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ON GREEN AND GREEN-LAZARFELD CONJECTURES FOR SIMPLE COVERINGS OF ALGEBRAIC CURVES

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Let X be a smooth genus g curve equipped with a simple morphism $f : X \to C$, where C is either the projective line or more generally any smooth curve whose gonality is computed by finitely many pencils. Here we apply a method developed by Aprodu to prove that if g is big enough then X satisfies both Green and Green-Lazarsfeld conjectures. We also partially address the case in which the gonality of C is computed by infinitely many pencils.

1. Introduction

Let X be a smooth complex curve of genus g. For any spanned $L \in \mathrm{Pic}(X)$ and all integers i, j let $K_{i,j}(X,L)$ denote the Koszul cohomology groups introduced in [10]. Green's conjecture states that $K_{p,1}(X,\omega_X) = 0$ if and only if $p \geq g - \mathrm{Cliff}(X) - 1$, where $\mathrm{Cliff}(X)$ is the Clifford index of X, while Green-Lazarsfeld conjecture (see [11], Conjecture (3.7)) predicts that for every line bundle L on X of sufficiently large degree $K_{p,1}(X,L) = 0$ if and only if $p \geq r - \mathrm{gon}(X) + 1$, where r is the (projective) dimension of L and $\mathrm{gon}(X)$ is the gonality of X.

Both Green and Green-Lazarsfeld conjectures have been verified for the general curve of genus g (see [16], [17], [7], [3]) and for the general d-gonal

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curve of genus g (see [15] for $d \leq g/3$, [16], Corollary 1 on p. 365, for $d \geq g/3$, [7], [4]). In particular, [4] shows that both conjectures are satisfied for any smooth d-gonal curve verifying a suitable linear growth condition on the dimension of Brill-Noether varieties of pencils. Such a condition holds for the general d-gonal curve, but for special curves it turns out to be rather delicate (see [14], Statement T, and [6], Proposition 1.3).

Here we consider the case in which X is a multiple covering. Let $h : A \to B$ be a covering of degree ≥ 2 between smooth and connected projective curves. The covering h is said to be *simple* if it does not factor non-trivially, i.e. for any smooth curve D such that there are morhisms $h_1 : A \to D$ and $h_2 : D \to B$ with $h = h_2 \circ h_1$ the morphism h_2 is an isomorphism. Every covering of prime order is simple. By applying [4], Theorem 2, and [12], Theorem 1, we are going to prove the following result.

Theorem 1.1. *Let X be a smooth genus g curve equipped with a simple morphism $f : X \to C$ of degree $m \geq 2$, where C is a smooth curve of genus q whose gonality z is computed by finitely many g^1_z. Assume* $g \geq \max\{1+mq+(m-1)(2mz-5), 1+mq+(m-1)(mz-1), 4mz-9, 3mz-6\}$. *Then X satisfies both Green and Green-Lazarsfeld conjectures.*

In the special case $q = 0$, the corresponding notion of simple linear series is classical (see for instance [1]) and Green's conjecture has already been established for $m \geq 5$ in [5], Theorem 4.9, by exploiting [8] instead of [12]. By the way, for $q = 0$ our previous statement simplifies as follows.

Corollary 1.2. *Let X be a smooth genus g curve carrying a simple g^1_m of degree $m \geq 3$. If $g \geq 1+(m-1)(2m-5) \geq 4$ then X satisfies both Green and Green-Lazarsfeld conjectures.*

If instead $q > 0$ and we drop the assumption that the gonality of C is computed by finitely many pencils, we obtain with the same method the following partial result.

Proposition 1.3. *Let X be a smooth genus g curve equipped with a simple morphism $f : X \to C$ of degree $m \geq 2$, where C is a smooth curve of genus $q \geq 1$ and gonality $z \geq 2$. Assume $g \geq 1+mq+(m-1)(2mz-5)$. If $m = 2$ assume also $g \geq 8z-9$. Then $K_{p,1}(X, \omega_X) = 0$ for any $p \geq g-mz+2$ and $K_{r-mz+2,1}(X,L) = 0$ for every line bundle L on X with $h^0(X,L) = r+1$ and* $\deg(L) \geq 3g$.

