LE MATEMATICHE
Vol. LIII (1998) – Supplemento, pp. 83–97

BOUNDS FOR THE REGULARITY OF MONOMIAL IDEALS

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1. Introduction.

Regularity of an ideal I is defined to be the minimal number r such that the i-th syzygy module of I is generated by elements of degree $\leq i + r$ for all $i \geq 0$. It is denoted by reg I. The regularity of an ideal can be considered as a refined notion of the maximal degree of minimal generators of I as a measure of the complexity of Gröbner basis computations and it is important both from the computational and theoretical point of view.
It is usually difficult to determine the regularity of I without knowing the shifts of the minimal free resolution of I explicitly. But in order to obtain a minimal free resolution using computer algebra systems such as Macaulay, a Gröbner basis has to be computed in each step. So from the computational point of view it is not useful to acquire the regularity from a known minimal free resolution. What we need are some kind of general estimates for regularity using other invariants of the ideal I. Using Gröbner basis theory, we have reg I = reg Gin(I), where Gin (I) is a generic initial ideal of I with respect to the reverse lexicographic order (see, e.g. [4]). Then to estimate reg I, it is enough to consider the monomial ideal Gin (I). Thus estimating the regularity of monomial ideals is the first step toward studying general homogeneous ideals. In this paper, we hence focus on the regularity of monomial ideals.
After preparing some terminology and known facts on simplicial complexes and

Stanley-Reisner rings in § 2, we give the estimate

$$\text{reg } I \leq \text{ arith-deg } I$$

for a monomial ideal I in § 3. This estimates refines the inequality

$$(\text{maximal degree of minimal generators of } I) \leq \text{ arith-deg } I$$

in Sturmfels-Trung-Vogel [14]. After finishing our paper we learned that this bound for the regularity had independently been proved by Hoa and Trung in 1997 using different methods [8].
In § 4 we generalize a certain inequality which is conjectured by Eisenbud, and obtained in [16]. The class which is considere in [16] is pure simplicial complexes connected in codimension 1. Introducing a correcting term we generalize the estimate to the class of pure simplicial complexes with possibly more than one connected component, each of which connected in codimension 1. This class contains all Buchsbaum complexes, while the class considered in [16] does not contain all of these.
In the final section we give some examples showing that all bounds are sharp and each is best in some situations.

We would like to thank Professors D. Eisenbud and S. Popescu for their helpful and encouraging suggestions, advice and comments. We would also like to thank Professors A. Ragusa, G. Paxia and all other organizers of PRAGMATIC 97 for their hospitality and EUROPROJ for the financial support that made PRAGMATIC 97 possible.

2. Preliminaries.

First we fix some notation and recall some facts about simplicial complexes and Stanley-Reisner rings. The reference for the following is [12], if not denoted otherwise.

A simplicial complex Δ on the vertex set $V = \{x_1, \ldots, x_n\}$ is a collection of subsets of V, such that

1) $\{x_i\} \in \Delta$ for every $x_i \in V$ and
2) if $\tau \subset \sigma$ and $\sigma \in \Delta$, then $\tau \in \Delta$.

An element $\sigma \in \Delta$ is called an *i-face* of Δ, if the number of vertices of σ, $\#\sigma$, is $i + 1$. The number of i-faces of Δ is denoted by f_i. The maximal faces of a simplicial complex will also be called *facets*:

$$d - 1 := \dim \Delta := \max\{\#\sigma - 1 : \sigma \in \Delta\} = \max\{i : f_i \neq 0\}.$$

Let $A = k[x_1, \dots, x_n]$ be a polynomial ring over a field k, where the variables x_i correspond to the vertices $x_i \in V$. Then

$$I_\Delta := (x_{i_1} \cdots x_{i_r} : i_1 < \cdots < i_r, \{x_{i_1}, \dots, x_{i_r}\} \notin \Delta)$$

is the square-free monomial ideal corresponding to the simplicial complex Δ. The ring $k[\Delta] := A/I_\Delta$ is called the Stanley-Reisner ring of Δ. Its Hilbert-series can be written in the following way:

$$F(k[\Delta], \lambda) = \sum_{i=0}^{\infty} \dim_k(k[\Delta]_i)\lambda^i$$

$$= 1 + \sum_{i=1}^{d} \frac{f_{i-1}\lambda^i}{(1-\lambda)^i}$$

Hence $d = \dim(k[\Delta]) = \dim \Delta + 1$ and $\deg(k[\Delta]) = f_{d-1}$.

