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AN EXPLICIT DESCRIPTION OF SOME SURFACES OF DEGREE 8 IN $\mathbb{P}^5$

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1. Introduction.

Many mathematicians have studied the classification by the degree d of embedded smooth projective varieties (see for instance [1], [3], [5], [6], [8], [9]). For $d \leq 6$ Inonescu gave a complete list ([5],[6]). Later on the same author ([7]) broadened the classification of smooth projective varieties up to degree 8. In particular, he constructed ([7]; 4.2) a smooth projective surface of degree 8 in $\mathbb{P}^5$ using Reider's Theorem in the following way:

Let X be a geometrically ruled surface over a curve of genus 2 with invariant $e = -2$ and let $H \equiv C_0 + 3F$ where C_0 and F are the generators of the Picard group of X. Then, by Reider's Theorem H is very ample and therefore embeds X as a smooth projective surface in $\mathbb{P}^5$.

It is well known that if X is a geometrically ruled surface over a curve C, then there exists a locally free sheaf $\mathcal{E}$ of rank two on C such that $X = \mathbb{P}(\mathcal{E})$.

This gives rise the following question:

Which are the vector bundles $\mathcal{E}$ over a curve C of genus 2 such that $X = \mathbb{P}(\mathcal{E})$ is the surface described by Ionescu and when does $O_{\mathbb{P}(\mathcal{E})}(1)$ embed X in $\mathbb{P}^5$ as a smooth surface of degree 8?

The aim of this note is to explain a method for the study of this question. This allows us to describe this vector bundles. Moreover, it may be able to give us an explicit description of these surfaces.

Interest in classical questions of this sort continues to the present day, but in our days we can use new methods. Lately, computer programs as *Macaulay2* have emerged as important tools in the study of geometrical problems.

Our idea is to construct a family of vector bundles, rewrite the construction in terms of graded modules and finally, use *Macaulay2* to analyze the properties of the modules and rings so constructed. If these verify some "good" properties, we are able to claim that we have constructed a right family of vector bundles and to give the ideal defining the surface corresponding to each vector bundle of the family.

Next we outline the structure of this note. In Section 2 we construct a "good" family of rank two vector bundles on a curve of genus two in the sense that they are the candidates for to be the solution of the question. In the first part of the Section 3 we translate the construction in terms of modules. Mainly we will associated to each vector bundle $\mathcal{E}$ of the family constructed in the section 2, a module E. We will describe a routine in *Macaulay2* that determines this module and the ideal I in the coordinate ring S of $\mathbb{P}^5$ of the image of $X = \mathbb{P}(\mathcal{E})$ by the line bundle $O_{\mathbb{P}(\mathcal{E})}(1)$. In the second part of Section 3 we will study the very ampleness of $O_{\mathbb{P}(\mathcal{E})}(1)$ in terms of the ring S/I. Finally, in section 4 we will explain how to realize all the process and how to check the very ampleness with the support of *Macaulay2*. We hope that in a short period of time we will be able to publish these routines and explicit examples of Ionescu's construction.

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2. Construction of vector bundles on a curve.

Notations. Basically we employ the same definitions one can find in the book of D. Eisenbud and the book of R. Hartshorne in the references. For convenience we recall some of the notations that we will use in the sequel:

1. C will be a smooth, irreducible projective curve of genus $g = g(C) = 2$, K_C denotes the canonical divisor on C and $O_C(D)$ denotes the line bundle associated to the divisor D on C.
2. $H^i(\mathcal{F}) := H^i(C, \mathcal{F})$ denotes the i-th cohomology group of the sheaf $\mathcal{F}$ on C and $h^i(\mathcal{F}) = h^i(C, \mathcal{F})$ denotes its dimension.
3. $H^i_I(M)$ denotes the i-th local cohomology group of a graded module M with support in I.