Finally, if $f : X \to C$ is not simple, then $f = f_s \circ \cdots \circ f_1$ with $s \geq 2$ and each f_i a simple covering. One could hope to apply Theorem 1.1 to each covering f_i, but the numerical restrictions on the intermediate curves make such an iterative approach effective only in very few cases.

2. The proofs

Remark 2.1. Let $u : X' \to C'$ be a degree m morphism between smooth curves with X' of genus g and C' of genus q. Let $v : X' \to \mathbb{P}^1$ be a degree x morphism such that the associated morphism $(u,v) : X' \to C' \times \mathbb{P}^1$ is birational onto its image. Then $g \leq mq + (m-1)(x-1)$ (Castelnuovo-Severi inequality, see for instance [13], Corollary at p. 26). Notice that (u,v) is not birational onto its image if and only if there are a smooth curve C'' (namely, the normalization of $(u,v)(X')$) and morphisms $w : X' \to C''$, $u_1 : C'' \to C'$ and $v_1 : C'' \to \mathbb{P}^1$ such that $\deg(w) \geq 2$, $u = u_1 \circ w$ and $v = v_1 \circ w$. If u is simple, then u_1 must be an isomorphism and (u,v) is not birational onto its image if and only if there is a morphism $\eta = v_1 \circ u_1^{-1} : C' \to \mathbb{P}^1$ such that $v = \eta \circ u$. Hence in the set-up of Theorem 1.1 if $g \geq 1 + mq + (m-1)(mz-1)$ then X has gonality mz and for every $L \in \mathrm{Pic}^{mz}(X)$ such that $h^0(X,L) = 2$ there is $R \in \mathrm{Pic}^z(C)$ such that $h^0(C,R) = 2$ and $L \cong f^*(R)$.

Proof of Theorem 1.1. By Remark 2.1, X has gonality mz and $\dim(W^1_{mz}(X)) = 0$. M. Aprodu proved that X has Clifford index $mz-2$ and satisfies both Green and Green-Lazarsfeld conjectures if $\dim(W^1_{mz+t}(X)) \leq t$ for every integer t such that $0 \leq t \leq g - 2mz + 2$ ([4], Theorem 2). Since the function $x \to \dim(W^1_x(X))$ is strictly increasing in the interval $[\mathrm{gon}(X), g-1]$, it is sufficient to prove $\dim(W^1_{g-mz+2}(X)) = g - 2mz + 2$. Assume $\dim(W^1_{g-mz+2}(X)) > g - 2mz + 2$, i.e. $\dim(W^1_{g-mz+2}(X)) = g - mz - j$ for some integer $j \leq mz - 3$. We have $j \geq 0$ by H. Martens' Theorem ([14], see for instance [2], IV., Theorem (5.1)). Notice also that $g \geq 4j + 3$ and $2j + 2 \leq g - mz + 2 \leq g - 1 - j$, hence a theorem of R. Horiuchi yields $\dim(W^1_{2j+2}(X)) = j$ ([12], Theorem 1). Let now Γ be any irreducible component of $W^1_{2j+2}(X)$ such that $\dim(\Gamma) = j$. Since f is simple and $g - mq > (m-1)(2mz-5) \geq (m-1)(2j+1)$, by Remark 2.1 there are an integer $y \leq \lfloor (2j+2)/m \rfloor$, a non-empty open subset Φ of Γ and an open subset Ψ of $W^1_y(C)$ such that every element of Φ is the pull-back of an element of Ψ plus $2j + 2 - my$ base points. Thus $j = \dim(\Gamma) = \dim(\Psi) + 2j + 2 - my \leq \dim(W^1_y(C)) + 2j + 2 - my$. We have $\Phi \neq \emptyset$, so $\Psi \neq \emptyset$ and $y \geq z$. Hence $\dim(W^1_y(C)) \leq \dim(W^1_z(C)) + 2(y-z)$ ([9], Theorem 1). Since by assumption $\dim(W^1_z(C)) = 0$, by putting everything together we get $j \leq 2(y-z) + 2j + 2 - my \leq 2j + 2 - mz$, i.e. $j \geq mz - 2$, contradiction. □

The following auxiliary result provides a suitable generalization of [4], Theorem 2, by repeating almost verbatim the same proof.