Now let

$$0 \longrightarrow \oplus_j A(-j)^{\beta_{rj}} \longrightarrow \cdots \longrightarrow \oplus_j A(-j)^{\beta_{1j}} \longrightarrow A \longrightarrow k[\Delta] \longrightarrow 0$$

be a graded minimal free resolution of $k[\Delta]$ over A. The minimal length of such a resolution of $k[\Delta]$ is denoted by $\operatorname{pd} k[\Delta]$. We denote by $\Delta_W := \{\sigma \in \Delta : \sigma \subset W\}$ the restriction of Δ on the vertex set W. Then the graded Betti-numbers of $k[\Delta]$ are given by Hochster's formula [9]

$$\beta_{ij} = \sum_{\#W=j, W\subset V} \dim \widetilde{H}_{j-i-1}(\Delta_W; k),$$

where $\widetilde{H}_l(\Delta_W; k)$ is the l-th reduced homology of Δ_W.
Given a graded minimal free resolution of $k[\Delta]$, the regularity of $k[\Delta]$ is defined as the maximum over all $j - i$ for which $\beta_{ij} \neq 0$. Therefore the regularity of a Stanley-Reisner ring can be computed by Hochester's formula:

$$\operatorname{reg} k[\Delta] = \max\{i + 1 : \exists W \subset V : \widetilde{H}_i(\Delta_W; k) \neq 0\}.$$

This formula is not very useful for explicit computations because of the possibly huge number of subsets W, that has to be considered. For computing regularity it is in some cases easier to use a formula by [15], which is applicable in the situation of $\operatorname{codim} k[\Delta] \geq 2$ and uses the Alexander dual complex $\Delta^* := \{V \setminus \sigma : \sigma \notin \Delta\}$ of the simplicial complex Δ:

$$(*) \qquad \operatorname{reg} I_\Delta - \operatorname{indeg} I_\Delta = \dim k[\Delta^*] - \operatorname{depth} k[\Delta^*].$$

Here $\operatorname{indeg} I_\Delta$ denotes the minimum of all i for which we have $(I_\Delta)_i \neq 0$.

3. A Bound for the Regularity of a Monomial Ideal.

Our goal in this section is to prove a conjecture of Bayer and Mumford [1] (in the form stated in [6]) that the regularity of a square-free monomial ideal is bounded by the geometric degree of this ideal. For this proof we will use facts about Stanley-Reisner rings, especially a formula for regularity by Terai [15]. After this we will use polarization to generalize this bound to the arithmetic degree of a non-square-free monomial ideal. See [14] for the definitions and basic properties of geometric and arithmetic degree.

Theorem 3.1. *Let I_Δ be a square-free monomial ideal corresponding to a simplicial complex Δ. Let $k[\Delta]$ denote its Stanley-Reisner ring; suppose* $\operatorname{codim} k[\Delta] \geq 2$. *Then we have*

$$\operatorname{reg} I_\Delta \leq \text{geom-deg}\, I_\Delta.$$

Proof. Denote the minimal number of generators of I_{Δ^*} by $\mu(\Delta^*)$. This number is just the number of maximal faces of Δ by the definition of the Alexander dual complex. Therefore we need to show that

$$\operatorname{reg} I_\Delta \leq \mu(\Delta^*).$$

By formula $(*)$ for the regularity this is just

$$\operatorname{indeg} I_\Delta + \dim k[\Delta^*] - \operatorname{depth} k[\Delta^*] \leq \mu(\Delta^*).$$

