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GRADED BETTI NUMBERS OF GENERAL FINITE SUBSETS OF POINTS ON PROJECTIVE VARIETIES

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Introduction.

Let $X \subset \mathbb{P}^n_k$ be a projective variety over an algebraically closed field. The goal of the present paper is to study the graded Betti numbers of general points on X. The underlying idea is that these numbers should carry information about the geometry of the embedding of X.

The Betti numbers $b_{i,j}$ are the numerical invariants of X given by the minimal free resolution of its homogeneous coordinate ring. $b_{i,j}$ is equal to the dimension over k of $\mathrm{Tor}^i(S_X, k)_{i+j}$. We will picture these numbers in the so-called Betti diagram, in which $b_{i,j}$ appears at the intersection of the i^{th} column with the j^{th} row.

The first point is that if we have a large enough set of general points on our variety X then the Betti diagram of the points consists of two parts: at the top we recover the Betti diagram of X and there is also an additional part at the bottom. This fact, which has already been proved in [6], gives a precise meaning to the assertion that the set of points gives information about the whole variety.

The main results in this paper deal with the bottom part in the Betti diagram of the points. We will see that for general points this residual part consists of only two nontrivial rows. We can tell exactly which are these rows. Moreover, if they are let's say the j^{th} and the $(j+1)^{\mathrm{th}}$ rows, we have explicit expressions for $b_{i+1,j} - b_{i,j+1}$ in terms of the Hilbert polynomial of X and the dimension of the ambient space.

The first question one could ask is how do the numbers in these rows vary when we vary the number of points. In the case of curves the situation is especially nice. We prove a conjecture of L'vovsky from [6] saying that we have periodicity. Namely if d is the degree of X, by adding d general points the bottom part of the Betti diagram moves with one row down.

Let's consider a simple example, that of a rational quartic in $\mathbb{P}^3$ given parametrically by $\mathbb{P}^1 \ni (u, v) \longrightarrow (u^4, u^3v, uv^3, v^4) \in \mathbb{P}^3$. The Betti diagram of X is:

$$\begin{array}{lcccc} 0: & 1 & - & - & - \\ 1: & - & 1 & - & - \\ 2: & - & 3 & 4 & 1 \end{array}$$

We will prove that for $n \geq 13\,(= \deg X \cdot \operatorname{reg} X + 1)$, if we take a set X_n of n points on X, the Betti diagram of X_n does not depend on the particular set of points chosen. Here are the Betti diagrams of X_n for $n = 15$ and $n = 19$, as they will come out from Propositions 1.3, 1.5 and 3.1 below.

$$\begin{array}{llcccc} n = 15 & 0: & 1 & - & - & - \\ & 1: & - & 1 & - & - \\ & 2: & - & 3 & 4 & 1 \\ & 3: & - & 2 & 2 & - \\ & 4: & - & - & 2 & 2 \end{array}$$

$$\begin{array}{llcccc} n = 19 & 0: & 1 & - & - & - \\ & 1: & - & 1 & - & - \\ & 2: & - & 3 & 4 & 1 \\ & 3: & - & - & - & - \\ & 4: & - & 2 & 2 & - \\ & 5: & - & - & 2 & 2 \end{array}$$

We notice that the first three lines in the diagrams of the points give the diagram of the curve and that the third and the fourth row shift with one row down when we add 4 points.

The main idea in proving this result is that if we add the points of a hyperplane section of the curve the Betti diagram changes in the way described above. We will show that the bottom lines of the Betti diagram of the points are the same for two sets which are linearly equivalent. It follows that by adding a hyperplane section we get in fact points with general Betti numbers.

The same kind of analysis can be done if we start with an additional subscheme $Y \subset \mathbb{P}^n$ with $X \not\subset Y$. For $\Gamma \subset X$ a set of points we can compare the Betti diagrams for $\Gamma \cup Y$ and $X \cup Y$. In fact, the above arguments prove the results in the general setting.

In higher dimensions the picture is more involved. If P_X is the Hilbert polynomial of X, for certain values of the number of points, namely $\gamma = P_X(r)$, $r \gg 0$, we have explicit expressions for the Betti numbers in the bottom part of the diagram. They are polynomials of degree equal to dim $X - 1$. If we fix a distance from these values, let's say k (i.e. $\gamma = P_X(r) + k$) and we compute differences between corresponding Betti numbers for γ and $\gamma + 1$ then the result is independent of r for r large enough.

When X is a curve the above statement gives the periodicity result. Unsurprisingly, the idea of the proof is again to show that we can pass from $P_X(r) + k$ to $P_X(r+1) + k$ points by adding points in a hyperplane section. However, unlike in the one-dimensional case, we are far from being able to give precise bounds for where periodicity starts.

The second question we will consider in the paper is related to the Minimal Resolution Conjecture (MRC). We will see that this general situation is up to a point similar to that of general points in $\mathbb{P}^n$. More precisely, we have just two rows in the bottom part of the Betti diagram, let's say the j^{th} and the $(j+1)^{\text{th}}$, and we know the expression for $b_{i+1,j} - b_{i,j+1}$ for all i. In $\mathbb{P}^n$ (MRC) says that for general sets of points $b_{i+1,j} \cdot b_{i,j+1} = 0$ for all i. We will consider the same question when we replace $\mathbb{P}^n$ with the variety X.

In $\mathbb{P}^n$ the conjecture is known to hold for any number of points if n is small (n=2,3,4) and for all n if the number of points is very large with respect to n. On the other hand there are counterexamples in each $\mathbb{P}^n$ for all $n \geq 6$, $n \neq 9$ (see [2] for the counterexamples and also for the history of the problem).

In fact we will carry over the discussion only in the case of smooth curves. Here if we are looking to numbers of points larger than a certain bound depending on the degree, the genus and the regularity of the curve X, by the periodicity theorem we have essentially d diagrams, where d is the degree of X. Using the expression for general Betti numbers for points on curves in Proposition 2.1, one sees that all these diagrams satisfy (MRC) iff the bundle $\mathcal{M} = \Omega_{\mathbb{P}^n}(1)|_X$ on X satisfies a certain vanishing property. More precisely, for any vector bundle $\mathcal{E}$ on X, we will consider a general line bundle $\mathcal{L}$ such that $-\text{rank}(\mathcal{E}) + 1 \leq \chi(\mathcal{E} \otimes \mathcal{L}) \leq 0$. We say that $\mathcal{E}$ satisfies the property (V_1) if for such $\mathcal{L}$, $\mathrm{H}^0(\mathcal{E} \otimes \mathcal{L}) = 0$. Then X satisfies (MRC) (for every number of points large enough) iff all the exterior powers of the bundle $\mathcal{M}$ defined above satisfy (V_1). We will see that this property implies something only slightly weaker than semistability. The property is strongly connected with a question studied by Raynaud in [8], namely whether every semistable vector bundle $\mathcal{E}$ with $\chi(\mathcal{E}) \leq 0$ has $\mathrm{H}^0(\mathcal{E} \otimes \mathcal{L}) = 0$, for $\mathcal{L}$ a general line bundle of degree zero. The answer to this question is no in general, for every curve of genus $g \geq 2$, Raynaud constructed counterexamples. However it is true in some cases like rank($\mathcal{E}$) $= 2$, or rank($\mathcal{E}$) $= 3$ and X is general. This will translate in our case in the fact that points on a plane curve satisfy (MRC) and that the same is true for a general curve of genus g embedded in $\mathbb{P}^3$ such that the corresponding bundle $\Omega_{\mathbb{P}^n}(1)$ is semistable.

One case which can be easily understood independently of the theory mentioned above is that of a smooth rational curve, on which M splits as a direct sum of line bundles. The result is that X satisfies (MRC) in the sense discussed above iff the degrees of the components of $\mathcal{M}$ are "as close as possible". Let's consider a specific example. We will see that when deg $X = n$ or $n+1$ we have always (MRC) (even if X is elliptic). Therefore the first interesting case is that of a smooth rational quintic in $\mathbb{P}^3$. Then X satisfies (MRC) iff $\mathcal{M}$ has splitting type $(-1, -2, -2)$ (which happens for the general rational quintic) and it doesn't satisfy (MRC) iff $\mathcal{M}$ has splitting type $(-1, -1, -3)$ (which is equivalent to the fact that X lies on a smooth quadric).

The paper is organized as follows. In the first paragraph we prove the basic results about the shape of the Betti diagram for general points on a variety. The second paragraph deals with periodicity for curves. We prove here also a duality result relating the $(r-1)^{\text{th}}$ row in the Betti diagram for $P_X(r-1)+\alpha$ general points on the smooth curve X $(0 \le \alpha \le \deg X)$ to the r^{th} row in the Betti diagram for $P_X(r) - \alpha$ general points. In the next paragraph we apply the expression for Betti numbers as the dimension of the cohomology of a complex involving general line bundles to study (MRC) on curves. We study the connection between our condition and the one studied by Raynaud and derive some examples. In the last paragraph we return to the periodicity problem and prove the extension to the higher dimensional case.

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Notations and Conventions.

The ground field k will be algebraically closed and of arbitrary characteristic. $\mathbb{P}^n = \mathbb{P}(V)$ will be the projective n-space of hyperplanes in the $(n+1)$-dimensional vector space V over k. $\mathcal{O}_{\mathbb{P}^n}(1)$ will denote the tautological quotient bundle on $\mathbb{P}^n$. X will be a projective variety and P_X and H_X will denote its Hilbert polynomial and Hilbert series, respectively. For any closed subscheme $Y \subset \mathbb{P}^n$, $I(Y) \subset S = k[X_0, \ldots, X_n]$ will denote the saturated ideal of Y and $S(Y) := S/I(Y)$. $\mathcal{I}_Y$ will denote the sheaf of ideals of Y in $\mathbb{P}^n$ and when Y will be a subscheme of another projective scheme $Z \subset \mathbb{P}^n$, $\mathcal{I}_{Y/Z}$ will denote the sheaf of ideals of Y in Z . For any finitely generated graded S-module M with minimal resolution $F_\bullet$, where $F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{b_{i,j-i}}$ for $i \geq 0$, the Betti diagram of M has at the intersection of the j^{th} row with the i^{th} column the graded Betti number $b_{i,j}$. The index of the last nontrivial row in the Betti diagram of M is called the regularity of M and is denoted by reg M. When $M = I \subset S$ is a saturated ideal and $\mathcal{I}$ is the associated sheaf of ideals, then reg $I \leq m$ iff $\mathrm{H}^i(\mathcal{I}(m-i)) = 0$ for every $i \geq 1$. For a proof of this fact, see for example [1] , Theorem 20.18.

If $Y \subset \mathbb{P}^n$ is any closed subscheme, the Betti diagram of $S(Y)$ is called the Betti diagram of Y, the Betti numbers are denoted with $b_{i,j}(Y)$ while the regularity of $I(Y)$ is called the regularity of Y. Therefore, reg $S(Y) =$ reg $Y - 1$.

