamma'$ cut on V a net $\mathcal{F}$ of divisors of type (1,2).

Claim. *The general member of this net is smooth.*

Proof of the claim. By Bertini, the singular set is contained in $\Pi \cup \Gamma'$. Furthermore, as in the proof of Lemma 3 of the last section, the general member of $\mathcal{F}$ is smooth at the points of Γ' (the codimension of Γ' equals the dimension of $\mathcal{F}$). Finally, recall that, if a divisor $(1, 2)$ is singular at some point $p \in V$, then the divisor contains the unique line of V through p not contained in Π_p (the unique plane of V containing p). Hence, if two such surfaces are both singular at a point, then they share a line which, by hypothesis, is not contained in the base locus of $\mathcal{L}$. Thus if we choose surfaces which are singular at different points of Π, then general linear combination of such divisors is smooth at all points of Π.
Therefore, the general member of $\mathcal{F}$ is a smooth del Pezzo quintic surface which can be identified with $S_{\Gamma_{\mathcal{L}}}$ for a suitable pencil $\mathcal{L}$. The family $\mathcal{F}$ cuts on $S_{\Gamma_{\mathcal{L}}}$ a pencil of curves whose moving part is $\mathcal{L}$ and whose base locus is Γ'. We conclude the proof by the proposition proved in the first section. □

5. A result concerning the Gale transform of sets in higher Veronese surfaces.

Consider a set of 7 points in $\mathbb{P}^3$. By Goppa's theorem we know that the Gale transform is contained in a conic iff:

a) the original set is contained in a quadric cone;
b) the Gale transformation consists in a projection from the vertex of the cone.

In this section we are going to generalize this simple fact to sets of points contained in higher Veronese surfaces.

Let us introduce the following notations. Let $\mathcal{V}_d$ denote the Veronese surface of degree d^2 and let $\Gamma \subset \mathcal{V}_d$ be a general set of points. As above, we will denote by Γ' the Gale transform of Γ.
Consider two smooth plane curves C and C' such that $C \cap C' = \Gamma \cup \overline{\Gamma}$, where $\overline{\Gamma}$ is a general set of $\deg(C)\deg(C') - \deg(\Gamma)$ points. Let $X_{C,C'}$ be the rational surface obtained by mapping the blow up $\mathbb{P}^2(\overline{\Gamma})$ via the linear series $|H^0(\mathcal{I}_{\overline{\Gamma}}(\deg(C) + \deg(C') - d - 3))|$ Let $n = \deg(C) + \deg(C') - d - 3$.

Lemma. *The Gale transform Γ' of Γ is contained in the surface $X_{C,C'}$.*

Proof. Goppa's theorem shows that Γ' is the image of Γ with respect to the linear series $|(d(C) - d - 3)l + \Gamma|$ on C. But $|(d(C) - d - 3)l + \Gamma| \simeq |(d(C) + d(C') - d - 3)l - \overline{\Gamma}|$ and the lemma follows. $\square$

Remark. In general, $h := h^0(\mathcal{I}_{\overline{\Gamma}}(n))$ is bigger than s, thus Γ' spans a s-dimensional subspace W of $\mathbb{P}^h$.

Theorem. *If $d > \frac{(d+2)(d+1)}{2}$, then Γ' is contained in a cone $\mathcal{K}$ over a Veronese surface $\mathcal{V}_d$ of degree d^2, and the Gale transformation of Γ' onto Γ is induced by the projection from the vertex of $\mathcal{K}$.*

Proof. We choose the degrees of the curves C and C' such that $\delta := n - d \geq \max\{d(C), d(C')\}$ and we consider a general curve D in $|H^0(\mathcal{I}_{\overline{\Gamma}}(\delta))|$. ($D$ is smooth by the condition on the degrees.)
The subseries of $|H^0(\mathcal{I}_{\overline{\Gamma}}(n))|$ consisting of curves containing D defines a linear space K of codimension $\frac{(d+2)(d+1)}{2} + 1$ in $\mathbb{P}^h$, and it is clear that its intersection with $W := \operatorname{span}(\Gamma')$ is proper. Such a subseries cuts on $X_{C,C'}$ a linear system whose moving part is the proper transform of the complete linear series of plane curves of degree d. Set $K := \overline{K} \cap W \subset W \simeq \mathbb{P}^s$ and consider a general linear subspace $Z \subset \mathbb{P}^s$ of dimension r. Consider the projection $\pi_{\overline{K}}$ from $\overline{K}$ to Z. As its restriction over $X_{C,C'}$ is the complete linear series of plane curves of degree d, we have that $\pi_{\overline{K}}(X_{C,C'}) \simeq \mathcal{V}_d$, hence $X_{C,C'}$ is contained in the cone $\overline{\mathcal{K}}$ over $\pi_{\overline{K}}(X_{C,C'})$. But $X_{C,C'}$ contains Γ', which is then contained in the cone $\mathcal{K}$ with base $\pi_{\overline{K}}(X_{C,C'}) \simeq \mathcal{V}_d$ and vertex K. Finally, the composition of $\pi^{-1}_{C,C'}$ with $\pi_{\overline{K}}$ is the identity on the plane hence the projection from the vertex of $\mathcal{K}$ pulls back Γ' to Γ. $\square$

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