But $\operatorname{indeg} I_\Delta = \min\{\#\sigma : \sigma \notin \Delta\} = \min\{n - \#(V \setminus \sigma) : V \setminus \sigma \in \Delta^*\} = \operatorname{embdim} k[\Delta^*] - \dim k[\Delta^*]$. So it is left to prove that

$$\operatorname{embdim} k[\Delta^*] - \operatorname{depth} k[\Delta^*] \leq \mu(\Delta^*).$$

In this formula the left side is just the projective dimension of $k[\Delta^*]$ by the Auslander-Buchsbaum theorem. But for the Taylor-resolution (see e.g. [4], Exercise 17.11) of a monomial ideal the lenght of the resolution is just the minimal number of generators. So $\operatorname{pd} k[\Delta^*] \leq \mu(\Delta^*)$ is certainly true. □

In the previous proof we have also shown the following equality which is also interesting itself:

Corollary 3.2. *Using the same notation and hypothesis as above, we have:*

$$\operatorname{reg} I_\Delta = \operatorname{pd} k[\Delta].$$

Let $S = k[x_1, \dots, x_n]$ denote the polynomial ring in n variables, let $M = \{x_1, \dots, x_n\}$ be the set of variables. Let $I = (f_1, \dots, f_s)$ be a (not necessarily square-free) monomial ideal in S.

Definition 3.3. ([14]). *Let 2^M be the set of all subsets of M. We denote by* $\operatorname{supp}(m)$ *the set of variables of the monomial m. A pair (m, Z) consisting of a monomial $m \in S$ and a set $Z \in 2^M$ is called admissible if $Z \cap \operatorname{supp}(m) = \emptyset$. We define a partial order $\leq$ on the set of all admissible pairs by*

$$(m, Z) \leq (m', Z') \Longleftrightarrow m \text{ divides } m' \text{ and } \operatorname{supp}(\frac{m'}{m}) \cup Z' \subseteq Z.$$

An admissible pair (m, Z) is called standard *with respect to I if*

i.) $m \cdot k[Z] \cap I = \{0\}$ *and*
ii.) (m, Z) *is minimal with respect to $\leq$ in the set of pairs satisfying i.).*

The number of all standard admissible pairs of I will be denoted by $\operatorname{std}(I)$.

In the following we would like to compare the standard admissible pairs of $I = (f_1, \dots, f_s)$ and of a square-free monomial ideal formed by polarizing I. For this purpose the conditions i.) and ii.) of the above definition will be substituted by equivalent conditions that make the comparison easier. (Without loss of generality we may assume that none of the generators divides another generator of I).
If we write an admissible pair as $(x_{i_1}^{\alpha_{i_1}} \cdots x_{i_r}^{\alpha_{i_r}},\ Z = \{x_j : j \notin \{i_1, \dots, i_r\}\})$, $\alpha_i \geq 0$, the conditions for an admissible pair to be standard can be reformulated as follows:

1.) $\forall l \leq s\ \exists j(l) \in \{i_1, \dots, i_r\} : x_{j(l)}^{\alpha_{j(l)}+1}$ divides f_l
2.) $\forall Z' \supset Z,\ Z' \neq Z,\ \exists l \leq s\ \exists a \in S$ monomial: $a \cdot f_l \in x_{j_1}^{\alpha_{j_1}} \cdots x_{j_p}^{\alpha_{j_p}} \cdot k[Z']$, where $\{x_{j_1}, \dots, x_{j_p}\} = M \setminus Z'$.

The equivalence of the conditions 1.) and i.) is easy to check:
If there is no such $j(l)$, then we can find a monomial $a \in S$, such that $a \cdot f_l \in x_{i_1}^{\alpha_{i_1}} \cdots x_{i_r}^{\alpha_{i_r}} \cdot k[Z]$ and therefore the condition i.) cannot be satisfied. On the other hand we will never find a monomial $a \in S$ such that $a \cdot f_l \in x_{i_1}^{\alpha_{i_1}} \cdots x_{i_r}^{\alpha_{i_r}} \cdot k[Z]$, if there is a monomial $b \in S$ such that $b \cdot x_{j(l)}^{\alpha_{j(l)}+1} = f_l$. In a similar way also the equivalence of 2.) and ii.) can be checked:

If for some set $Z' \supset Z$, $Z' \neq Z$ we cannot find a generator f_l of I and a monomial $a \in S$ such that $a \cdot f_l \in x_{j_1}^{\alpha_{j_1}} \cdots x_{j_p}^{\alpha_{j_p}} \cdot k[Z']$, then the pair $(x_{j_1}^{\alpha_{j_1}} \cdots x_{j_p}^{\alpha_{j_p}}, Z')$ satisfies the condition 1.) and is less than our pair with respect to our order $\leq$. If conversely we can find such l and a for all $Z' \supset Z$, $Z' \neq Z$, then our pair is indeed minimal.

Now we are interested in the square-free monomial ideal I_{Pol} corresponding to I by polarization in the following way:

We replace each factor $x_i^{\alpha_i}$ of the minimal generators of the monomial ideal I by $x_{(i,1)} \cdots x_{(i,\alpha_i)}$. We always use the lowest possible second index. We have $I_{Pol} \subset S_{Pol} = k[x_{(1,1)}, \dots, x_{(1,\beta_1)}, \dots, x_{(n,\beta_n)}]$, where β_i is the maximum over all exponents of x_i in the generators of I. Our new set of variables will be denoted by $M_{Pol} = \{x_{(1,1)}, \dots, x_{(n,\beta_n)}\}$.
Since I_{Pol} is generated by square-free monomials, all standard admissible pairs of I_{Pol} must be of the structure $(1, Z_{Pol})$ for a suitable subset $Z_{Pol} \subset M_{Pol}$ ([14], Lemma 3.5). By our special choice of the polarization we achieved also that whenever $x_{(j,p)}$ is a factor of a minimal generators of I_{Pol}, so is $x_{(j,\tilde{p})}$ for all $\tilde{p} \leq p$. Therefore none of the subsets Z_{Pol} can lack more than one variable with the same first index.
Hence we can write a standard admissible pair of I_{Pol} as $(1, Z_{Pol} = \{x_{(j,p)} : (j, p) \notin \{(i_1, p_1), \dots, (i_r, p_r)\}\})$, where $i_1 < \dots < i_r$. By the same considerations as before we can now reformulate the conditions for an admissible pair of the above structure to be standard:

1_{Pol}.) $\forall l \leq s \ \exists (j_l, p_l) \in M_{Pol} \setminus Z_{Pol} : x_{(j_l, p_l)}$ divides $f_{l\ Pol}$
2_{Pol}.) $\forall Z'_{pol} \supset Z_{Pol},\ Z'_{Pol} \neq Z_{Pol},\ \exists l \leq s : f_{l\ Pol} \in k[Z'_{Pol}]$.

Remark 3.4. *Since I_{Pol} is a square-free monomial ideal, it is corresponding to some simplicial complex Δ. From this point of view, the condition 1_{Pol}.) means that $Z_{Pol} \in \Delta$. 2_{Pol}.) is equivalent to saying that Z_{Pol} is maximal among the faces of Δ under condition 1_{Pol}.).*

Lemma 3.5. *There is a 1 to 1 correspondence between the standard admissible pairs of I and those of I_{Pol}.*

Proof. Claim: A pair $(x_{i_1}^{\alpha_{i_1}} \cdots x_{i_r}^{\alpha_{i_r}},\ Z = \{x_i : i \notin \{i_1, \dots, i_r\}\})$ is a standard admissable pair of I iff $(1, Z_{Pol} = \{x_{(i,j)} : (i, j) \notin \{(i_1, \alpha_{i_1} + 1), \dots, (i_r, \alpha_{i_r} + 1)\}\})$ is a standard admissible pair of I_{Pol}.
The equivalence of the two conditions 1.) and 1_{Pol}.) is a direct consequence of the fact that $x_{(j,p)}$ divides $f_{l\ Pol}$, iff x_j^p divides f_l by our construction of I_{Pol}.
The condition 2.) just means that the factor x_i of f_l has at most the exponent α_i for each $i \in \{i_1, \dots, i_r\}$ or in other words after polarization $x_{(i,\alpha_i+1)}$ cannot be a factor of $f_{l\ Pol}$. Hence condition 2.) implies condition 2_{Pol}.). If conversely $x_{(i,\alpha_i+1)}$ is not a factor of $f_{l\ Pol}$ the exponent of x_i in f_l cannot have exceeded α_i before polarization. □

As shown by Sturmfels, Trung and Vogel ([14], Lemma 3.3) $\text{std}(I) = \text{arith-deg}(I)$. Thus our last lemma shows:

Corollary 3.6. arith-deg I = artith-deg I_{Pol}.