We will use freely the computation of $b_{i,j}(M)$ by Koszul cohomology for any finitely generated graded S-module M. Namely, $b_{i,j}(M)$ is the dimension over k of the homology of the following complex of k-vector spaces:

$$\wedge^{i+1} V \otimes M_{j-1} \longrightarrow \wedge^{i} V \otimes M_j \longrightarrow \wedge^{i-1} V \otimes M_{j+1}$$

(see [3] for details).

The few notions about vector bundles which appear in the third paragraph without definition can be found in [8].

1. General Sets of Points on Projective Varieties.

Proposition 1.1. *Let $X \subset \mathbb{P}^n$ be a projective variety with* dim $X \geq 1$. *Then if Γ is a general set of γ (distinct) points on X, the Hilbert function of $S(\Gamma)$ is given by:*

$$H_{S(\Gamma)}(r) = \min\{H_{S(X)}(r), \gamma\}.$$

Proof. An equivalent formulation is that

$$\dim_k I(\Gamma)/I(X) = \max\{H_{S(X)}(r) - \gamma, 0\}$$

and this follows easily by induction on the number of points. Notice that because reg $\Gamma \leq \gamma$, we have to put only a finite number of conditions (namely, for $r \leq \gamma - 1$) and the proposition is proved. $\square$

We will say that a set Γ of distinct points on X is in general position on X if its Hilbert function is given by the formula in Proposition 1.1.

Corollary 1.2. *Let $X \subset \mathbb{P}^n$ be a projective variety with* $\dim X \geq 1$ *and* $Y \subset \mathbb{P}^n$ *an arbitrary closed subscheme such that* $X \not\subset Y$. *If* $\Gamma \subset X$ *is a set of* γ *points in general position on* X, *with* $\gamma \geq H_{S(X)}(r)$, *then* $I(\Gamma \cup Y)_j = I(X \cup Y)_j$ *for every* $j \leq r$. *Moreover, if* X *is a curve of degree* d *and* $\gamma \geq j \cdot d + 1$ *the same conclusion holds for any subscheme of degree* γ.

Proof. Only the last assertion has to be justified, but it is a consequence of Bezout's theorem. □

The next result shows that if we have a large number of general points on a variety then we can recover the Betti diagram of the variety from the Betti diagram of the points .

Proposition 1.3.

i) Let $X \subset \mathbb{P}^n$ *be a projective variety with* $\dim X \geq 1$ *and* $Y \subset \mathbb{P}^n$ *an arbitrary closed subscheme such that* $X \not\subset Y$. *For every* $r \geq 0$, *if* $\Gamma \subset X$ *is a subset of* γ *general points, where* $\gamma \geq H_{S(X)}(r+1)$, *then*

$$b_{i,j}(\Gamma \cup Y) = b_{i,j}(X \cup Y)$$

for every i *and every* $j \leq r$.

ii) If X *is a curve of degree* d *and* $\Gamma \subset X$ *is a subscheme of degree* $\gamma \geq d(r+1)+1$, *then*

$$b_{i,j}(\Gamma \cup Y) = b_{i,j}(X \cup Y)$$

for every i *and every* $j \leq r$.

In particular, if $r = \operatorname{reg}(X \cup Y)$, we get that the first $r+1$ rows of the Betti diagram of $\Gamma \cup Y$ give the Betti diagram of $X \cup Y$.

Proof. i) and ii): Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \wedge^{i+1} V \otimes S(X \cup Y)_{j-1} \arrow[r] \arrow[d] & \wedge^{i} V \otimes S(X \cup Y)_{j} \arrow[r] \arrow[d] & \wedge^{i-1} V \otimes S(X \cup Y)_{j+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \wedge^{i+1} V \otimes S(\Gamma \cup Y)_{j-1} \arrow[r] & \wedge^{i} V \otimes S(\Gamma \cup Y)_{j} \arrow[r] & \wedge^{i-1} V \otimes S(\Gamma \cup Y)_{j+1} \arrow[r] & 0
\end{tikzcd}$$

where the vertical morphisms are the natural ones. Because the above horizontal sequences compute $b_{i,j}(S(X \cup Y))$ and $b_{i,j}(S(\Gamma \cup Y))$, respectively and because the vertical maps are identities for $j \leq r$ (by Corollary 1.2), this concludes the proof of both (i) and (ii). □

We express now the Betti numbers in the bottom part of the diagram of Γ as Betti numbers corresponding to a graded module which depends only on the ideal sheaf of Γ in X and the ideal sheaf of $Y \cap X$ in X.

Suppose that we are in one of the cases (i) or (ii) above, with $r = m - 1$, $m = \operatorname{reg}(X \cup Y)$. Consider the following exact sequence:

$$0 \longrightarrow I(\Gamma \cup Y)/I(X \cup Y) \longrightarrow S(X \cup Y) \longrightarrow S(\Gamma \cup Y) \longrightarrow 0,$$

and let $F_\bullet$ and $G_\bullet$ be the minimal free resolutions of $I(\Gamma \cup Y)/I(X \cup Y)$ and $S(X \cup Y)$, respectively. Let $u_\bullet : F_\bullet \longrightarrow G_\bullet$ be a morphism of complexes of graded modules extending the inclusion in the above exact sequence. Then we have:

Proposition 1.4. *With the above notations, the cone $C(u_\bullet)$ of $u_\bullet$ is the minimal free resolution of $S(\Gamma \cup Y)$. Furthermore, if $\Gamma \cap Y = \emptyset$, then*

$$b_{i,j}(\Gamma \cup Y) = b_{i-1,j+1}\Big(\bigoplus_{l \geq 0} \mathrm{H}^0(\mathcal{I}_{\Gamma/X} \otimes \mathcal{I}_{Y \cap X/X}(l))\Big)$$

for every i and every $j \geq m$.

Proof. From the long exact sequence in homology for the cone of $u_\bullet$ we deduce that $C(u_\bullet)$ is a (free) resolution of $S(\Gamma \cup Y)$. On the other hand, for every i, $F_i = \bigoplus_{j \geq m+1} S(-i-j)^{\beta_{i,j}}$ (because $(I(\Gamma \cup Y)/I(X \cup Y))_l = 0$ for every $l \leq m$) and $G_i = \bigoplus_{j \leq m-1} S(-i-j)^{\beta'_{i,j}}$, (because $S(X \cup Y)$ is $(m-1)$-regular). This shows that $u_\bullet$ is given by matrices with entries in the ideal $(X_0, \ldots, X_n)$, and therefore $C(u_\bullet)$ is minimal. The only assertion which still has to be checked is that $(I(\Gamma \cup Y)/I(X \cup Y))_l = \mathrm{H}^0(\mathcal{I}_{\Gamma/X}(l))$, for every $l \geq m$. Let's consider the following diagram with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & I(X \cup Y)_l \arrow[r] \arrow[d] & I(\Gamma \cup Y)_l \arrow[r] \arrow[d] & (I(\Gamma \cup Y)/I(X \cup Y))_l \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^0(\mathcal{I}_{X \cup Y}(l)) \arrow[r] & \mathrm{H}^0(\mathcal{I}_{\Gamma \cup Y}(l)) \arrow[r] & \mathrm{H}^0(\mathcal{I}_{\Gamma \cup Y/X \cup Y}(l)) \arrow[r] & 0
\end{tikzcd}$$

for $l \geq m - 1$ (the exactness of the bottom sequence follows from the fact that $I(X \cup Y)$ is m-regular, and therefore $\mathrm{H}^1(\mathcal{I}_{X \cup Y}(l)) = 0$ for $l \geq m - 1$). The first two vertical maps in the diagram are isomorphisms because $I(X \cup Y)$ and $I(\Gamma \cup Y)$ are saturated ideals. By the 5-lemma we get that the third vertical map is also an isomorphism. Moreover, because Y does not meet Γ, $\mathcal{I}_{\Gamma \cup Y/X \cup Y} = \mathcal{I}_{\Gamma/X} \otimes \mathcal{I}_{Y \cap X/X}$, concluding the proof of the proposition. □

Remark. *In the statement of Propositions 1.3 and 1.4 above, when Y is the empty set, the condition that $\gamma \geq H_{S(X)}(m)$ (where $m = \operatorname{reg}(X)$) can be written also as $\gamma \geq P_X(m)$, where P_X is the Hilbert polynomial of X, using the fact that $P_X(l) = H_{S(X)}(l)$, for $l \geq m-1$.*

We have proved therefore that for a large number of general points on a variety the Betti diagram of the points contains as its first rows the Betti diagram of the variety. From now on, we will study the residual part. For the sake of simplicity we will restrict ourselves to the case where Y is the empty set, although it is easy to obtain analogous results when Y is nonempty. We will consider the additional scheme Y only in the second paragraph, in order to prove the periodicity theorem in its strong form, as it was stated in [6].

The following result shows that the residual part consists of at most two rows, and in some cases, only one row. In addition, it computes the regularity of the set of points.

Proposition 1.5. *Let $X \subset \mathbb{P}^n$ a projective variety with* $\dim X \geq 1$, $\operatorname{reg} X = m$ *and with Hilbert polynomial P_X. Let $\Gamma \subset X$ be a set of γ points in general position, where $P_X(r-1) \leq \gamma \leq P_X(r) - 1$ and $r \geq m+1$. Then:*

i) $b_{i,j}(\Gamma) = 0$, for every i and every j such that $m \leq j \leq r-2$ and $b_{1,r-1}(\Gamma) \neq 0$;

ii) $b_{i,j}(\Gamma) = 0$, for every i and every j such that $j \geq r+1$. Moreover, $b_{i,r}(\Gamma) = 0$ for every i, iff $\gamma = P_X(r-1)$.

Proof. i): By Corollary 1.2 we get that $(I(\Gamma)/I(X))_l \neq 0$ iff $l \geq r$. On the other hand Proposition 1.4, gives $b_{i,j}(\Gamma) = b_{i-1,j+1}(I(\Gamma)/I(X))$ for every i and every $j \geq m$, so i) is proved.
(ii): $b_{i,j}(\Gamma) = 0$ for every i and every $j \geq s$, for some $s \geq r$ iff $b_{i,j}(\bigoplus_l \mathrm{H}^0(\mathcal{I}_{\Gamma/X}(l))) = 0$ for every i and every $j \geq s+1$ (by Proposition 1.4). But the last fact is equivalent to $\operatorname{reg}(\bigoplus_l H^0(\mathcal{I}_{\Gamma/X}(l))) \leq s$ and therefore to $\mathrm{H}^1(\mathcal{I}_{\Gamma/X}(s-1)) = 0$, since $\dim \Gamma = 0$ and $s \geq r \geq \operatorname{reg} X + 1$. Consider the following exact sequence:

$$0 \longrightarrow \mathcal{I}_{\Gamma/X} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_\Gamma \longrightarrow 0$$

and the first part of the long exact sequence in cohomology associated:

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(\mathcal{I}_{\Gamma/X}(s-1)) \longrightarrow \mathrm{H}^0(\mathcal{O}_X(s-1)) \longrightarrow \mathrm{H}^0(\mathcal{O}_\Gamma(s-1)) \longrightarrow \\ \longrightarrow \mathrm{H}^1(\mathcal{I}_{\Gamma/X}(s-1)) \longrightarrow \mathrm{H}^1(\mathcal{O}_X(s-1)) = 0. \end{aligned}$$

Because $\mathrm{h}^0(\mathcal{O}_\Gamma(s-1)) = \gamma$ and $\mathrm{h}^0(\mathcal{O}_X(s-1)) = P_X(s-1)$ ($s \geq r \geq m+1$) we see that $\mathrm{H}^1(\mathcal{I}_{\Gamma/X}(s-1)) = 0$ iff $\dim_k(I(\Gamma)/I(X)) = P_X(s-1) - \gamma$, which by Proposition 1.1 is equivalent to the fact that $\gamma \leq P_X(s-1)$, i.e. what we had to prove. □

Remark. *When X is a curve, the same proof gives the statement for any set of γ points if we assume that $\gamma \geq m \cdot \deg X + 1$.*

Since we know the Hilbert function of I_Γ/I_X for any Γ as in Proposition 1.5 and because the Betti numbers in the two bottom rows of the diagram of Γ are equal to corresponding Betti numbers of I_Γ/I_X, we can provide an explicit expression for $b_{i+1,r-1}(\Gamma) - b_{i,r}(\Gamma)$. We will give this expression in the next proposition and we will see that it is polynomial in r of degree equal to $\dim X - 1$. The formula comes from the computation of the Hilbert function of I_Γ/I_X using the minimal free resolution.

