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ON FRACTIONAL DEVIATION OPERATORS

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The so called *fractional deviation operators* are introduced. This class of integral transforms appears naturally from the study of iteration of fractional integrals of Riemann-Liouville type. Since B. Ross'formulation on fractional iteration process, among other problems selected by T. Osler [5] toward 1974, several authors have been working on this subject. In particular, are worth mentioning contributions of B. Rubin [7] that allowed an intrinsic connection between fractional integrals with different limits of integration (Love's question, see [5] also) and the corresponding Ross' problem for Chen fractional integrals, handled by A. Nahushev [4] and M. Salahitdinov, with broad applications to non local boundary value problems. In this article we consider *deviation operators* as integral transforms, their connection with operators of Rubin type and mapping properties between classical and weighted Lebesgue spaces.

1. Preliminaries.

Definition 1. *Let $c, x \in R$, $c \leq x$, $f \in L^1_{loc}[c, +\infty]$, $\gamma \in R^+$. By the Riemann - Liouville transform of f of order γ and lower limit of integration c we mean the function $I^{\gamma}_{c+} f$ given as*

$$I^{\gamma}_{c+} f(x) = \int_c^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} f(t)\, dt.$$

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Proposition 1. ([1]) *With the above notation:*

a) The R - L transform is well defined and

$$I_{c+}^{\gamma}: L_{loc}^{1}[c,+\infty] \to L_{loc}^{1}[c,+\infty].$$

b) For $\gamma_1, \gamma_2 \in R^+$ the following semigroup property holds

$$I_{c+}^{\gamma_1} \,{}_o\, I_{c+}^{\gamma_2} = I_{c+}^{\gamma_1+\gamma_2}.$$

Definition 2. *Let a, b, γ be real numbers, $a < b, \gamma \in R^+$, $f \in L^1[a,b]$. We also denote*

$$ {}_{a,b}I_{+}^{\gamma} f(y) = \int_a^b \frac{(y-t)^{\gamma-1}}{\Gamma(\gamma)} f(t)\, dt\ , \quad y > b,$$
$$ {}_{a,b}I_{-}^{\gamma} f(x) = \int_a^b \frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} f(t)\, dt\ , \quad x < a.$$

Definition 3. *Let $a < c < b$, $\alpha, \beta \in R^+$. By the fractional deviation operator ${}_{a,b}D_{c+}^{\alpha,\beta}$ of R - L class (a, b, α, β) and lower limit c we mean the map*

$$ {}_{a,b}D_{c+}^{\alpha,\beta}: L_{loc}^{1}[a,+\infty] \to L_{loc}^{1}[b,+\infty],$$

$$ {}_{a,b}D_{c+}^{\alpha,\beta} \doteq I_{b+}^{\beta} \,{}_o\, I_{a+}^{\alpha} - I_{c+}^{\alpha+\beta}.$$

Definition 4. *Let $\alpha, \beta \in R^+$, $x^1 < x^2 < x^3$. We'll write*

$$F_{\alpha,\beta}(x^1,x^2,x^3) = F_{\alpha,\beta}^{x^2}(x^1,x^3)$$

$$\doteq \int_{x^2}^{x^3} (x-x^1)^{\alpha-1}(x^3-x)^{\beta-1}\, dx.$$

([1]) See [3].

It is easy to see that $F_{\alpha,\beta}$ is well defined and in terms of the incomplete beta function

$$B_{\alpha,\beta}(x) = \int_x^{+\infty} s^{\alpha-1}(1+s)^{-\alpha-\beta}\, ds,$$

where $x > 0$, the following formulae hold:

$$B_{\alpha,\beta}(0+) = Be(\alpha, \beta), \tag{1}$$

$$B_{\alpha,\beta}(x) + B_{\beta,\alpha}(x^{-1}) = Be(\alpha, \beta), \tag{2}$$

$$F_{\alpha,\beta}(x^1, x^2, x^3) = \left(x^3 - x^1\right)^{\alpha+\beta-1} B_{\alpha,\beta}\left(\frac{x^2 - x^1}{x^3 - x^2}\right), \tag{3}$$

$$F_{\alpha,\beta}(px^1, px^2, px^3) = p^{\alpha+\beta-1} F_{\alpha,\beta}(x^1, x^2, x^3),\ p\text{-positive}, \tag{4}$$

$$F_{\alpha,\beta}(x^1, x^2, x^3) = F^0_{\alpha,\beta}\left(x^1 - x^2, x^3 - x^2\right). \tag{5}$$