Remark 3.7. *The inequality*

$$\text{arith-deg}\, I \leq \text{arith-deg}\, I_{Pol}$$

can also be seen as a special case of ([10], Theorem 2.1]*).*

For square-free monomial ideals the arithmetic degree coincides with the geometric degree. So combining this corollary and the bound for the regularity of a square-free monomial ideal, we have shown:

Theorem 3.8. *Let $I \subset k[x_1, \dots, x_n]$ be a monomial ideal and suppose* $\operatorname{codim} k[x_1, \dots, x_n]/I \geq 2$*. Then the regularity of I is bounded by its arithmetic degree, i.e.,*

$$\operatorname{reg} I \leq \text{arith-deg}\, I\,.$$

4. A better bound for some square-free monomial ideals.

In this section, we generalize a theorem in [16] that is an affirmative answer for a certain conjecture of Eisenbud [6], which is a monomial version of Eisenbud-Goto Conjecture [5]. The class which we consider includes the class of Buchsbaum Stanley-Reisner rings, while the class considered in [16] does not include that class completely.

Theorem 4.1. ([16]). *Let k be a field and let Δ be a pure simplicial complex connected in codimension 1. Then we have*

$$\operatorname{reg} I_\Delta \leq \deg k[\Delta] - \operatorname{codim} k[\Delta] + 1.$$

Corollary 4.2. *Let k be a field and let Δ be a simplicial complex consisting of connected components $\Delta_1, \dots, \Delta_s$, each of these pure and connected in codimension 1. Then we have*

$$\operatorname{reg} I_\Delta \leq \max_{1\leq i\leq s} \{\deg k[\Delta_i] - \operatorname{codim} k[\Delta_i] + 1\}.$$

Theorem 4.3. *Let k be a field, let Δ be a pure simplicial complex allowing more than one connected component, each of them connected in codimension one. Then we get the following bound for the regularity:*

$$\deg k[\Delta] - \operatorname{codim} k[\Delta] + \dim \widetilde{H}_0(\Delta; k) \cdot \dim \Delta + 1 \geq \operatorname{reg} I_\Delta .$$

We give a proof simplified by suggestions of Eisenbud. For reader's convenience, we overlap some parts of the proof in [16].
Proof. Let V be the vertex set of Δ. Put $\#(V) = n$ and $\dim k[\Delta] = d$. We prove the theorem by induction on the number f_{d-1} of facets in Δ.

First if $\operatorname{codim} k[\Delta] \leq 1$, then $k[\Delta]$ is a hypersurface. In this case the theorem is clear.

Suppose $\operatorname{codim} k[\Delta] \geq 2$ and $f_{d-1} \geq 2$. Then there exists a facet $\sigma \in \Delta$ such that

$$\Delta' := \Delta \setminus \{\tau \in \Delta \mid \text{ for any facet } \rho(\neq \sigma) \in \Delta; \tau \not\subset \rho\}$$

is pure and connected in codimension 1. Denote by V' the vertex set of Δ' and by f'_{d-1} the number of facets in Δ'. There are three cases.