Proposition 1.6. *Let $X \subset \mathbb{P}^n$ be a projective variety with $d = \dim X \geq 1$, $\operatorname{reg} X = m$ and with Hilbert polynomial P_X. Let γ be an integer such that $P_X(r-1) \leq \gamma \leq P_X(r) - 1$, for some r, with $r \geq m+1$ and let Γ be a set of γ points on X in general position. Then:*

$$b_{i+1,r-1}(\Gamma) - b_{i,r}(\Gamma) = \sum_{l=0}^{d-1}(-1)^l \binom{n-l-1}{i-l} \Delta^{l+1} P_X(r+l) - \\ - \binom{n}{i}(\gamma - P_X(r-1)).$$

Proof. From Proposition 1.1 we get that

$$H(s) = \begin{cases} 0, & \text{if } s \leq r-1; \\ P_X(s) - \gamma, & \text{if } s \geq r, \end{cases}$$

where $H = H_{I_\Gamma/I_X}$ is the Hilbert function of I_Γ/I_X.

We need the following lemma.

Lemma. *With the above notations, for any $q \geq 1$,*

$$\Delta^q H(s) = \begin{cases} 0, & \text{if } s \leq r-1; \\ \sum_{j=0}^{i}(-1)^j \binom{q-i-2+j}{j} \Delta^{i+1-j} P_X(r+i-j) + \\ \quad + (-1)^{i-1}\binom{q-1}{i}\alpha, & \text{if } s = r+i, \\ & 0 \leq i \leq q-1; \\ \Delta^q P_X(s), & \text{if } s \geq r+q, \end{cases}$$

where $\alpha = \gamma - P_X(r-1)$ and we use the convention $\binom{-1}{0} = 1$.

Proof of the lemma. We will proceed by induction on q. For $q = 1$, the assertion is trivial.

If the assertion is true for q, it is clear that $\Delta^{q+1}H(s) = 0$ for $s \leq r - 1$ and $\Delta^{q+1}H(s) = \Delta^{q+1}P_X(s)$ for $s \geq r + q$. If $s = r + i$, $1 \leq i \leq q - 1$, then:

$$\begin{aligned}
\Delta^{q+1}H(s) &= \Delta^q H(r+i) - \Delta^q H(r+i-1) = \\
&= \sum_{j=0}^{i} (-1)^j \binom{q-i-2+j}{j} \Delta^{i+1-j} P_X(r+i-j) + (-1)^{i-1}\binom{q-1}{i}\alpha \\
&- \sum_{j=0}^{i-1} (-1)^j \binom{q-i-1+j}{j} \Delta^{i-j} P_X(r+i-1-j) - (-1)^{i-2}\binom{q-1}{i-1}\alpha \\
&= \sum_{j=0}^{i} (-1)^j \left(\binom{q-i-2+j}{j} + \binom{q-i-2+j}{j-1}\right) \Delta^{i+1-j} P_X(r+i-j) \\
&\qquad + (-1)^{i-1}\left(\binom{q-1}{i} + \binom{q-1}{i-1}\right)\alpha \\
&= \sum_{j=0}^{i} (-1)^j \binom{q-i-1+j}{j} \Delta^{i+1-j} P_X(r+i-j) + (-1)^{i-1}\binom{q}{i}\alpha.
\end{aligned}$$

We have also $\Delta^{q+1}H(r) = \Delta P_X(r) - \alpha$ and

$$\begin{aligned}
\Delta^{q+1}H(r+q) &= \Delta^q H(r+q) - \Delta^q H(r+q-1) \\
&= \Delta^q P_X(r+q) - \left(\Delta^q P_X(r+q-1) + (-1)^q \alpha\right) \\
&= \Delta^{q+1} P_X(r+q) + (-1)^{q+1}\alpha,
\end{aligned}$$

which completes the induction step. $\square$

We return to the *proof of the proposition*. Using the notations in the lemma, we have:

$$\Delta^{d+1}H(s) = \begin{cases} 0, & \text{if } s \leq r-1\,; \\ \sum_{j=0}^{k}(-1)^j \binom{d-k-1+j}{j} \Delta^{k+1-j} P_X(r+k-j) + & \\ \quad + (-1)^{k-1}\binom{d}{k}\alpha, & \text{if } s = r+k, \\ & \ 0 \leq k \leq d\,; \\ 0, & \text{if } s \geq r+d+1, \end{cases}$$

where we used the fact that deg $P_X = d$ and therefore $\Delta^{d+1} P_X = 0$.

On the other hand, by Propositions 1.4 and 1.5, if $F_\bullet$ is the minimal free resolution of I_Γ/I_X, then $F_i = S(-r-i)^{b_{i+1,r-1}(\Gamma)} \oplus S(-r-i-1)^{b_{i+1,r}(\Gamma)}$, for any i, $0 \le i \le n-1$ and $F_n = 0$. Therefore

$$\sum_{s\ge 0} \Delta^{d+1} H(s) t^s = \frac{1}{(1-t)^{n-d}} \sum_{i=0}^{n-1} (-1)^i \left(b_{i+1,r-1}(\Gamma) t^{r+i} + b_{i+1,r}(\Gamma) t^{r+i+1} \right).$$

This can be rewritten as

$$(1-t)^{n-d} \sum_{s\ge 0} \Delta^{d+1} H(s) t^s = \sum_{i=0}^{n} (-1)^i \left(b_{i+1,r-1}(\Gamma) - b_{i,r}(\Gamma) \right) t^{r+i},$$

which implies:

$$b_{i+1,r-1}(\Gamma) - b_{i,r}(\Gamma) = (-1)^i \sum_{k=0}^{d} (-1)^{i-k} \binom{n-d}{i-k} \Delta^{d+1} H(r+k).$$

Using the expression for $\Delta^{d+1} H(r+k)$ we rewrite this as :

$$\begin{aligned}
b_{i+1,r-1}(\Gamma) - b_{i,r}(\Gamma) &= \\
&\sum_{k=0}^{d} (-1)^k \binom{n-d}{i-k} \left(\sum_{j=0}^{k} (-1)^j \binom{d-k-1+j}{j} \Delta^{k+1-j} P_X(r+k-j) \right) \\
&\qquad + \sum_{k=0}^{d} (-1)^k \binom{n-d}{i-k} (-1)^{k-1} \binom{d}{k} \alpha \\
&= \sum_{0\le j\le k\le d} (-1)^{k-j} \binom{d-(k-j)-1}{j} \binom{n-d}{i-k} \Delta^{k+1-j} P_X(r+k-j) \\
&\qquad - \left(\sum_{k=0}^{d} \binom{n-d}{i-k} \binom{d}{k} \right) \alpha \\
&= \sum_{0\le l\le d} (-1)^l \Delta^{l+1} P_X(r+l) \left(\sum_{j=0}^{d-l} \binom{d-l-1}{j} \binom{n-d}{i-j-l} \right) \\
&\qquad - \left(\sum_{k=0}^{d} \binom{n-d}{i-k} \binom{d}{k} \right) \alpha.
\end{aligned}$$

But
$$\sum_{j=0}^{d-l}\binom{d-l-1}{j}\binom{n-d}{i-j-l}=\binom{n-l-1}{i-l},$$
identity obtained by comparing the coefficients of X^{i-l} in $(X+1)^{d-l-1}(X+1)^{n-d}=(X+1)^{n-l-1}$. We have also
$$\sum_k\binom{n-d}{i-k}\binom{d}{k}=\binom{n}{i},$$
by comparing the coefficients of X^i in $(X+1)^{n-d}(X+1)^d=(X+1)^n$. This completes the proof of the proposition. $\square$

Remark. *The above statement also remains true for any set of points if X is a curve and $\gamma \geq m\cdot \deg X+1$.*

We are especially interested in the values of the Betti numbers for general sets Γ of γ points on X, with $\gamma \geq P_X(m)$, $m=\operatorname{reg} X$. Since Betti numbers are upper semicontinuous functions, they get their minimum on an open subset of $X^\gamma \setminus \bigcup_{i\neq j}\{x|x_i=x_j\}$ (because reg Γ is bounded we have to deal with only a finite set of Betti numbers once γ is fixed). We will say that Γ has general Betti numbers if $b_{i,j}(\Gamma)$ is equal to the minimal value $b_{i,j}(\gamma)$ for every i and j. Since we can compute the Hilbert function of Γ from its Betti numbers, it follows that if Γ has general Betti numbers then it is in general position on X.

Propositions 1.5 and 1.6 show that our situation bears a certain similarity with the particular case $X=\mathbb{P}^n$ which was thoroughly studied (see [2] for a detailed account of the problem). In that case, from Proposition 1.4 above follows that when $\binom{r-1+n}{n}\leq\gamma\leq\binom{r+n}{n}$, $r\geq 1$ then besides $b_{0,0}=1$, the only nontrivial entries in the Betti diagram of general sets of points in $\mathbb{P}^n$ are in the lines $r-1$ and r. Then the Minimal Resolution Conjecture (MRC) asserts that $b_{i+1,r-1}(\gamma)\cdot b_{i,r}(\gamma)=0$ for every i. In [2] there are constructed counterexamples to (MRC), the simplest of which is that of 11 points in $\mathbb{P}^6$.