We begin this section recalling the construction given by Ionescu in [7]:

Let X be a geometrically ruled surface over a curve of genus 2 with invariant $e = -2$ and let $H \equiv C_0 + 3F$ where C_0 and F are generators of $\mathrm{Pic}(X)$. Then by Reider's Theorem H is very ample and therefore it embeds X in $\mathbb{P}^5$ as a smooth projective surface of degree $H^2 = 8$.

We will say that a smooth projective surface X is of Ionescu's type if it is a ruled surface over a curve of genus 2 with invariant $e = -2$. We will say that a rank two vector bundle $\mathcal{E}$ on a curve C verifies Ionescu's conditions if $X := \mathbb{P}(\mathcal{E})$ is of Ionescu's type and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ embeds X in $\mathbb{P}^5$ as a smooth projective surface of degree 8.

The aim of this section is to construct rank two vector bundles $\mathcal{E}$ verifying Ionescu's condition. Before starting the construction of such vector bundles, we quote three well known results as we need ([4]; V, 2).

Lemma 2.1. *If $\pi : X \longrightarrow C$ is a ruled surface, it is possible to write $X = \mathbb{P}(\mathcal{E}')$ where $\mathcal{E}'$ is a locally free sheaf on C with the property that $H^0(\mathcal{E}') \neq 0$ but for all invertible sheaves $\mathcal{O}_C(D)$ on C with $\deg(D) < 0$, we have $H^0(\mathcal{E}'(D)) = 0$. Moreover in this case the invariant e of X is given by $e = -\deg(\mathcal{E}')$. In this case we say $\mathcal{E}'$ is normalized.*

Lemma 2.2. *If $\mathcal{E}$ and $\mathcal{E}'$ are two locally free sheaves of rank two on C, then $\mathbb{P}(\mathcal{E}')$ and $\mathbb{P}(\mathcal{E})$ are isomorphic as ruled surfaces over C if and only if there is an invertible sheaf $\mathcal{O}_C(D)$ on C such that*

$$\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{O}_C(D).$$

Lemma 2.3. *Let $\mathcal{E}$ be a coherent sheaf of rank two on C and let D be a divisor on C. Then*

$$c_1(\mathcal{E}(D)) = c_1(\mathcal{E}) + 2\deg(D).$$

Assume that $\mathcal{E}$ is a rank two vector bundle on a curve C verifying Ionescu's condition. From the definition of Ionescu's condition $h^0(\mathcal{E}) = 6$.

On the other hand, consider the natural map

$$\pi : \mathbb{P}(\mathcal{E}) \longrightarrow C$$

and H such that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{O}_X(H)$. By definition of the Chern classes we have:

$$\pi^* c_0(\mathcal{E}) H^2 - \pi^* c_1(\mathcal{E}) H + \pi^* c_2(\mathcal{E}) = 0.$$

Since $\mathcal{E}$ is a vector bundle on a curve, $c_2(\mathcal{E}) = 0$ and therefore,

$$8 = H^2 = \pi^* c_1(\mathcal{E}) H = c_1(\mathcal{E}) = \deg(\mathcal{E})$$

where the first equality follows from the fact that $\mathcal{E}$ verifies Ionescu's condition.

So, by Lemma 2.1 and Lemma 2.2, $\mathcal{E}$ is a rank two vector bundle on C such that

$$h^0(\mathcal{E}) = 6; \quad \deg(\mathcal{E}) = 8$$

and there exists a normalized rank two vector bundle $\mathcal{E}'$ on C such that

$$\mathcal{E} \cong \mathcal{E}' \otimes O_C(D)$$

where $D \in \mathrm{Pic}(C)$ has degree 3. In fact, $X := \mathbb{P}(\mathcal{E})$ has invariant $e = -2 = -\deg(\mathcal{E}')$ and by Lemma 2.3:

$$8 = \deg(\mathcal{E}) = c_1(\mathcal{E}) = c_1(\mathcal{E}' \otimes O_C(D)) = \deg(\mathcal{E}') + 2\deg(D) = 2 + 2\deg(D).$$