Case (i). $V \neq V'$. Put $V \setminus V' = \{v\}$. For $W \subset V$ with $v \notin W$ we have $\Delta_W \cong \Delta'_W$. On the other hand, for $W \subset V$ with $v \in W$, Δ_W has the same homotopy type with $\Delta'_{W\setminus\{v\}}$. Since

$$\operatorname{reg} I_\Delta = \max\{i + 2 \mid \widetilde{H}_i(\Delta_W; k) \neq 0 \text{ for some } W \subset V\},$$

we have

$$\begin{aligned} \operatorname{reg} I_\Delta &= \operatorname{reg} I_{\Delta'} \\ &\leq f'_{d-1} - (n-1-d) + (d-1)\dim \widetilde{H}_0(\Delta'; k) + 1 \\ &= f_{d-1} - (n-d) + (d-1)\dim \widetilde{H}_0(\Delta; k) + 1. \end{aligned}$$

Case (ii). $V = V'$. We have $\operatorname{reg} I_\Delta = \operatorname{pd} k[\Delta^*]$ by Corollary 2.2. If we prove $\operatorname{pd} k[\Delta^*] \leq \operatorname{pd} k[(\Delta')^*] + 1$, we have

$$\begin{aligned} \operatorname{reg} I_\Delta &\leq \operatorname{reg} I_{\Delta'} + 1 \\ &\leq f'_{d-1} - (n-d) + (d-1)\dim \widetilde{H}_0(\Delta'; k) + 2 \\ &= f_{d-1} - (n-d) + (d-1)\dim \widetilde{H}_0(\Delta; k) + 1. \end{aligned}$$

Then we have only to prove

$$\operatorname{pd} k[\Delta^*] \leq \operatorname{pd} k[(\Delta')^*] + 1.$$

Put $k[\Delta^*] = k[(\Delta')^*]/(m)$, where $m = \Pi_{x_i \in V \setminus \sigma} x_i$. If we show that

$$\operatorname{pd} k[(\Delta')^*] \geq \operatorname{pd} (I_{(\Delta')^*} + (m))/I_{(\Delta')^*},$$

then the mapping cone guarantees that

$$\operatorname{pd} k[\Delta^*] \leq \operatorname{pd} k[(\Delta')^*] + 1$$

by [4], Exercise A.3.30. But now we have

$$\begin{aligned}(I_{(\Delta')^*} + (m))/I_{(\Delta')^*} &\cong (m)/((m) \cap I_{(\Delta')^*}) \\ &\cong (m)/((m) \cap (m_1, \dots, m_t)) \\ &\cong (m)/(\operatorname{lcm}(m, m_1), \dots, \operatorname{lcm}(m, m_t)) \\ &\cong A/(m'_1, \dots, m'_t) \oplus_A (m),\end{aligned}$$

where $I_{(\Delta')^*} = (m_1, \dots, m_t)$, $m'_i = \frac{\operatorname{lcm}(m,m_i)}{m}$ and $A = k[x_i \mid x_i \in V]$. Hence, we have only to show

$$\operatorname{pd} k[(\Delta')^*] \geq \operatorname{pd} A/(m'_1, \dots, m'_t).$$

Now we have $k[(\Delta')^*]_m \cong A_m/(m'_1, \dots, m'_t)A_m$. Hence we have

$$\operatorname{pd} k[(\Delta')^*] \geq \operatorname{pd} k[(\Delta')^*]_m = \operatorname{pd} A_m/(m'_1, \dots, m'_t)A_m = \operatorname{pd} A/(m'_1, \dots, m'_t).$$

Case (iii). $V' \subset V$, $\#V' = \#V - d$.
This situation corresponds to taking away the last simplex of a connected component. Therefore we have:

$$\begin{aligned}\deg k[\Delta] &= \deg k[\Delta'] + 1 \\ \operatorname{codim} k[\Delta] &= \operatorname{codim} k[\Delta'] + d \\ \dim \widetilde{H}_0(\Delta; k) &= \dim \widetilde{H}_0(\Delta'; k) + 1.\end{aligned}$$

Then we have

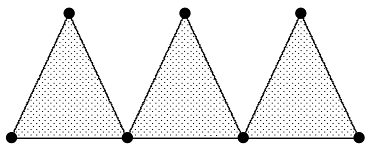
$$\begin{aligned}\operatorname{reg} I_\Delta = \operatorname{reg} I_{\Delta'} &\leq \deg k[\Delta'] - \operatorname{codim} k[\Delta'] + (d-1) \dim \widetilde{H}_0(\Delta'; k) + 1 \\ &= \deg k[\Delta] - \operatorname{codim} k[\Delta] + \dim \Delta + (d-1) \dim \widetilde{H}_0(\Delta; k) + 1. \qquad \square\end{aligned}$$