Similarly, using the previous notations, we will say that for an arbitrary variety X and γ with $P_X(r-1)\leq\gamma\leq\mathbb{P}_X(r)$, $r\geq m+1$, (MRC) holds for γ on X if $b_{i+1,r-1}(\gamma)\cdot b_{i,r}(\gamma)=0$ for every i. An other way to express this is given by Proposition 1.5. From the formula for $b_{i+1,r-1}(\gamma)-b_{i,r}(\gamma)=Q_{i,r}(\gamma)$ we get the following lower bounds for the Betti numbers: if $Q_{i,r}(\gamma)\geq 0$ then $b_{i,r}(\gamma)\geq 0$, $b_{i+1,r-1}(\gamma)\geq Q_{i,r}(\gamma)$ while if $Q_{i,r}(\gamma)\leq 0$ then $b_{i+1,r-1}(\gamma)\geq 0$, $b_{i,r}(\gamma)\geq -Q_{i,r}(\gamma)$. Then (MRC) for γ simply says that all the general Betti numbers are equal to these lower bounds.

Let's consider now a couple of trivial examples. We will return to this problem in the third paragraph.

Example 1. $\gamma = P_X(r-1)$, $r \geq m+1$. *In this case (MRC) holds for every variety X and any set Γ in general position, since by Proposition 1.5, $b_{i,r}(\Gamma) = 0$ for every i.*

Example 2. $\gamma = P_X(r)-1$, $r \geq m+1$. *(MRC) holds for any nondegenerate variety X and any set Γ in general position on X. Indeed in this case $(I_\gamma/I_X)_t = 0$ for $t \leq r-1$, $\dim_k(I_\Gamma/I_X)_r = 1$ and because of nondegeneracy there are no linear relations on the generator of $(I_\Gamma/I_X)_r$. Therefore $b_{1,r-1}(\gamma) = 1$ and $b_{i,r-1}(\gamma) = 0$ for $i \geq 2$, so that (MRC) holds.*

The following proposition and its proof will be used in the study of Betti diagrams of points on varieties X with $\dim X \geq 2$. It describes what happens with the bottom two lines in the diagram of a set Γ as above when we add an other point: the numbers in the $(r-1)^{\text{th}}$ row are decreasing, while the numbers in the r^{th} row are increasing. Moreover, the consecutive differences are decreasing or increasing at the same time.

Proposition 1.7. *Let $X \subset \mathbb{P}^n$ be a projective variety with* $\dim X \geq 2$. *Let m be the regularity of X and P_X its Hilbert polynomial. Let γ be a number such that $P_X(r-1) \leq \gamma \leq P_X(r)-1$, for some $r \geq m+1$. Let Γ be a set of γ points on X and $P \in X \setminus \Gamma$ such that both Γ and $\Gamma \cup P$ are in general position. Then:*

i) $b_{i,r-1}(\Gamma) \geq b_{i,r-1}(\Gamma \cup P)$ and $b_{i,r}(\Gamma) \leq b_{i,r}(\Gamma \cup P)$ for every i.

ii) If $\gamma \leq P(r)-2$ and $Q \in X \setminus (\Gamma \cup P)$ is such that $\Gamma \cup Q$ and $\Gamma \cup P \cup Q$ are also in general position, then $b_{i,r-1}(\Gamma) - b_{i,r-1}(\Gamma \cup P) \geq b_{i,r-1}(\Gamma \cup Q) - b_{i,r-1}(\Gamma \cup P \cup Q)$ and $b_{i,r}(\Gamma \cup P) - b_{i,r}(\Gamma) \leq b_{i,r}(\Gamma \cup P \cup Q) - b_{i,r}(\Gamma \cup Q)$.

Proof. Consider the following commutative diagram with exact columns:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\wedge^i V \otimes \mathrm{H}^0(\mathcal{I}_{\Gamma\cup P/X}(r)) \arrow[r, "f'"] \arrow[d] & \wedge^{i-1} V \otimes H^0(\mathcal{I}_{\Gamma\cup P/X}(r+1)) \arrow[r, "f''"] \arrow[d] & \wedge^{i-2} V \otimes H^0(\mathcal{I}_{\Gamma\cup P/X}(r+2)) \arrow[d] \\
\wedge^i V \otimes \mathrm{H}^0(\mathcal{I}_{\Gamma/X}(r)) \arrow[r, "g'"] \arrow[d] & \wedge^{i-1} V \otimes \mathrm{H}^0(\mathcal{I}_{\Gamma/X}(r+1)) \arrow[r, "g''"] \arrow[d] & \wedge^{i-2} \otimes \mathrm{H}^0(\mathcal{I}_{\Gamma/X}(r+2)) \arrow[d] \\
\wedge^i V \otimes k(P) \arrow[r, "h'"] \arrow[d] & \wedge^{i-1} V \otimes k(P) \arrow[r, "h''"] \arrow[d] & \wedge^{i-2} V \otimes k(P) \arrow[d] \\
\wedge^i V \otimes \mathrm{H}^1(\mathcal{I}_{\Gamma\cup P/X}(r)) & \wedge^{i-1} V \otimes \mathrm{H}^1(\mathcal{I}_{\Gamma\cup P/X}(r+1)) & \wedge^{i-2} V \otimes \mathrm{H}^1(\mathcal{I}_{\Gamma\cup P/X}(r+2))
\end{tikzcd}$$

Because $\Gamma \cup P$ is $(r+1)$-regular and X is $(r-1)$-regular ($m \leq r-1$), we get that $\mathrm{H}^1(\mathcal{I}_{\Gamma\cup P/X}(j)) = 0$ for every $j \geq r$ and therefore the above diagram is a short exact sequence of complexes. Let $B_{i+1,r-1}(\Gamma \cup P) := \ker f'$, $B_{i+1,r-1}(\Gamma) := \ker g'$, $B_{i,r}(\Gamma \cup P) := \ker f''/\operatorname{Im} f'$, $B_{i,r}(\Gamma) := \ker g''/\operatorname{Im} g'$. Because $\mathrm{H}^0(\mathcal{I}_{\Gamma\cup P/X}(r-1)) = (I_{\Gamma\cup P}/I_X)_{r-1} = 0$ and $\mathrm{H}^0(\mathcal{I}_{\Gamma/X}(r-1)) = (I_\Gamma/I_X)_{r-1} = 0$, we get that $\dim_k B_{i+1,r-1}(\Gamma \cup P) = b_{i+1,r-1}(\Gamma \cup P)$ and $\dim_k B_{i+1,r-1}(\Gamma) = b_{i+1,r-1}(\Gamma)$. We have also $\dim_k B_{i,r}(\Gamma \cup P) = b_{i,r}(\Gamma \cup P)$ and $\dim_k B_{i,r}(\Gamma) = b_{i,r}(\Gamma)$. To get the above relations, we used Proposition 1.4 and the computation of Betti numbers via Koszul cohomology.

In addition, we have $\mathrm{H}^1(\mathcal{I}_{\Gamma\cup P/X}(j)) = 0$ for $j \geq r$. Indeed, let's consider the short exact sequence of sheaves :

$$0 \longrightarrow \mathcal{I}_{\Gamma\cup P/X}(j) \longrightarrow \mathcal{O}_X(j) \longrightarrow \mathcal{O}_{\Gamma\cup P}(j) \longrightarrow 0$$

and the first terms of its associated long exact sequence in cohomology:

$$0 \longrightarrow \mathrm{H}^0(\mathcal{I}_{\Gamma\cup P/X}(j)) \longrightarrow \mathrm{H}^0(\mathcal{O}_X(j)) \longrightarrow \mathrm{H}^0(\mathcal{O}_{\Gamma\cup P}(j)) \longrightarrow$$

$$\longrightarrow \mathrm{H}^1(\mathcal{I}_{\Gamma\cup P/X}(j)) \longrightarrow \mathrm{H}^1(\mathcal{O}_X(j)) = 0$$

$\mathrm{H}^1(\mathcal{O}_X(j)) = 0$ since $j \geq r \geq \operatorname{reg}(X) + 1$. But $\operatorname{card}(\Gamma \cup P) = \gamma + 1$, so that $\dim_k \mathrm{H}^0(\mathcal{O}_{\Gamma\cup P}(j)) = \gamma + 1$. On the other hand, using Proposition 1.1 and the fact that $\mathrm{H}^0(\mathcal{I}_{\Gamma\cup P}(j)) = (I_{\Gamma\cup P/X})_j$ and $\mathrm{H}^0(\mathcal{O}_X(j)) = S(X)_j$ for $j \geq r \geq \operatorname{reg}(X) + 1$ (see, for example, the proof of Proposition 1.4), we get:

$$\dim_k \mathrm{H}^0(\mathcal{O}_X(j)) - \dim_k \mathrm{H}^0(\mathcal{I}_{\Gamma\cup P/X}(j)) = \gamma + 1.$$

In conclusion, $\mathrm{H}^1(\mathcal{I}_{\Gamma\cup P/X}(j)) = 0$, for $j \geq r$.

The third row in the diagram is just a part of the Koszul complex given by the surjective map which is the evaluation at P, $v : V = \mathrm{H}^0(\mathcal{O}(1)) \longrightarrow k(P)$. Therefore, if $W := \ker v$, $\dim_k W = n$, $\ker h' = \wedge^i W$ and $\ker h'' = \operatorname{Im} h'$. The long exact sequence in homology of the above exact sequence of complexes can be written as follows:

$$0 \longrightarrow B_{i+1,r-1}(\Gamma \cup P) \longrightarrow B_{i+1,r-1}(\Gamma) \longrightarrow \wedge^i W \longrightarrow B_{i,r}(\Gamma \cup P) \longrightarrow$$
$$\longrightarrow B_{i,r}(\Gamma) \longrightarrow 0.$$

This proves that $b_{i+1,r-1}(\Gamma \cup P) \leq b_{i+1,r-1}(\Gamma)$ and $b_{i,r}(\Gamma \cup P) \geq b_{i,r}(\Gamma)$, for every i.

ii) We will consider the same diagram as at the beginning of the proof of i) but for $\Gamma \cup Q$, instead of Γ. The natural inclusions $\mathcal{I}_{\Gamma\cup Q\cup P} \subset \mathcal{I}_{\Gamma\cup P}$ and

$\mathcal{I}_{\Gamma\cup Q} \subset \mathcal{I}_\Gamma$ induce a morphism of exact sequences of complexes and therefore a commutative diagram for the long exact sequences in homology:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_{i+1,r-1}(\Gamma\cup P) & \xrightarrow{u_1} & B_{i+1,r-1}(\Gamma) & \longrightarrow & \wedge^i W & \longrightarrow \\
 & & \uparrow & & \uparrow & & & \\
0 & \longrightarrow & B_{i+1,r-1}(\Gamma\cup Q\cup P) & \xrightarrow{w_1} & B_{i+1,r-1}(\Gamma\cup Q) & \longrightarrow & \wedge^i W & \longrightarrow
\end{array}$$

$$\begin{array}{cccccc}
\longrightarrow & B_{i,r}(\Gamma\cup P) & \xrightarrow{u_2} & B_{i,r}(\Gamma) & \longrightarrow & 0 \\
 & \uparrow & & \uparrow & & \\
\longrightarrow & B_{i,r}(\Gamma\cup Q\cup P) & \xrightarrow{w_2} & B_{i,r}(\Gamma\cup Q) & \longrightarrow & 0
\end{array}$$

where the third vertical isomorphism is the identity. Therefore $\dim_k(\ker u_2) \leq \dim_k(\ker w_2)$ and $\dim_k(\operatorname{coker} u_1) \geq \dim_k(\operatorname{coker} w_1)$, which completes the proof of the proposition. $\square$

2. The Periodicity Theorem.

In this section we will be concerned with general sets of points on an integral curve X of degree d and arithmetic genus g in $\mathbb{P}^n$. Let H_X be the Hilbert function of X. First, we will draw some conclusions from the results of the previous section. Y will be an arbitrary closed subscheme of P^n with $X \not\subset Y$ and let $m = \operatorname{reg}(X \cup Y)$.