By Riemann-Roch Theorem,

$$h^0(\mathcal{E}) - h^1(\mathcal{E}) = \chi(\mathcal{E}) = \deg(\mathcal{E}) + rk(\mathcal{E})(1-g) = 6.$$

Therefore, $\mathcal{E}$ is a rank two vector bundle on C such that

$$h^0(\mathcal{E}) = 6 \quad \text{and} \quad h^1(\mathcal{E}) = 0.$$

We will construct a family $\mathcal{F}$ of rank two vector bundles $\mathcal{E}$ on a curve C of genus 2 such that

$$h^0(\mathcal{E}) = 6 \quad \text{and} \quad h^1(\mathcal{E}) = 0. \tag{1}$$

Notice that every $\mathcal{E} \in \mathcal{F}$ such that the line bundle $O_{\mathbb{P}(\mathcal{E})}(1)$ is very ample, verifies Ionescu's condition. So, first of all we will construct the family $\mathcal{F}$ and later on we will explain how to check the very ampleness condition.

Any locally free sheaf of rank two on a curve C is an extension of invertible sheaves ([4]; V, Corollary 2.7). We consider $\mathcal{G}$ the irreducible family of rank two vector bundles $\mathcal{E}$ on C given by a non trivial extension:

$$e: \quad 0 \longrightarrow O_C(D_1) \longrightarrow \mathcal{E} \longrightarrow O_C(D_2) \longrightarrow 0$$

where D_1 and D_2 vary in $\mathrm{Pic}(C)$.

The next step is to study the necessary conditions that D_1 and D_2 have to verify in order to obtain a subfamily $\mathcal{F} \subset \mathcal{G}$ of rank 2 vector bundles $\mathcal{E}$ verifying (1).

Assume that $\mathcal{E} \in \mathcal{G}$ verifies (1) and consider the exact cohomology sequence:

$$0 \longrightarrow H^0(O_C(D_1)) \longrightarrow H^0(\mathcal{E}) \longrightarrow H^0(O_C(D_2)) \longrightarrow$$
$$\longrightarrow H^1(O_C(D_1)) \longrightarrow H^1(\mathcal{E}) \longrightarrow H^1(O_C(D_2)) \longrightarrow 0$$

associated to the exact sequence:

$$e: \quad 0 \longrightarrow O_C(D_1) \longrightarrow \mathcal{E} \longrightarrow O_C(D_2) \longrightarrow 0.$$

Since $h^1(\mathcal{E}) = 0$ we have $h^1(O_C(D_2)) = 0$. On the other hand, by the additivity of the Euler Characteristic,

$$(2) \qquad 6 = \chi(\mathcal{E}) = \chi(O_C(D_1)) + \chi(O_C(D_2)) = \deg(D_1) + \deg(D_2) - 2$$

where the last equality follows from Riemann-Roch Theorem.

Twisting by $O_C(-D_1)$ the exact sequence e and taking cohomology we obtain the long exact sequence:

$$0 \longrightarrow H^0(O_C) \longrightarrow H^0(\mathcal{E}(-D_1)) \longrightarrow H^0(O_C(D_2 - D_1)) \longrightarrow \cdots$$

and from this we get $h^0(\mathcal{E}(-D_1)) > 0$.

We have seen that $\mathcal{E} \cong \mathcal{E}' \otimes O_C(D)$ where $D \in \mathrm{Pic}(C)$ has degree 3 and $\mathcal{E}'$ is normalized.

Since $\mathcal{E}'$ is normalized and $0 < h^0(\mathcal{E}(-D_1)) = h^0(\mathcal{E}'(D - D_1))$ it follows that $\deg(D - D_1) \geq 0$ and we obtain:

$$(3) \qquad \deg(D_1) \leq 3; \quad \deg(D_2) \geq 5$$

where the second inequality follows from (2).