Corollary 4.4. *Let Δ be a Buchsbaum complex. Then we have*

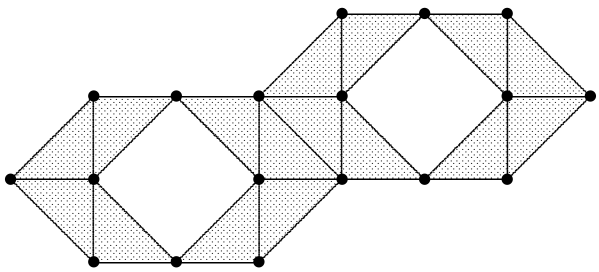
$$\deg k[\Delta] - \operatorname{codim} k[\Delta] + \dim \widetilde{H}_0(\Delta; k) \cdot \dim \Delta + 1 \geq \operatorname{reg} I_\Delta.$$

The condition that each connected component has to be connected in codimension 1 cannot be dropped by introducing a correcting term as we did when allowing more than one connected component, at least not without making the bound much weaker for most situations. If the simplicial complex is no longer connected in codimension 1, we can have various situations each leading to a different correcting term, e.g.

1.) There is a maximal face such that only one vertex of it is also a vertex of another maximal face. When taking away this maximal face in the process we loose not only one but $\dim \Delta$ vertices. So the formula needs to be corrected by adding $\dim \Delta - 1$ to the bound for each time this situation appears:



2.) There is a maximal face such that we loose just one vertex but also a $(\dim \Delta - 1)$-cycle. In this step the regularity drops by $\dim \Delta - 1$, if it was the last cycle at all that we lost. In this situation we can correct by adding $\dim \Delta - 1$ just once:



These two cases are the extremes, everything in between is also possible. Therefore a useful bound in this situation requires more knowledge about the structure of the simplicial complex. We can of course use the biggest possible correcting term "$+(\deg k[\Delta] - 1)(c - 1)$" if we know that the complex is connected in codimension c. But the formula then gives a bound that is far above $\deg k[\Delta]$ in most situations.

In one special situation we can also give an exact formula for the regularity in terms of degree, codimension and homology of the simplicial complex:

Remark 4.5. *Let Δ be a 1-dimensional simplicial complex. Then we have*

$$\operatorname{reg} I_\Delta = \deg k[\Delta] - \operatorname{codim} k[\Delta] + \dim \widetilde{H}_0(\Delta, k) - \max\{0, \dim \widetilde{H}_1(\Delta, k) - 1\}.$$

Proof. By Hochster's formula we know that for curves the regularity does not exceed 3 and that it is equal to 2 iff $\dim \widetilde{H}_1(\Delta, k) = 0$. Furthermore we know that $\deg k[\Delta] = f_1$ and that codim k $[\Delta] = f_0 - 2$. So we can prove the above formula by checking each of the two cases explicitly:

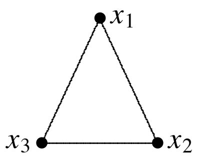
case 1: $\operatorname{reg} I = 2$
$\deg k[\Delta] - \operatorname{codim} k[\Delta] + \dim \widetilde{H}_0(\Delta, k) - \max\{0, \dim \widetilde{H}_1(\Delta, k) - 1\} = f_1 - f_0 + 2 + \dim \widetilde{H}_0(\Delta, k) - 0.$
Using the Euler-Poincaré formula this equals to 2 which is just the regularity of I

case 2: $\operatorname{reg} I = 3$
$\deg k[\Delta] - \operatorname{codim} k[\Delta] + \dim \widetilde{H}_0(\Delta, k) - \max\{0, \dim \widetilde{H}_1(\Delta, k) - 1\} = f_1 - f_0 + 2 + \dim \widetilde{H}_0(\Delta, k) - \dim \widetilde{H}_1(\Delta, k) + 1.$
Using again the Euler-Poincaré formula this is just 3. □

5. Examples.

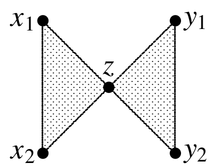
In this section we will show some examples to prove that the bounds computed in the previous sections are sharp and that each of those two bounds is in some situations the best. We will also include a third bound $\dim k[\Delta] + 1 \geq \operatorname{reg} I_\Delta$ which is just a consequence of the fact that $\operatorname{reg} I_\Delta = \max\{i + 2 : \exists W \subset V : \widetilde{H}_i(\Delta_W; k) \neq 0\}$.