Proposition 2.1. *With the above notations, if $\Gamma \subset X$ is an effective Cartier divisor of degree γ not meeting Y and such that* $\deg \gamma \geq d \cdot m + 1$ *then:*

i) $b_{i,j}(\Gamma \cup Y) = b_{i,j}(X \cup Y)$ for any i and every $j \leq m-1$. Moreover $b_{i,j}(\Gamma \cup Y)$ equals the dimension over k of the homology of the complex of vector spaces:

$$\wedge^i V \otimes \mathrm{H}^0(\mathcal{O}(-\Gamma) \otimes \mathcal{I}_{Y\cap X/X} \otimes \mathcal{O}_X(j)) \longrightarrow$$
$$\longrightarrow \wedge^{i-1} V \otimes \mathrm{H}^0(\mathcal{O}(-\Gamma) \otimes \mathcal{I}_{Y\cap X/X} \otimes \mathcal{O}_X(j+1)) \longrightarrow$$
$$\longrightarrow \wedge^{i-2} V \otimes \mathrm{H}^0(\mathcal{O}(-\Gamma) \otimes \mathcal{I}_{Y\cap X/X} \otimes \mathcal{O}_X(j+2))$$

for any i and every $j \geq m$. The same result is true if Γ is in general position on X with $(\operatorname{Sing}(X) \cup Y) \cap \Gamma = \emptyset$ *and* $\deg \Gamma \geq H_X(m)$.

ii) If H is a hyperplane section of X not meeting Y then $b_{i,j}((\Gamma + H) \cup Y) = b_{i,j-1}(\Gamma \cup Y)$ for any i and every $j \geq m$, while $b_{i,m}((\Gamma + H) \cup Y) = 0$ for every i.

Proof. i) : We have just to apply Propositions 1.3 and 1.4. The expression for $b_{i,j}(\Gamma \cup Y) = b_{i-1,j+1}(\bigoplus_l \mathrm{H}^0(\mathcal{I}_{\Gamma/X} \otimes \mathcal{I}_{Y\cap X/X}(l)))$ is obtained by computing the Betti numbers via Koszul cohomology. We remark that in this case $\mathcal{I}_{\Gamma/X} = \mathcal{O}(-\Gamma)$. When Γ is in general position we conclude using the same results.

ii) : The statement follows from (i) once we notice that the above complex computing $b_{i,j}(\Gamma \cup Y)$ remains unchanged if we replace Γ by $\Gamma + H$ and j by $j + 1$. The fact that $b_{i,m}(\Gamma \cup Y) = 0$ for every i follows from Proposition 1.3 ii). □

In analogy with the notation in the previous paragraph, we will denote by $b_{i,j}(\gamma; Y)$ the minimal value of the Betti numbers for $\Gamma \cup Y$, where $\Gamma \subset X$ is a set of γ points (value which is obtained when Γ is general). Let m_1 be $\max\{m, \operatorname{reg} X\}$.

Theorem 2.2. (Periodicity Theorem). *With the above notations, if* $\gamma \geq \max\{d \cdot m_1 + 1 - g, g\}$*, then*

$$b_{i,j+1}(\gamma + d; Y) = b_{i,j}(\gamma; Y)$$

for every i *and every* $j \geq m + 1$ *and* $b_{i,m}(\gamma + d; Y) = 0$ *for every* i.

Proof. The last assertion follows directly from Proposition 1.3 i). We use the fact that $H(m_1) = d \cdot m_1 + 1 - g$, because $m_1 \geq \operatorname{reg} X$.

From Proposition 2.1 i), we see that $b_{i,j}(\Gamma \cup Y)$ does not depend on Γ, but only on $\mathcal{O}(-\Gamma)$, when Γ is an effective Cartier divisor. But because $\gamma \geq g$, it is the same thing considering general sets of γ points or general line bundles on X of degree $-\gamma$ (the map $(X \setminus \mathrm{Sing}(X))^\gamma \longrightarrow \mathrm{Pic}^{-\gamma}(X)$, given by $(x_1, \ldots, x_\gamma) \longrightarrow \mathcal{O}(-x_1 - \ldots - x_\gamma)$ is dominant). Therefore $b_{i,j}(\gamma; Y)$ is equal to the dimension over k of the homology of the following complex of k-vector spaces:

$$\wedge^i V \otimes \mathrm{H}^0(\mathcal{L} \otimes \mathcal{O}_X(j) \otimes \mathcal{I}_{Y\cap X/X}) \longrightarrow \wedge^{i-1} V \otimes \mathrm{H}^0(\mathcal{L} \otimes \mathcal{O}_X(j+1) \otimes \mathcal{I}_{Y\cap X/X}) \to$$

$$\to \wedge^{i-2} V \otimes \mathrm{H}^0(\mathcal{L} \otimes \mathcal{O}_X(j+2) \otimes \mathcal{I}_{Y\cap X/X})$$

for any i and any $j \geq m$, where $\mathcal{L}$ is a general line bundle of degree $-\gamma$. The above complex remains unchanged by replacing $\mathcal{L}$ with $\mathcal{L} \otimes \mathcal{O}_X(-1)$ and j by $j + 1$.

On the other hand, the morphism $\phi : \mathrm{Pic}^{-\gamma-d}(X) \longrightarrow \mathrm{Pic}^{-\gamma}(X)$ given by multiplication with $\mathcal{O}_X(1)$ is an isomorphism and therefore the image by ϕ of the open set of $\mathrm{Pic}^{-\gamma-d}(X)$ on which can be computed $b_{i,j}(\gamma + d; Y)$ will intersect the open subset of $\mathrm{Pic}^{-\gamma}(X)$ on which can be computed $b_{i,j-1}(\gamma; Y)$. From the above facts we get the conclusion of the theorem. □

Remark 1. *From the proof of Theorem 2.2 we see also that if $Y \cap X$ is an effective Cartier divisor on X, then the set of residual parts which appear does not depend on Y. Indeed, if* $\deg(Y \cap X) = e$*, for a general line bundle $\mathcal{L}$ of degree $-\gamma$, $\mathcal{L} \otimes \mathcal{I}_{Y\cap X/X}$ is a general line bundle of degree $-\gamma - e$ and therefore*

$$b_{i,j}(\gamma; Y) = b_{i,j}(\gamma + e)$$

for every i, every $j \geq m$ and γ large enough.

Remark 2. *With similar proofs, it is possible to generalize the Periodicity Theorem, as well as the assertions in Theorem 1.3 to reduced but possibly reducible curves. Suppose, for example, that $X \subset \mathbb{P}^n$ is a reduced curve with irreducible components $X_1, \dots, X_t$ of degrees $d_1, \dots, d_t$, arithmetic genera $g_1, \dots g_t$, and Hilbert series $H_1, \dots, H_t$. Let $Y \subset \mathbb{P}^n$ be any closed subscheme such that $X_l \not\subset Y$, for every l, $1 \leq l \leq t$ and let $m = \operatorname{reg}(X \cup Y)$. For each sequence of t numbers $\gamma_1, \dots, \gamma_t$ and each i and j, let $b_{i,j}(\gamma_1, \dots, \gamma_t; Y)$ be the Betti number $b_{i,j}(\Gamma \cup Y)$, where $\Gamma = \bigcup_{1\leq l\leq t} \Gamma_l$ is a set of points of X such that for each l, $1 \leq l \leq t$, $\Gamma_l \subset X_l$ is a general subset of γ_l points. If $\gamma_l \geq H_l(m)$ for every l, then*

$$b_{i,j}(\gamma_1, \dots \gamma_t; Y) = b_{i,j}(X \cup Y)$$

for every i and every $j \leq m - 1$. In addition, if we suppose that for some l, we also have $\gamma_l \geq \max\{g_l, H_l(m_l)\}$, where $m_l = \max\{m, \operatorname{reg} X_l\}$ then

$$b_{i,j}(\gamma_1, \dots, \gamma_l + d_l, \dots, \gamma_t) = b_{i,j-1}(\gamma_1, \dots, \gamma_l, \dots, \gamma_t)$$

for every i and every $j \geq m + 1$.

Remark 3. *Another conclusion which can be drawn from the above results is that in order to compute the numbers $b_{i,j}(\gamma; Y)$, for γ large enough, it is possible to replace a set of points with another one such that the associated divisors are linearly equivalent. In particular, if X is a smooth rational curve, any set of points (in fact, any divisor of degree γ) gives the right numbers.*

We are interested in further properties of the Betti numbers $b_{i,j}(\gamma)$, for $j \geq m = \operatorname{reg} X$ and $\gamma \geq \max\{d \cdot m + 1 - g, g\}$ in the case of a curve X of regularity m and Hilbert polynomial $P_X(T) = dT + 1 - g$. In fact we will concentrate on the two lines at the bottom of the Betti diagrams (see Proposition 1.5). Namely we will consider γ with $P_X(r-1) \leq \gamma \leq P_X(r)$ for some $r \geq m + 1$ and we want to study $b_{i,r-1}(\gamma)$ and $b_{i,r}(\gamma)$. We will assume in addition that $P_X(r-1) \geq g$ (which is a very mild assumption), in order to be able to use general line bundles instead of general points. By Theorem 2.2, there are essentially d pairs of rows to study. The next proposition shows that the $(r-1)^{\text{th}}$ row for γ points is the same as the r^{th} row for $P_X(r-1)+P_X(r)-\gamma$ points, but the entries appear in reverse order. The idea is to use our way of computing $b_{i,j}(\gamma)$, Serre duality and the fact that if a line bundle $\mathcal{L}$ is general then $\omega_X \otimes \mathcal{L}^{-1}$ is general, too.

Proposition 2.3. *Suppose that X is smooth. Then, with the above notations, $b_{i,r-1}(\gamma) = b_{n+1-i,r}(P(r)+P(r-1)-\gamma)$ for every i.*

Proof. Using Theorem 1.3 (iii), we see as in the proof of Theorem 2.2 that for $j \geq m$,

$$b_{i,j}(\gamma) = b_{i-1,j+1}(M),$$

where $M = \bigoplus_t \mathrm{H}^0(\mathcal{L} \otimes \mathcal{O}_X(t))$ for a general line bundle of degree $-\gamma$, $\mathcal{L}$. Let $F_\bullet$ be a minimal free resolution of M. We remark that the sheafification of M is just $\mathcal{L}$, so that using local duality over S and the fact that M is Cohen-Macaulay, we get that $\mathrm{Hom}(F_\bullet, S(-n-1))$ is a minimal resolution of

$$\operatorname*{Ext}_S^{n-1}(M, S(-n-1)) \cong (\mathrm{H}^2_m(M))' \cong (\bigoplus_t \mathrm{H}^1(\mathcal{L}(t)))'.$$

Using Serre duality on X, we get: $N := (\bigoplus_t \mathrm{H}^1(\mathcal{L}(t)))' \cong \bigoplus_t \mathrm{H}^0(\omega_X \otimes \mathcal{L}^{-1}(t))$. Therefore, for $j \geq m$, $b_{i,j}(\gamma) = b_{i-1,j+1}(M) = b_{n-i,1-j}(N)$.