Definition 5. *Let $a < b$, $\alpha, \beta \in R^+$. We introduce the operators*

$${}_{a,b}E_+^{\alpha,\beta} : L^1[a,b] \to L^1_{loc}[b, +\infty],$$

$${}_{a,b}E_+^{\alpha,\beta} f(y) \doteq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b F^b_{\alpha,\beta}(u, y) f(u)\, du, \quad f \in L^1[a,b],\ b < y,$$

and

$${}_{a,b}E_-^{\alpha,\beta} : L^1[a,b] \to L^1_{loc}[-\infty, a],$$

$${}_{a,b}E_-^{\alpha,\beta} f(x) \doteq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b F^a_{\alpha,\beta}(x, v) f(v)\, dv, \quad f \in L^1[a,b],\ x < a.$$

Definition 6. *Let $-\infty \le a < b < +\infty$, $\gamma \in R^+$, $f \in L^1[a,b]$. By a Rubin type transform $R^\gamma_{a,b} f$ of f we mean the function given as*

$$R^\gamma_{a,b} f(x) = \frac{\sin(\pi\gamma)}{\pi} \int_a^b \left(\frac{b-t}{x-b}\right)^\gamma \frac{f(t)}{x-t}\, dt, \quad x > b.$$

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2. On deviation operators.

Theorem 1. *Let* $a < c < b,\ \alpha,\ \beta,\ \gamma \in R^+,\ f \in L^1_{loc}\,[a, +\infty]$. *Then*

$$ {}_{a,b}D^{\alpha,\beta}_{c+} f = {}_{a,b}\, E^{\alpha,\beta}_{+} f - {}_{c,b}\, I^{\alpha+\beta}_{+} f. \tag{6}$$

$$ {}_{a,b}E^{\alpha,\beta}_{+} f + \left({}_{-b,-a}E^{\beta,\alpha}_{-} f^{\sim} \right)^{\sim} = {}_{a,b}\, I^{\alpha+\beta}_{+} f, \tag{7}$$

where $f^{\sim}(x) = f(-x)$ *and* $x \le a$.

$$ \left({}_{a,b}I^{\gamma}_{-} f \right)^{\sim} = {}_{-b,-a}\, I^{\gamma}_{+} f^{\sim}. \tag{8}$$

$$ \left({}_{a,b}D^{\alpha,\beta}_{c+} f \right)^{\sim} = {}_{-c,-a}\, I^{\alpha+\beta}_{-} f^{\sim} - {}_{-b,-a}\, E^{\beta,\alpha}_{-} f^{\sim}. \tag{9}$$

In general, the above identities must be interpreted in the almost everywhere sense.

Proof. With fixed $d > b \ge e \ge a$ we obtain

$$ \| I^{\gamma}_{e+} \|_{L^1[b,d]} \le \frac{(d-e)^{\gamma}}{\Gamma(\gamma+1)} \| f \|_{L^1[e,d]} \tag{10}$$

and therefore

$$ \| {}_{a,b}D^{\alpha,\beta}_{c+} f \|_{L^1[b,d]} \le \tag{11}$$

$$ \le \left[\frac{(d-a)^{\alpha}}{\Gamma(\alpha+1)} \cdot \frac{(d-b)^{\beta}}{\Gamma(\beta+1)} + \frac{(d-c)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right] \| f \|_{L^1[a,d]}.$$

On the other hand:

$$ \| {}_{c,b}I^{\gamma}_{+} f \|_{L^1[b,d]} \le \frac{(d-c)^{\gamma}}{\Gamma(\gamma+1)} \| f \|_{L^1[c,b]}, \tag{12}$$

$$ \| {}_{a,b}E^{\alpha,\beta}_{+} f \|_{L^1[b,d]} \le \frac{(d-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \| f \|_{L^1[a,b]}. \tag{13}$$

From the above relations we deduce that ${}_{a,b}D^{\alpha,\beta}_{c+}$ and ${}_{a,b}E^{\alpha,\beta}_{+} - {}_{c,b}\, I^{\alpha+\beta}_{+}$ maps $L^1[a, d]$ linearly and continuously into $L^1[b, d]$ for each $d > b$. It is now expedient to observe that $I^{\alpha}_{a+}g \in C[a,d]$ whenever $g \in C[a,d]$ ([3], Ch. 1, Corollary 2). By identifying $I^{\alpha}_{a+}g$ with its restriction over $[b,d]$ we obtain $\mathrm{I}^{\beta}_{b+}(I^{\alpha}_{a+}g) \in C[b,d]$. Analogously $I^{\alpha+\beta}_{c+}g \in C[b,d]$ by identifying g with its restriction to $[c,d]$. Now ${}_{a,b}D^{\alpha,\beta}_{c+}g \in C[b,d]$ and from a direct application of Fubini's Theorem we obtain (6).In the general case, given $f \in L^1[a,d]$ and a positive number ζ we may write $f = g + h$, $g \in C[a,d]$ and $\|h\|_{L^1[a,d]} \le \zeta$. Using (11) – (13) we have