Notice that since $\deg(K_C) = 2g(C) - 2 = 2$, the above inequalities on the degrees of D_1 and D_2 give us:

$$\dim \mathrm{Ext}^1(O_C(D_2), O_C(D_1)) = h^1(O_C(D_1 - D_2)) =$$
$$= h^0(O_C(D_2 - D_1 + K_C))^* > 0$$

and therefore, we have non trivial extensions.

Let $\mathcal{F} \subset \mathcal{G}$ be the subfamily of rank two vector bundles $\mathcal{E}$ on C given by a non trivial extension:

$$e: \quad 0 \longrightarrow O_C(D_1) \longrightarrow \mathcal{E} \longrightarrow O_C(D_2) \longrightarrow 0$$

with $D_1, D_2 \in \mathrm{Pic}(C)$ verifying:

$$\deg(D_1) \leq 3 \quad \text{and} \quad \deg(D_2) \geq 5;$$
$$6 = \deg(D_1) + \deg(D_2) - 2;$$
$$h^1(O_C(D_1)) = 0.$$

Claim. *$\mathcal{F}$ is a non-empty family and each $\mathcal{E} \in \mathcal{F}$ verifies* (1).

Proof of the Claim. If D_1 has degree 3, by Riemman-Roch Theorem

$$h^1(O_C(D_1)) = 0$$

and in this case $\deg(D_2) = 5$, which implies $h^1(O_C(D_2)) = 0$. The exact cohomology sequence:

$$\begin{aligned} 0 \longrightarrow H^0(O_C(D_1)) \longrightarrow H^0(\mathcal{E}) \longrightarrow H^0(O_C(D_2)) \longrightarrow \\ \longrightarrow H^1(O_C(D_1)) \longrightarrow H^1(\mathcal{E}) \longrightarrow H^1(O_C(D_2)) \longrightarrow 0 \end{aligned}$$

associated to the exact sequence:

$$e : \ 0 \longrightarrow O_C(D_1) \longrightarrow \mathcal{E} \longrightarrow O_C(D_2) \longrightarrow 0$$

with D_1 of degree 3 and D_2 of degree 5, gives us:

$$h^1(\mathcal{E}) = 0.$$

On the other hand, using the exact sequence which defines $\mathcal{E}$ we get:

$$\deg(\mathcal{E}) = c_1(\mathcal{E}) = \deg(D_1) + \deg(D_2) = 8.$$

Therefore by Riemann-Roch Theorem:

$$h^0(\mathcal{E}) = \chi(\mathcal{E}) = \deg(\mathcal{E}) + \operatorname{rank}(\mathcal{E})(1 - g(C)) = 6$$

which proves our claim.

Remark. It is well known that there is a one-to-one correspondence between sections $\sigma : C \longrightarrow X = \mathbb{P}(\mathcal{E})$ and surjections $\mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$, where $\mathcal{L}$ is an invertible sheaf of C given by $\mathcal{L} = \sigma^* O_{\mathbb{P}(\mathcal{E})}(1)$ ([4]; V, Proposition 2.6). On the other hand since $\deg(D_2) \geq 2g(C) + 1$, $O_C(D_2)$ is very ample and by this correspondence there is a section such that $O_C(D_2) = \sigma^* O_{\mathbb{P}(\mathcal{E})}(1)$. Therefore, one guess that $O_{\mathbb{P}(\mathcal{E})}(1)$ given by the above construction is not so far to be very ample.