But $N(-2r+1) = \bigoplus_t \mathrm{H}^0((\omega_X \otimes \mathcal{L}^{-1} \otimes \mathcal{O}_X(-2r+1)) \otimes \mathcal{O}_X(t))$ and $\omega \otimes \mathcal{L}^{-1} \otimes \mathcal{O}_X(-2r+1)$ is a general line bundle on X of degree $(2g-2)+\gamma-(2r-1)d = -(P(r)+P(r-1)-\gamma)$. Therefore

$$b_{i,j}(\gamma) = b_{n-i,1-j}(N) = b_{n-i,-j+2r}(N(-2r+1)) = \\ = b_{n-i+1,2r-1-j}(P(r)+P(r-1)-\gamma),$$

if $2r-1-j \geq m$ (using the same argument from the beginning of the proof). Taking $j=r$ and $j=r-1$, we get the assertion of the proposition. □

3. The Minimal Resolution Conjecture for Points on Curves.

In this paragraph we will assume that $X \subset \mathbb{P}^n$ is a nondegenerate smooth curve of degree d, genus g and regularity m. Let $P_X(T) = dT + 1 - g$ be the Hilbert polynomial of X. We will study whether X satisfies (MRC) for γ points (as defined in the first paragraph) for every γ, with $\gamma \geq \max\{P_X(m), g\}$. If this happens, we will say briefly that (MRC) holds for $\gamma \gg 0$. We consider first the case when X has small degree.

Proposition 3.1. *With the above notations, if X has degree n or $n+1$ then X satisfies (MRC) for all values of $\gamma \geq P_X(m)$ (in this case $m \leq 3$ and $g \leq 1$).*

Proof. If $\gamma = P_X(r-1)+\alpha$ with $r \geq m+1$ and $0 \leq \alpha \leq d$ then Proposition 1.6 gives $b_{i+1,r-1}(\gamma) - b_{i,r}(\gamma) = d\binom{n-1}{i} - \alpha\binom{n}{i}$.

But $d\binom{n-1}{i} \geq \alpha\binom{n}{i}$ iff $di \leq n(d-\alpha)$. Therefore (MRC) says that if $di \leq n(d-\alpha)$ then $b_{i,r}(\gamma) = 0$ and if $di \geq n(d-\alpha)$ then $b_{i+1,r-1}(\gamma) = 0$.

We will use the Linear Syzygy Theorem (see [4] for related definitions and proof). We are interested in the linear part in the resolution of I_Γ/I_X, where Γ is a set of points on X, computing $b_{i,j}(\gamma)$. I_Γ/I_X is 1-generic, i.e., if h is in S_1 and $0 \neq u \in I_\gamma/I_X$ then $hu \neq 0$. This is implied by the fact that X is a nondegenerate variety. Then, since $(I_\Gamma/I_X)_t = 0$ for $t \leq r-1$, the Linear Syzygy Theorem says that $b_{i,r}(I_\Gamma/I_X) = 0$ for $i \geq \dim_k(I_\Gamma/I_X)_r = P_X(r) - \gamma = d - \alpha$. We know that $b_{i-1,r-1}(\Gamma) = b_{i,r}(I_\Gamma/I_X)$.

Suppose that $di \geq n(d-\alpha)$. If $d = n$, then $i \geq d-\alpha = n-\alpha$ and the above facts imply that $b_{i+1,r-1}(\gamma) = 0$. If $d = n+1$, then $n\alpha \geq n^2+n-ni-i$. If $i = n$, $b_{i+1,r-1}(\Gamma) = 0$ for trivial reasons. If $i \leq n-1$ then since α is an integer, we have $\alpha \geq n+1-i = d-i$. Therefore we can apply the same argument as before to get $b_{i+1,r-1}(\Gamma) = 0$.

Suppose now that $di \leq n(d-\alpha)$. By Proposition 2.3, $b_{i,r}(\gamma) = b_{n+1-i,r-1}(P(r)-\alpha)$. As $P_X(r) - \alpha = P_X(r-1) + (d-\alpha)$ and $d(n-i) \geq n(d-(d-\alpha))$, by what we have just proved we get $b_{i,r}(\gamma) = 0$. This concludes the proof of (MRC) for X. $\square$

Using a standard Koszul cohomology argument we get necessary and sufficient conditions for X to satisfy (MRC) for every $\gamma \gg 0$. Suppose that X is embedded in $\mathbb{P}(V)$ by the linear system $V \subset \mathrm{H}^0(X, \mathcal{L})$, where $\mathcal{L} = \mathcal{O}_X(1)$. Let $\mathcal{M}_V$ be the kernel of the evaluation map

$$\phi : V \otimes \mathcal{O}_X \longrightarrow \mathcal{L}$$

i.e. $\mathcal{M}_V = \Omega_{\mathbb{P}^n}(1)|_X$. For any real number x we will denote by $[x]$ the integer n characterized by $n \leq x < n+1$. We have:

Proposition 3.2. *With the above notations, X satisfies (MRC) for every $\gamma \gg 0$ iff*

$$\mathrm{H}^0(\wedge^i \mathcal{M}_V \otimes \mathcal{F}) = 0$$

for every i and for a general line bundle $\mathcal{F}$ of degree $g - 1 + \left[\frac{di}{n}\right]$.

Proof. (MRC) says that for any $\gamma = P_X(r-1+\alpha)$, $0 \le \alpha \le d-1$, $r \ge m+1$, if $d\binom{n-1}{i} \le \binom{n}{i}\alpha$, then $b_{i+1,r-1}(\gamma) = 0$ and if $d\binom{n-1}{i} \ge \binom{n}{i}\alpha$, then $b_{i,r}(\gamma) = 0$.

As in the proof of Proposition 3.1, using Proposition 2.3 we see that in fact it is enough to consider just the first condition.

We compute now $b_{i+1,r-1}(\gamma)$. As in the proof of Theorem 2.2, this can be computed as the dimension over k of the homology of the following complex:

$$\wedge^{i+1} V \otimes \mathrm{H}^0(\mathcal{P} \otimes \mathcal{L}^{r-1}) \longrightarrow \wedge^i V \otimes \mathrm{H}^0(\mathcal{P} \otimes \mathcal{L}^r) \longrightarrow \wedge^{i-1} V \otimes \mathrm{H}^0(\mathcal{P} \otimes \mathcal{L}^{r+1}),$$

where $\mathcal{P}$ is a general line bundle of degree $-\gamma$. Therefore $\mathcal{F} := \mathcal{P} \otimes \mathcal{L}^r$ is a general line bundle of degree $-\gamma + rd = d + g - \alpha - 1$. In particular, $\deg(\mathcal{P} \otimes \mathcal{L}^{r-1}) \le g - 1$ and therefore $\mathrm{H}^0(\mathcal{P} \otimes \mathcal{L}^{r-1}) = 0$. This implies that $b_{i+1,r-1}(\gamma) = \dim_k(\ker f)$, where

$$f : \wedge^i V \otimes \mathrm{H}^0(\mathcal{F}) \longrightarrow \wedge^{i-1} V \otimes \mathrm{H}^0(\mathcal{F} \otimes \mathcal{L}).$$

The short exact sequence defining $\mathcal{M}_V$:

$$0 \longrightarrow \mathcal{M}_V \longrightarrow V \otimes \mathcal{O}_X \longrightarrow \mathcal{L} \longrightarrow 0$$

induces a short exact sequence:

$$0 \longrightarrow \wedge^i \mathcal{M}_V \longrightarrow \wedge^i V \otimes \mathcal{O}_X \stackrel{u}{\longrightarrow} \wedge^{i-1} \mathcal{M}_V \otimes \mathcal{L} \longrightarrow 0$$

and an inclusion:

$$\wedge^{i-1} M_V \otimes \mathcal{L} \stackrel{j}{\longrightarrow} \wedge^{i-1} V \otimes \mathcal{L} \longrightarrow 0.$$

This gives $f = \mathrm{H}^0(j \otimes 1_{\mathcal{F}}) \circ \mathrm{H}^0(u \otimes 1_{\mathcal{F}})$. Therefore $b_{i+1,r-1}(\gamma) = \dim_k \mathrm{H}^0(\wedge^i \mathcal{M}_V \otimes \mathcal{F})$ and (MRC) says that if $d\binom{n-1}{i} \le \binom{n}{i}\alpha$ then $H^0(\wedge^i \mathcal{M}_V \otimes \mathcal{F}) = 0$.

But $d\binom{n-1}{i} \le \binom{n}{i}\alpha$ is equivalent to $\alpha \ge d - \frac{id}{n}$, so that (MRC) is equivalent to the assertion that if $\alpha \ge d - \frac{id}{n}$ then $\mathrm{H}^0(\wedge^i \mathcal{M}_V \otimes \mathcal{F}) = 0$ for a general line bundle $\mathcal{F}$ of degree $d + g - \alpha - 1$. But this is equivalent to the assertion in the statement of the proposition. $\square$

One notices that with the above notations $\deg(\mathcal{M}_V) = -d$, $\operatorname{rank}(\mathcal{M}_V) = n$ so that the slope is $\mu(\mathcal{M}_V) = \frac{\deg(\mathcal{M}_V)}{\operatorname{rank}(\mathcal{M}_V)} = -\frac{d}{n}$.

For any real number x we will denote by $\{x\}$ the integer n defined by $n - 1 < x \le n$. Therefore $[-x] = -\{x\}$. In the light of Proposition 3.2, we make the following definition.

Definition. *Let $\mathcal{E}$ be a vector bundle on X with slope $\mu(\mathcal{E})$. We say that $\mathcal{E}$ satisfies the first generic vanishing condition* (V_1) *if* $\mathrm{H}^0(\mathcal{E} \otimes \mathcal{L}) = 0$ *for $\mathcal{L}$ a general line bundle of degree $g - 1 - \{\mu(E)\}$.*

We say that $\mathcal{E}$ satisfies (V_i) if $\wedge^i \mathcal{E}$ satisfies (V_1) and furthermore that $\mathcal{E}$ satisfies (V) if it satisfies (V_i) for every i.

Remark 1. *Since* $\dim_k \mathrm{H}^0(\mathcal{E} \otimes \mathcal{L})$ *is an upper semicontinuous function, it is equivalent to say that* $\mathrm{H}^0(\mathcal{E} \otimes \mathcal{L}) = 0$ *for some $\mathcal{L}$, or for general $\mathcal{L}$ in a certain degree.*

Remark 2. *With the notations in the definition, by Riemann-Roch formula we see that* $\chi(\mathcal{E} \otimes \mathcal{L}) = \operatorname{rank}(\mathcal{E})\big(\mu(\mathcal{E}) + \deg(\mathcal{L}) + 1 - g\big)$ *and therefore that the degree of $\mathcal{L}$ which appears in the definition is the largest one that makes* $\chi(\mathcal{E} \otimes \mathcal{L}) \leq 0$.