Lemma 2.2. *Fix an integer $n \geq 1$ and let C be a smooth d-gonal curve of genus g such that $\dim G^1_{d+m} \leq n - 1 + m$ for all m with $n - 1 \leq m \leq g - 2d + n + 1$. Then $K_{g-d+n,1}(C, \omega_C) = 0$ and $K_{r-d+n,1}(C,L) = 0$ for every line bundle L on C with $h^0(C,L) = r+1$ and $\deg(L) \geq 3g$.*

Proof. Define integers k, ν as follows:

$$\begin{aligned} k &= g-d+n & (1)\\ \nu &= 2k-g & (2) \end{aligned}$$

and let X be the stable curve obtained from C by identifying $\nu+1$ pairs of general points on C. In particular, let p,q be a pair of points on C identified to a node on X. If $K_{k,1}(C,\omega_C(p+q))=0$ then according to [7], Theorem 2.1, for every effective divisor E of degree $e\geq 1$ we have $K_{k+e,1}(C,\omega_C(p+q+E))=0$. Thus if L is any line bundle on C of degree $x\geq 3g$, then $h^0(C,L-\omega_C(p+q))\geq 1$ and $K_{k+x-2g,1}(C,L)=0$. On the other hand, by [7], Lemma 2.3 and [16], p. 367, we have $K_{k,1}(C,\omega_C)\subseteq K_{k,1}(C,\omega_C(p+q))\subseteq K_{k,1}(X,\omega_X)$, therefore in order to prove our statement we may assume $K_{k,1}(X,\omega_X)\neq 0$ and look for a contradiction. By (2), X has genus $2k+1$, hence by [3], Proposition 8, there exists a torsion-free sheaf F on X with $\deg(F)=k+1$ and $h^0(X,F)\geq 2$. Let s with $0\leq s\leq \nu+1$ be the number of nodes at which F is *not* locally free. If $f: X'\to X$ is the partial normalization of X at all such nodes, then $F=f_*(L)$, where $L=f^*(F)/\mathrm{Tors}(f^*(F))$ is a line bundle on X' with $\deg L=k+1-s$ and $h^0(X',L)=h^0(X,F)\geq 2$. By taking the pull-back of L on C, we obtain a g^1_{k+1-s} not separating $\nu+1-s$ pairs of general points on C, hence it follows that $\dim G^1_{k+1-s}(C)\geq \nu+1-s$.

In order to reach a contradiction, assume first $0\leq s\leq g-2d+2$ (notice that if $g=2r-1$ and $d=r+1$ this case does not occur). ¿From (1) we obtain $k+1-s=d-2d+g+n+1-s$ with $n-1\leq -2d+g+n+1-s\leq g-2d+n+1$. Hence our numerical hypotheses imply that

$$\dim G^1_{k+1-s}(C)\leq g-2d+2n-s\leq \nu-s.$$

Assume now $s>g-2d+2$. We claim that also in this case

$$\dim G^1_{k+1-s}(C)=\max_r\{2(r-1)+\dim W^r_{k+1-s}(C)\}<\nu+1-s.$$

Indeed, we have

$$\begin{aligned} \dim W^r_{k+1-s}(C) &\leq \dim W^1_{k+1-s-(r-1)}(C)\leq\\ &\leq \dim W^1_d(C)+2(k+1-s-(r-1)-d)\leq\\ &\leq 1+2(k+1-s-(r-1)-d) \end{aligned}$$

where the second inequality is provided by [9], Theorem 1. Hence from (2) it follows that $\dim W^r_{k+1-s}(C)<\nu+1-s-2(r-1)$ for any r, as claimed. □

Proof of Proposition 1.3. We argue as in the proof of Theorem 1.1 by applying Lemma 2.2 with $n=2$ instead of [4], Theorem 2. This time we need to prove $\dim(G^1_{g-mz+3}(X)) \leq g-2mz+4$, therefore we assume by contradiction $\dim(G^1_{g-mz+3}(X)) = g-2mz+1-j$ with $j \leq mz-4$. Once again the numerical hypotheses of [12], Theorem 1, are easily checked, hence we get $j \leq \dim(W^1_z(C))+2j+2-mz$. Since in any case $\dim(W^1_z(C)) \leq 1$ by [9], Theorem 1, we obtain the desired contradiction $j \geq mz-3$. □

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