For each $\mathcal{E}$ in $\mathcal{F}$ we consider the morphism corresponding to the sheaf $O_{\mathbb{P}(\mathcal{E})}(1)$ on $X = \mathbb{P}(\mathcal{E})$ defined by the sections $s_0, \cdots, s_5 \in H^0(O_{\mathbb{P}(\mathcal{E})}(1))$:

$$\phi : X \longrightarrow \tilde{X} \subset \mathbb{P}(H^0(O_{\mathbb{P}(\mathcal{E})}(1))^*) = \mathbb{P}^5.$$

We want to find vector bundles $\mathcal{E} \in \mathcal{F}$ such that $O_{\mathbb{P}(\mathcal{E})}(1))$ is very ample (i.e. $\mathcal{E}$ verifies Ionescu's condition) or, equivalently, vector bundles such that ϕ embeds X as a smooth surface $\tilde{X}$ of degree eight in $\mathbb{P}^5$. To this end, we will translate the above construction in terms of extensions of modules and we will associated to the vector bundle $\mathcal{E}$ a module E. Then, using some functions that we have defined in *Macaulay2*, we will compute the coordinate ring of $\tilde{X}$ and we will explain how to check the very ampleness of $O_{\mathbb{P}(\mathcal{E})}(1)$ in terms of this ring.

3. Development of the problem.

3.1. Algebraic Construction.

The goal of this subsection is to translate in terms of modules the construction that we have seen in Section 2. Mainly, we will associate to each extension of sheaves an extension of modules. In particular we will associated to each vector bundle given by such extension a module. To this end, first of all we recall some well known constructions which give a relation between sheaves and modules.

Almost all the facts that we explain here hold in a more general situation but we prefer restrict ourselves in the context of the problem that we are studying.

Let C be a smooth, irreducible, projective curve of genus two and $O_C(D)$ be a line bundle on C associated to the divisor D. Consider R the coordinate ring of C and denote by $I(D) \subset R$ the ideal associated to the divisor D.

It is well known that there exists a canonical graded module representing $O_C(D)$, given by:

$$M_D := H^0_*(O_C(D)) := \oplus_{n\geq 0} H^0(O_C(D)(n)).$$

On the other hand given M a graded R-module we can consider $\tilde{M}$ the sheaf of modules associated to M ([4]; II.5 for more details). In particular if $I(D)$ is the ideal associated to D then,

$$O_C(D) = \widetilde{I(D)}^*$$

where $*$ denotes the dual of a module.

Next we will recall the definition of local cohomology and we will see how this last two constructions can be related.

Definition 3.1.1. *Let R be a ring, I an ideal of R, and M a R-module. We define the zeroeth local cohomology module of M with support in I to be the set of all elements of M which are annihilated by some power of I:*

$$H^0_I(M) = \cup_n (0 :_M I^n)$$

where $(0 :_M I^n)$ denotes the set of elements of M annihilated by I^n. We define the higher local cohomology groups as the right derived functors of H^0_I.

The following theorem will be very useful for us in the sequel.

Theorem 3.1.2. *Let R be a polynomial ring and Q a maximal ideal. Take M a graded R-module and consider $\tilde{M}$ the corresponding sheaf associated to M. Then there is the exact sequence:*

$$0 \longrightarrow H^0_Q(M) \longrightarrow M \longrightarrow H^0_*(\tilde{M}) \longrightarrow H^1_Q(M) \longrightarrow 0.$$

Moreover, if $\operatorname{depth}(M) > 1$ *then* $H^0_Q(M) = H^1_Q(M) = 0$ *and therefore*

$$M = H^0_*(\tilde{M}).$$

If C is a smooth curve, as we are assuming, for every divisor D on C we have:

$$\operatorname{depth}(I(D)^*) > 1.$$

Hence, by the above theorem we obtain the following relation between the two constructions that we have introduced:

$$M_D := H^0_*(O_C(D)) = H^0_*(\widetilde{I(D)^*}) = I(D)^*. \tag{4}$$

Putting all together we see that the module associated to the line bundle $O_C(D)$ on C is nothing more than $I(D)^* = \operatorname{Hom}(I(D), R)$. Moreover, we can describe the ideal $I(D)$ as follows.