Remark 3. *Any line bundle satisfies condition* (V) *since* $\mathrm{H}^0(\mathcal{L}) = 0$ *for a general line bundle of degree $g - 1$.*

Remark 4. *Since for any vector bundle $\mathcal{E}$ and any i, $0 \leq i \leq \mathrm{rk}(E)$, $\mu(\wedge^i \mathcal{E}) = i\mu(E)$, we can reformulate Proposition 3.2 as follows: X satisfies (MRC) for γ large enough iff $\mathcal{M}_V$ satisfies condition (V).*

Our definition is closely related to the definition made by Raynaud in [8]. He says that a vector bundle $\mathcal{E}$ satisfies the property $(\star)$ if $\dim_k \mathrm{H}^0(\mathcal{E} \otimes \mathcal{L}) \geq \max\{\chi(\mathcal{E}), 0\}$ for a general line bundle $\mathcal{L}$ of degree 0. We see that a bundle $\mathcal{E}$ satisfies (V_1) iff after a suitable normalization with a line bundle $\mathcal{L}$ such that $-\operatorname{rank}(\mathcal{E}) + 1 \leq \chi(\mathcal{E} \otimes \mathcal{L}) \leq 0$, $\mathcal{E} \otimes \mathcal{L}$ satisfies $(\star)$.

In [8] one studies the question whether all semistable bundle on a curve satisfies condition $(\star)$. The answer is no in general, but there are some cases when the answer is yes: when $\operatorname{rank}(\mathcal{E}) = 2$, when $g \leq 1$ or when $\operatorname{rank}(\mathcal{E}) = 3$ and $g = 2$ or X is a general curve (see [8] for the proof of these statements).

Because over a field of characteristic 0 the exterior powers of a semistable bundle are semistable ([7]), the above results imply via Proposition 3.2 the following:

Proposition 3.3. *Suppose that* $\operatorname{char} k = 0$ *and that $X \subset \mathbb{P}^n$ is a nondegenerate smooth curve of degree d and genus g such that $\Omega_{\mathbb{P}^n}|_X$ is semistable. If we are in one of the following situations:*

i) $g \leq 1$
ii) $n = 3$ *and* $g = 2$
iii) $n = 3$ *and X is general in the moduli space* $\mathcal{M}_g$,

then X satisfies (MRC) for $\gamma \gg 0$.

Using the fact cited above about rank 2 vector bundles and the fact that for a smooth nondegenerate plane curve X, $\Omega_{\mathbb{P}^2}|_X$ is semistable (see [2], Proposition 4.5) we get (MRC) for plane curves.

Proposition 3.4. *If $X \subset \mathbb{P}^2$ is a nondegenerate smooth curve, then X satisfies (MRC) for $\gamma \gg 0$.*

In the remaining of this paragraph we will discuss the property (V).

Proposition 3.5. *Let $\mathcal{E}$ be a vector bundle on X with* $\operatorname{rank}(\mathcal{E}) = r$ *and* $\deg(E) = d$.

i) If $\mathcal{L}$ is a line bundle, then $\mathcal{E}$ satisfies (V_i) iff $\mathcal{E} \otimes \mathcal{L}$ does.

ii) $\mathcal{E}$ satisfies (V_i) iff $\mathcal{E}^\star$ satisfies (V_{r-i}).

Proof. i) Since $\wedge^i(\mathcal{E} \otimes \mathcal{L}) = \wedge^i \mathcal{E} \otimes \mathcal{L}^i$, it is enough to prove the assertion for $i = 1$. By symmetry, it is enough to prove that if $\mathcal{E}$ satisfies (V_1) then so does $\mathcal{E} \otimes \mathcal{L}$.

We know that $\mathrm{H}^0(\mathcal{E} \otimes \mathcal{L}') = 0$ for $\mathcal{L}'$ a general line bundle of degree $g - 1 - \{\mu(\mathcal{E})\}$. But $\mu(\mathcal{E} \otimes \mathcal{L}) = \mu(\mathcal{E}) + \deg(L)$ and if we write $\mathcal{E} \otimes \mathcal{L}' = (\mathcal{E} \otimes \mathcal{L}) \otimes (\mathcal{L}' \otimes \mathcal{L}^{-1})$, since $\deg(\mathcal{L}' \otimes \mathcal{L}) = g - 1 - \{\mu(\mathcal{E})\} - \deg(\mathcal{L})$, this completes the proof.

ii) Follows immediately from i), since $\wedge^i \mathcal{E} \cong \wedge^{r-i} \mathcal{E}^\star \otimes \det(\mathcal{E})$. $\square$

The next proposition shows that property (V) implies something only slightly weaker than semistability. In the particular case when the slope is an integer, it implies semistability.

Proposition 3.6. *Let $\mathcal{E}$ be a vector bundle on X of rank r, degree d and slope $\mu(\mathcal{E})$ which satisfies condition (V). If $\mathcal{F}$ is a subbundle of $\mathcal{E}$ of rank r' and degree d', then*

$$d' \leq \{r' \cdot \mu(\mathcal{E})\}.$$

Proof. $\wedge^{r'} \mathcal{F}$ is a rank one subbundle of $\wedge^{r'} \mathcal{E}$ of degree d'. But because any line bundle of degree greater than or equal to g has sections, this implies that $\mathrm{H}^0(\wedge^{r'} \mathcal{E} \otimes \mathcal{L}) \neq 0$ for any line bundle $\mathcal{L}$, $\deg \mathcal{L} = g - d'$. Because $\mathcal{E}$ satisfies $(V_{r'})$, this gives $g - d' \geq g - \{r' \cdot \mu(\mathcal{E})\}$ and the assertion of the proposition. $\square$

The next result proves the converse of Proposition 3.6 for a vector bundle $\mathcal{E}$ which splits as a direct sum of line bundles. We get that $\mathcal{E}$ satisfies (V) iff the degrees of the line bundles are “as close as possible”. In particular, when the slope of $\mathcal{E}$ is an integer they have to be equal. We will derive then the condition for a smooth rational curve to satisfy (MRC).

Proposition 3.7. *Let $\mathcal{E}$ be $\mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r$, where $\mathcal{L}_i$ is a line bundle of degree d_i on X for every i, $1 \le i \le r$. We suppose $d_1 \ge \ldots \ge d_r$. Let μ be the slope of $\mathcal{E}$. Then the following are equivalent:*

i) $\mathcal{E}$ satisfies (V).

ii) For every subbundle $\mathcal{F}$ of $\mathcal{E}$ of rank r' and degree d', $d' \le \{r' \cdot \mu(\mathcal{E})\}$.

iii) $d_i = \{i\mu\} - \{(i-1)\mu\}$ for every i, $1 \le i \le r$.

Proof.

$$\wedge^i \mathcal{E} = \bigoplus_{1 \le j_1 < \ldots < j_i \le r} (\mathcal{L}_{j_1} \otimes \ldots \otimes L_{j_i}).$$

If $\mathcal{M}$ is any line bundle on X, $\mathrm{H}^0(\mathcal{M} \otimes \mathcal{L}) = 0$ for a general line bundle $\mathcal{L}$ of degree d iff $d + \deg(\mathcal{M}) \le g - 1$. This implies that $\wedge^i \mathcal{E}$ satisfies (V_1) iff $d_1 + \ldots + d_i + g - 1 - \{i\mu\} \le g - 1$.

Therefore $\mathcal{E}$ satisfies (V) iff $\sum_{j=1}^{i} d_j \le \{i\mu\}$, for every i, $1 \le i \le r$.

If $d_i = \{i\mu\} - \{(i-1)\mu\}$ for every i, $1 \le i \le r$, we clearly have $\sum_{j=1}^{i} d_j \le \{i\mu\}$, for every i, $1 \le i \le r$, which proves that iii) implies i).

We already know that i) implies ii). By applying ii) to $\mathcal{F} = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_i$, we get that $\sum_{j=1}^{i} d_j \le \{i\mu\}$ for every i, $1 \le i \le r$ and therefore in order to complete the proof of the proposition it is enough to deduce iii) from these relations.

For $i = 1$, we have $d_1 \le \{\mu\}$. But because $d_1 \ge d_i$, for every $i \ge 1$, we must have $rd_1 \ge \deg(\mathcal{E})$ i.e. $d_1 \ge \{\mu\}$ and therefore $d_1 = \{\mu\}$.

We continue by induction. Suppose that

$$d_j = \{j\mu\} - \{(j-1)\mu\},$$

for $1 \le j \le k$ and some $k \le r - 1$. Therefore

$$d_{k+1} \le \{(k+1)\mu\} - \sum_{j=1}^{k} d_j = \{(k+1)\mu\} - \{k\mu\}.$$

Suppose that $d_{k+1} \le \{(k+1)\mu\} - \{k\mu\} - 1$. Since $d_j \le d_{k+1}$ for $j \ge k+1$, we get $\deg(\mathcal{E}) - \sum_{j=1}^{k} d_j \le (r-k)d_{k+1}$ and therefore

$$\deg(\mathcal{E}) - \{k\mu\} \le (r-k)\big(\{(k+1)\mu\} - \{k\mu\} - 1\big).$$

This gives

$$(r-k-1)\{k\mu\} + \deg\, \mathcal{E} + r - k \le (r-k)\{(k+1)\mu\}.$$

Since for any x, $x \leq \{x\} < x + 1$, we get

$$(r - k - 1)k\mu + d + r - k < (r - k)(k + 1)\mu + r - k,$$

which gives the contradiction $0 < 0$. $\square$

Corollary 3.8. *Let $X \subset \mathbb{P}^n$ be a nondegenerate smooth rational curve of degree d. If $\Omega_{\mathbb{P}^n}(1)|_X \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$, with $a_1 \geq \ldots \geq a_n$, then X satisfies (MRC) for every γ large enough iff*

$$a_i = \left\lceil \frac{d(i-1)}{n} \right\rceil - \left\lceil \frac{di}{n} \right\rceil,$$

for every i, $1 \leq i \leq n$. In particular, if $n|d$ the condition is $a_1 = \ldots = a_n$.

Example. *Let X be a smooth nondegenerate rational curve of degree 5 in $\mathbb{P}^3$. Then $\Omega_{\mathbb{P}^3}(1)|_X \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_3)$, with $a_1 \geq a_2 \geq a_3$ and $a_1 + a_2 + a_3 = -5$. Since $\mathrm{H}^0(\Omega_{\mathbb{P}^3}|_X) = 0$, it follows that $a_i \leq -1$, for $1 \leq i \leq 3$. Therefore we have only two possibilities for (a_1, a_2, a_3), namely $(-1, -2, -2)$ and $(-1, -1, -3)$.*

The first type corresponds to the general smooth rational quintic in $\mathbb{P}^3$.