1. All bounds are sharp
   For the hollow n-simplex we have:



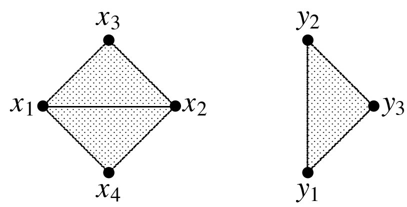
$I_\Delta = (x_1 \dots x_{n+1})$
$\deg k[\Delta] = n+1$
$\operatorname{codim} k[\Delta] = 1$
$\dim k[\Delta] = n$
$\operatorname{reg} I_\Delta = n+1$
Bound 1: $\deg k[\Delta] = n+1$
Bound 2: $\deg k[\Delta] - \operatorname{codim} k[\Delta] + 1 = n+1$
Bound 3: $\dim k[\Delta] + 1 = n+1$

2. Bound 1 is the best
For two n-simplices joined in one vertex we have:



$I_\Delta = (x_1y_1, \dots, x_1y_n, \dots, x_ny_n)$
$\deg k[\Delta] = 2$
$\operatorname{codim} k[\Delta] = n$
$\dim k[\Delta] = n+1$
$\operatorname{reg} I_\Delta = 2$
Bound 1: $\deg k[\Delta] = 2$
Bound 2: not applicable$-\Delta$ not connected in codimension 1
Bound 3: $\dim k[\Delta] + 1 = n+2$

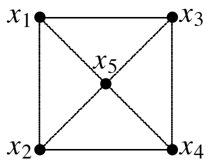
3. Bound 2 is the best even in the generalized version:



$I_\Delta = (y_1x_1, \dots, y_1x_{n+2}, \dots, y_{n+1}x_{n+2}, x_{n+1}x_{n+2})$
$\deg k[\Delta] = 3$
$\operatorname{codim} k[\Delta] = n+2$
$\dim \widetilde{H}_0(\Delta, k) = 1$
$\dim k[\Delta] = n+1$
$\operatorname{reg} I_\Delta = 2$

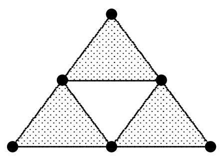
Bound 1: $\deg k[\Delta] = 3$
Bound 2: $\deg k[\Delta] - \operatorname{codim} k[\Delta] + 1 + \dim \widetilde{H}_0(\Delta, k) \cdot \dim \Delta = 2$
Bound 3: $\dim k[\Delta] + 1 = n + 2$

4. Bound 3 is the best



$I_\Delta = (x_1x_3, x_2x_4, x_1x_2x_5, x_2x_3x_5, x_3x_4x_5, x_1x_4x_5)$
$\deg k[\Delta] = 8$
$\operatorname{codim} k[\Delta] = 3$
$\dim k[\Delta] = 2$
$\operatorname{reg} I_\Delta = 3$
Bound 1: $\deg k[\Delta] = 8$
Bound 2: $\deg k[\Delta] - \operatorname{codim} k[\Delta] + 1 = 6$
Bound 3: $\dim k[\Delta] + 1 = 3$
This situation appears whenever there is a great number of cycles in the simplicial complex.

5. A situation where regularity, degree and codimension coincide



$\deg k[\Delta] = 3$
$\operatorname{codim} k[\Delta] = 3$
$\operatorname{reg} I_\Delta = 3$
This situation can be reproduced by any accumulation of $(n + 1)$ n-simplices such that we have a hollow n-simplex in the middle.

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