Since D is a divisor on a curve C we have:

$$D = \sum_{i=1}^{n} m_i p_i$$

where $p_1, \cdots, p_n$ are points on C. If P_i denotes the ideal of the point p_i in the coordinate ring of the curve, we get:

$$I(D) = P_1^{m_1} \cap \cdots \cap P_n^{m_n}.$$

Let us see how to associate to each extension of sheaves:

$$e: \quad 0 \longrightarrow O_C(D_1) \longrightarrow \mathcal{E} \longrightarrow O_C(D_2) \longrightarrow 0$$

with $\mathcal{E} \in \mathcal{F}$, an extension of modules. This extension of modules will allows us to compute explicitly the module $E := \oplus_{n\geq 0} H^0(\mathcal{E}(n))$ associated to the vector bundle $\mathcal{E}$ given by e.

By our assumption on the degree of D_2 and the fact that $h^1(O_C(D_1)) = 0$, for all $n \geq 0$, we have:

$$h^1(O_C(D_1)(n)) = h^1(O_C(D_2)(n)) = 0.$$

So, if we consider the cohomology sequence associated to e we obtain for all $n \geq 0$ the exact sequence:

$$0 \longrightarrow H^0(O_C(D_1)(n)) \longrightarrow H^0(\mathcal{E}(n)) \longrightarrow H^0(O_C(D_2)(n)) \longrightarrow 0$$

and therefore, we have the exact sequence of graded modules:

$$0 \longrightarrow \oplus_{n\geq 0} H^0(O_C(D_1)(n)) \longrightarrow \oplus_{n\geq 0} H^0(\mathcal{E}(n)) \longrightarrow$$
$$\longrightarrow \oplus_{n\geq 0} H^0(O_C(D_2)(n)) \longrightarrow 0,$$

or equivalently,

$$0 \longrightarrow M_{D_1} \longrightarrow \oplus_{n\geq 0} H^0(\mathcal{E}(n)) \longrightarrow M_{D_2} \longrightarrow 0$$

and this is an exact sequence of graded R-modules associated to the exact sequence e of sheaves.

Now we will see a more explicit description of the module $E := \oplus_{n\geq 0} H^0(\mathcal{E}(n))$ associated to the vector bundle $\mathcal{E}$. This is equivalent to take an element of the extension group $\mathrm{Ext}^1(M_{D_2}, M_{D_1})$.

For simplicity we denote by A the module M_{D_2} and by B the module M_{D_1}. Let us see how to obtain an element of $\mathrm{Ext}^1(A, B)$, i.e. the module E.

Take a free resolution of A:

$$\cdots \longrightarrow H_A \longrightarrow G_A \longrightarrow F_A \longrightarrow A \longrightarrow 0$$

and K the kernel of the map:

$$F_A \longrightarrow A \longrightarrow 0.$$

By definition of the Ext-group we have the exact sequence:

$$\cdots \longrightarrow \mathrm{Hom}(F_A, B) \longrightarrow \mathrm{Hom}(K, B) \longrightarrow \mathrm{Ext}^1(A, B) \longrightarrow 0.$$

Then, we only need to take an element of $\mathrm{Hom}(K, B)$, not in $\mathrm{Hom}(F_A, B)$. Now, if we consider a free resolution of B:

$$\cdots \longrightarrow H_B \longrightarrow G_B \longrightarrow F_B \longrightarrow B \longrightarrow 0$$

since there are epimorphisms:

$$G_A \longrightarrow K \longrightarrow 0;$$

$$F_B \longrightarrow B \longrightarrow 0;$$

it is enough to take an element $\phi \in \mathrm{Hom}(G_A, F_B)$. Moreover, if we consider the matrix,

$$m : G_A \oplus G_B \longrightarrow F_A \oplus F_B$$

given by

$$m = \begin{pmatrix} \phi_A & 0 \\ \phi & \phi_B . \end{pmatrix}$$

where

$$\phi_A : G_A \longrightarrow F_A;$$

$$\phi_B : G_B \longrightarrow F_B;$$

are given by the corresponding resolutions, we obtain:

$$E := \oplus_{n \geq 0} H^0(\mathcal{E}(n)) = \mathrm{Coker}(m).$$

Therefore, we have seen that if $\mathcal{E}$ is a vector bundle given by the extension:

$$0 \longrightarrow O_C(D_1) \longrightarrow \mathcal{E} \longrightarrow O_C(D_2) \longrightarrow 0$$

we can associated to $\mathcal{E}$ a graded R-module E. Moreover, we know explicitly how to construct this module.

Following this description we have defined a function in *Macaulay2* called *extensionModules* which given two modules A and B construct a module E such that the following sequence

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$$

is exact. We will apply this routine in the case where $A = M_{D_2}$ and $B = M_{D_1}$. Let us remark that in order to obtain A and B we only need to compute the ideal associated to the corresponding divisor that we have already described in the last section.

In the next subsection we will see how, with the help of *Macaulay2*, use this module E in order to check the very ampleness of the line bundle $O_{\mathbb{P}(\mathcal{E})}(1)$ where $\mathcal{E} \in \mathcal{F}$ is the vector bundle associated to E.

3.2. *How to check the very ampleness property.*

First of all, let us review the Jacobian criterion for regularity that later on we will use in order to check the smoothness of the variety image.

Theorem 3.2.1. *Let $S = K[x_1, \cdots, x_r]$ be a polynomial ring over a field K, let $I = (f_1, \cdots, f_s)$ be an ideal, and set $R = S/I$. Let $\mathcal{P}$ be a prime ideal of S containing I and write $K(\mathcal{P})$ for the residue class field at $\mathcal{P}$. Let c be the codimension of $I_{\mathcal{P}}$ in $S_{\mathcal{P}}$.*

a) *The Jacobian matrix J whose entries are the partial derivatives of the generators of I, taken modulo $\mathcal{P}$ has rank $\leq c$.*
b) *If $char(K) = p > 0$, assume also that $K(\mathcal{P})$ is separable over K. $R_{\mathcal{P}}$ is a regular local ring if only if J taken modulo $\mathcal{P}$ has rank equal to c.*

Now, we will state how to check if the vector bundle $\mathcal{E} \in \mathcal{F}$ so constructed is such that the linear series $O_{\mathbb{P}(\mathcal{E})}(1)$ is very ample. But before we want recall the following definition.

Definition 3.2.2. *If $Y \subset \mathbb{P}^n$ is an algebraic set of dimension r, we define the Hilbert Polynomial of Y to be the Hilbert Polynomial P_Y of its homogeneous coordinate ring $S(Y)$. (It is a polynomial of degree r). We define the degree of Y to be:*

$$\deg(Y) := r!\ (\mathit{leadingcoefficient\ of}\quad P_Y).$$

Lemma 3.2.3. *Let C be a smooth irreducible projective curve of genus 2 and $\mathcal{E}$ a rank two vector bundle on C with six sections and degree eight. Consider the map*

$$\phi : X = \mathbb{P}(\mathcal{E}) \longrightarrow \tilde{X} \subset \mathbb{P}^5$$

defined by six sections of $O_{\mathbb{P}(\mathcal{E})}(1)$. Let S be the coordinate ring of $\mathbb{P}^5$ and $I \subset S$ the ideal of $\tilde{X}$. Assume that $\tilde{X}$ is smooth, $\dim(S/I) = 3$ *and that* $\deg(\tilde{X}) = 8$. *Then ϕ defines an embedding or equivalently, $\tilde{X}$ is a smooth surface of degree eight in $\mathbb{P}^5$.*