The second one consists of exactly those smooth rational quintics which lie on a smooth quadric (see [3], Proposition 5). By the above corollary, X satisfies (MRC) for γ large enough iff we are in the first case. For example, the monomial curve $\mathbb{P}^1 \ni (u, v) \longrightarrow (u^5, u^4v, uv^4, v^5) \in \mathbb{P}^3$ doesn't satisfy (MRC) for some γ. In fact, from the proof of Proposition 3.2 one sees that these values are exactly those for which $\gamma \equiv 3, 4 \pmod 5$.

4. Periodicity in higher dimensions.

In this paragraph we will deal with varieties of dimension greater than one. The goal is to prove an analogue of Theorem 2.2 in this situation.

The results in the first paragraph show that in studying the Betti diagram for γ general points, it is natural to compare the behavior for values of γ between $P_X(r-1)$ and $P_X(r)$ with that for values between $P_X(r)$ and $P_X(r+1)$. Proposition 1.6 shows that the Betti numbers grow polynomially of degree $\dim X - 1$. However there is something that has a chance to remain constant, namely the differences between the corresponding Betti numbers for consecutive values of γ once we fix the distance from $P_X(r-1)$. Let's be more specific. For $\gamma = P_X(r-1)$ the Betti numbers in the bottom part of the diagram are determined and given by polynomial functions in r. Therefore, a complete understanding of the Betti diagrams would come from that of the differences $b_{i,j}(\gamma) - b_{i,j}(\gamma+1)$, for $P_X(r-1) \leq \gamma \leq P_X(r) - 1$. Moreover, since Proposition 1.6 relates the Betti numbers in the $(r-1)^{\text{th}}$ and the r^{th} row, it is enough to concentrate only on the $(r-1)^{\text{th}}$ row.

We think of $P_X(r-1)$ as the left margin for the range between $P_X(r-1)$ and $P_X(r)$ and fix a distance k from this margin. The main result of this paragraph is that the difference

$$b_{i,r-1}(P_X(r-1)+k) - b_{i,r-1}(P_X(r-1)+k+1)$$

does not depend on r, for $r \gg 0$. More precisely , we have:

Theorem 4.1. *Let $X \subset \mathbb{P}^n$ be a projective variety of dimension greater than one and let P_X be its Hilbert polynomial. Then there are integers $\alpha_{i,k}$ for every i and k with $1 \leq i \leq n$ and $k \geq 0$ such that*

$$b_{i,r-1}(P_X(r-1)+k) - b_{i,r-1}(P_X(r-1)+k+1) = \alpha_{i,k},$$

for every i, k and r, with $1 \leq i \leq n$, $k \geq 0$ and r large enough (depending on k).

Remark 1. *In the case when* $\dim X = 1$ *the statement is equivalent to the Periodicity Theorem (however, it does not say where periodicity starts).*

Remark 2. *Proposition 1.7 implies that for every i, $\{\alpha_{i,k}\}_{k\geq 0}$ is a decreasing sequence of positive integers and therefore, it is eventually constant.*

Remark 3. *From the proof of the theorem we will see that $\alpha_{i,k} \leq \binom{n}{i}$ for every i. On the other hand, it is easy to see, using Proposition 1.6, that (MRC) holds for every $\gamma \geq P_X(m)$ ($m = \operatorname{reg} X$) iff*

$$b_{i,r-1}(P_X(r-1)+k) - b_{i,r-1}(P_X(r)+k+1) = \min\{b_{i,r-1}(P_X(r-1)+k), \tbinom{n}{i}\}$$

for every i and every k, $0 \leq k \leq \Delta P_X(r)$, $r \geq m+1$. Therefore, if (MRC) holds for every $\gamma \geq P_X(m)$, then $\alpha_{i,k} = \binom{n}{i}$ for every k and every i.

The main steps in proving periodicity in the one dimensional case were to understand what happens when we add a hyperplane section and then to show that we can compute general Betti numbers in this way. We will consider a similar approach here. We will show at the same time the periodicity statement in Theorem 4.1 and the fact that from a point on we can compute general Betti numbers by adding points which lie in a hyperplane section.

We fix the notations for the rest of the paragraph. $X \subset \mathbb{P}^n$ is a projective variety of dimension $d \geq 2$, regularity m, Hilbert series H_X and Hilbert polynomial P_X. We fix a reduced and irreducible hyperplane section of X, $Y = X \cap H$. An equation of H will be denoted by h. Notice that we have $\operatorname{reg}(Y) \leq m$ and that $S(Y)_i = (\frac{S(X)}{hS(X)})_i$ for every $i \geq m$.

The first step in order to prove that in certain cases one can compute general Betti numbers by adding points in Y is to show that it is possible to add points in Y and have the union in general position on X. This is done in the following lemma.

Lemma 4.2. *With the above notations, suppose that γ is an integer with $P_X(r-1) \leq \gamma \leq P_X(r)$ and $\gamma_0 = \Delta P_X(r)$, for some $r \geq m+1$. If $\Gamma \subset X$ is a general set of γ points on X and $\Gamma_0 \subset Y$ is a general set of γ_0 points on Y then $\Gamma \cup \Gamma_0$ is in general position on X.*

Proof. Obviously, it is enough to show the existence of such Γ and Γ_0. The nonexistence of such sets can be interpreted as the fact that there is an $s \geq 1$ and $m_1, \ldots, m_k$ a basis of $S(X)_s$ such that for every $P_1, \ldots, P_\gamma \in X$ and every $P_{\gamma+1}, \ldots, P_{\gamma+\gamma_0} \in Y$, the matrix $(m_i(P_j))_{1 \leq i \leq k, 1 \leq j \leq \gamma+\gamma_0}$ does not have maximal rank. One can easily deduce a contradiction from this statement. When $s \leq r-1$, we have just to apply Proposition 1.1 to X. When $s \geq r$, by the above remark $H_Y(s) = \Delta P_X(s)$ and we apply Proposition 1.1 to both X and Y. □

Proof of Proposition 4.1. We have seen (Proposition 1.5) that (MRC) holds for any γ points in general position in X for $\gamma = P_X(r-1)$, $r \geq m+1$. In particular, using also Lemma 4.2 it follows that for $\gamma = P_X(r)$ we can compute the general Betti numbers by adding $\Delta P_X(r)$ general points in Y to $P_X(r-1)$ general points in X.

We will prove by induction on $k \geq -1$ both the existence of $\alpha_{i,k}$ in the statement of the theorem and the fact that for r large enough, if $\Gamma \subset X$ is a general set of $P_X(r-1)+k+1$ elements and $\Gamma_0 \subset Y$ is a general set of $\Delta P_X(r)$ elements, then $\Gamma \cup \Gamma_0$ computes general Betti numbers for $P_X(r)+k+1$ points on X.

For $k = -1$ the statement of the theorem is void while the second part of the assertion follows from the remark at the beginning of the proof.

For the induction step, suppose that we know both the assertions for k. Let r_k be such that these assertions are both valid for $r \geq r_k$ (we assume implicitly that $r \geq m+1$). Let's consider $r \geq r_k$.

Let $\Gamma \subset X \setminus Y$, $P \in X$, $\Gamma_0 \subset Y$ be such that Γ consists of $\gamma =$

$P_X(r-1)+k+1$ points, Γ_0 of $\gamma_0 = \Delta P_X(r)$ points, Γ, $\Gamma \cup P$ and $\Gamma \cup \Gamma_0$ compute general Betti numbers and $\Gamma \cup \Gamma_0 \cup P$ is in general position on X. We used the induction hypothesis and Lemma 4.2 to make sure we can choose such points.

Let's consider the commutative diagram from the beginning of the proof of Proposition 1.7 and the similar one where we replace Γ by $\Gamma \cup \Gamma_0$ and r by $r+1$.

There is a natural map:

$$\mathcal{I}_{\Gamma/X}(r) \longrightarrow \mathcal{I}_{\Gamma\cup\Gamma_0/X}(r+1) = \mathcal{I}_{\Gamma/X}(r+1) \otimes \mathcal{I}_{\Gamma_0/X}$$

which is induced by tensoring the natural inclusion $\mathcal{O}_X(-1) = \mathcal{I}_{Y/X} \subset \mathcal{I}_{\Gamma_0/X}$ with $\mathcal{I}_{\Gamma/X}(r+1)$.

Similarly, we have a map

$$\mathcal{I}_{\Gamma\cup P/X}(r) \longrightarrow \mathcal{I}_{\Gamma\cup\Gamma_0/X}(r+1)$$

and by taking global sections we get an induced map of diagrams. We therefore get a map of complexes from the long exact sequences associated (see the proof of Proposition 1.7):

$$\begin{tikzcd}
0 \arrow[r] & B_{i+1,r-1}(\Gamma \cup P) \arrow[r] \arrow[d] & B_{i+1,r-1}(\Gamma) \arrow[r, "\phi"] \arrow[d] & \wedge^i W \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & B_{i+1,r}(\Gamma \cup \Gamma_0 \cup P) \arrow[r] & B_{i+1,r}(\Gamma \cup \Gamma_0) \arrow[r, "\psi"] & \wedge^i W \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & B_{i,r}(\Gamma \cup P) \arrow[r] \arrow[d] & B_{i,r}(\Gamma) \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & B_{i,r+1}(\Gamma \cup \Gamma_0 \cup P) \arrow[r] & B_{i,r+1}(\Gamma \cup \Gamma_0) \arrow[r] & 0
\end{tikzcd}$$

where the vertical map in the middle is the identity.

Since $\operatorname{Im}\phi \subset \operatorname{Im}\psi$, we get

$$b_{i+1,r-1}(\Gamma) - b_{i+1,r-1}(\Gamma \cup P) \leq b_{i+1,r}(\Gamma \cup \Gamma_0) - b_{i+1,r}(\Gamma \cup \Gamma_0 \cup P) \leq \binom{n}{i},$$

which by the way we have chosen Γ, Γ_0 and P gives:

$$b_{i+1,r-1}(\gamma) - b_{i+1,r-1}(\gamma+1) \leq b_{i+1,r}(\gamma+\gamma_0) - b_{i+1,r}(\Gamma \cup \Gamma_0 \cup P) \leq$$

$$\leq b_{i+1,r}(\gamma \cup \gamma_0) - b_{i+1,r}(\gamma + \gamma_0 + 1) \leq \binom{n}{i}.$$

This shows that the sequence

$$\{b_{i+1,r-1}(P_X(r-1)+k+1) - b_{i+1,r-1}(P_X(r-1)+k+2)\}_{r \geq r_k}$$

is increasing and bounded and therefore constant for $r \geq r_{k+1}$, for some r_{k+1}. This implies that for $r \geq r_{k+1}$,

$$b_{i+1,r-1}(P_X(r-1)+k+1) - b_{i+1,r-1}(P_X(r-1)+k+2) = \alpha_{i,k+1}$$

and moreover, with the above notations $b_{i+1,r}(\Gamma \cup \Gamma_0 \cup P) = b_{i+1,r}(\gamma + \gamma_0 + 1)$, which completes the induction step. $\square$

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