Proof. Since $\tilde{X}$ is smooth and $\dim(S/I) = 3$, ϕ defines a map between two smooth surfaces. Moreover we have the following relation:

$$\deg(\mathcal{E}) = \deg(\phi) \deg(\tilde{X}).$$

By the assumption in the degree and this relation we have $\deg(\phi) = 1$. Therefore at least ϕ defines a regular map and it is an embedding if in X there are no (-1)-curves. If there are such curves in X, each of them has to be isomorphic to $\mathbb{P}^1$. Then we have a curve of genus 0 in X. On the other hand, there is a map $\pi : X \longrightarrow C$, where C is a curve of genus two. Therefore the curves of genus 0 cannot be in the fibers of π and we get a contradiction. Therefore, ϕ is an embedding. □

At the end of Subsection 3.1 we have seen how to compute the module E associated to the vector bundle $\mathcal{E}$ given by the extension:

$$e:\ 0 \longrightarrow O_C(D_1) \longrightarrow \mathcal{E} \longrightarrow O_C(D_2) \longrightarrow 0.$$

Using the *Macaulay2* function *vectorBundleImage* one can compute the ideal I of $\tilde{X}$ in the coordinate ring S of $\mathbb{P}^5$.

Using the Jacobian criterion we can check whenever $\tilde{X}$ is smooth. Since we know the ring S/I, we can compute its dimension and the Hilbert polynomial of $\tilde{X}$. Assume that this effective computations gives us the desired values. Then, by the lemma, I defines a smooth surface of degree eight in $\mathbb{P}^5$. This means that the vector bundle $\mathcal{E}$ given by e verifies Ionescu's condition and therefore we have an explicit example of a surface of Ionescu's type. We want remark that all of this computations are effective in *Macaulay2*.

4. Examples.

Next we outline the process that we will follow in order to obtain a vector bundle $\mathcal{E}$ verifying Ionescu's condition and the ideal of the corresponding surface of Ionescu's type.

First we fix a curve C of genus 2 and we consider the coordinate ring R of C. Then, we take the ideals $I(D_1)$ and $I(D_2)$ of D_1 and D_2 respectively, where $D_1, D_2 \in \mathrm{Pic}(C)$ are such that:

$$\begin{gathered}\deg(D_1) \leq 3 \text{ and } \deg(D_2) \geq 5;\\ 6 = \deg(D_1) + \deg(D_2) - 2;\\ h^1(O_C(D_1)) = 0.\end{gathered}$$

The following process programmed in *Macaulay2*, will allow us to check if the rank 2 vector bundle $\mathcal{E}$ given by and extension:

$$0 \longrightarrow O_C(D_1) \longrightarrow \mathcal{E} \longrightarrow O_C(D_2) \longrightarrow 0$$

verifies Ionescu's condition. Moreover, if it is the case, then we will obtain an ideal which defines a surface of Ionescu's type.

Following the above notations consider:

$$\begin{aligned}B &:= \mathrm{Hom}(I(D_1), R),\\ A &:= \mathrm{Hom}(I(D_2), R).\end{aligned}$$

Then

$$E := extensionModules(B, A)$$

is the module associated to the vector bundle $\mathcal{E}$ given by an extension class:

$$e: \ 0 \longrightarrow \mathcal{O}_C(D_2 + K_C) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_C(D_2) \longrightarrow 0.$$

Let $I := imageVariety(E)$ be the ideal of the image $\tilde{X}$. Finally consider

$$L := testSurface(I).$$

If the output of L is $\{3, 8, 6\}$ it means that $dim(S/I) = 3$, $\deg(\tilde{X}) = 8$ and that $\tilde{X}$ is smooth. Therefore, it means that I defines a smooth surface of Ionescu's type and that $\mathcal{E}$ given by e verifies Ionescu's condition.

If we are not successful and we obtain an other output of L, we can choose an other pair $\{D_1, D_2\}$ and repeat the process with the new ones. Moreover, we can change the curve C. In this way one can obtain different examples of surfaces of Ionescu's type.

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