$$
\begin{aligned}
&\|{}_{a,b}D^{\alpha,\beta}_{c+} f - \left({}_{a,b}E^{\alpha,\beta}_{+} - {}_{c,b}\, I^{\alpha+\beta}_{+}\right) f\|_{L^1[b,d]} = \\
&\qquad = \|{}_{a,b}D^{\alpha,\beta}_{c+} h - \left({}_{a,b}E^{\alpha,\beta}_{+} - {}_{c,b}\, I^{\alpha+\beta}_{+}\right) h\|_{L^1[b,d]} \\
&\qquad \le \|{}_{a,b}D^{\alpha,\beta}_{c+} h\|_{L^1[b,d]} + \|{}_{a,b}E^{\alpha,\beta}_{+} h\|_{L^1[b,d]} + \|{}_{c,b}I^{\alpha+\beta}_{+} h\|_{L^1[b,d]} \\
&\qquad \le \left[\frac{(d-a)^{\alpha}}{\Gamma(\alpha+1)} \cdot \frac{(d-b)^{\beta}}{\Gamma(\beta+1)} + \frac{(d-c)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right]\|h\|_{L^1[a,d]} + \\
&\qquad\quad + \frac{(d-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|h\|_{L^1[a,b]} + \frac{(d-c)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|h\|_{L^1[c,b]} \\
&\qquad \le \left[\frac{(d-a)^{\alpha}}{\Gamma(\alpha+1)} \cdot \frac{(d-b)^{\beta}}{\Gamma(\beta+1)} + \frac{(d-c)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \right. \\
&\qquad\quad \left. + \frac{(d-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{(d-c)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right]\zeta.
\end{aligned}
$$

Since ζ was arbitrary equality (6) became valid in the $\|\,\|_{L^1[b,d]}$-norm for each $d > b$ and hence it is also valid in the a.e. sense. The other identities may be proved in a similar way. □

Theorem 2. *Let $a, b, \gamma \in R$, $a < b$, $0 < \gamma < 1$, $f \in L^1[a,b]$. Then*

$$I^{\gamma}_{b+}\, R^{\gamma}_{a,b} f(y) = {}_{a,b}\, I^{\gamma}_{+} f(y) \qquad \text{a.e. } y > b. \tag{14}$$

Proof. In general, Rubin type transforms satisfy the following translation rule

$$\tau_{-b}\, R^{\gamma}_{a-b,0}\tau_b = R^{\gamma}_{a,b}, \tag{15}$$

and so we shall consider $R^{\gamma}_{c,0}$ with $-\infty \le c < 0$. If we write

$$\kappa(t,x) = \left(\frac{t}{x}\right)^{\gamma} \frac{1}{x+t} \tag{16}$$

with both x and t positive, then κ is a measurable homogeneous function of degree -1, and if $1 < p < \infty$:

$$\int_0^{+\infty} |\kappa(t,1)| t^{-1/p}\, dt = \int_0^{+\infty} |\kappa(1,x)| x^{-1/p'}\, dx = \chi\,, \tag{17}$$

where χ would be finite if $\gamma < 1/p$. In this case, from the Hardy - Littlewood - Pólya Theorem on boundedness of homogeneous operators [2], Theorem 319, the Rubin transform

$$R^{\gamma}_{-\infty,0} f(x) = \frac{\sin(\pi\gamma)}{\pi} \int_0^{+\infty} \left(\frac{t}{x}\right)^{\gamma} \frac{f(-t)}{x+t}\, dt, \quad x\text{-positive}, \tag{18}$$

becomes continuous between $L^p[-\infty, 0]$ and $L^p[0, +\infty]$ with

$$\left\| R^{\gamma}_{-\infty,0} f \right\|_{L^p[0,+\infty]} \leq \frac{\sin(\pi\gamma)}{\sin(\pi(\gamma + 1/p'))} \| f \|_{L^p[-\infty,0]}\,. \tag{19}$$

It is now easy to see that if $1 \leq p < 1/\gamma$ then $R^{\gamma}_{c,0}$, c-negative, maps linear and continuously $L^p[c, 0]$ on $L^p[0, +\infty]$ with

$$\left\| R^{\gamma}_{c,0} f \right\|_{L^p[0,+\infty]} \leq \frac{\sin(\pi\gamma)}{\sin(\pi(\gamma + 1/p'))} \| f \|_{L^p[c,0]}. \tag{20}$$

Moreover, under this conditions $R^{\gamma}_{c,0} f(x)$ will be defined and finite for every x-positive with the limit c finite or infinite.
Now for a given $f \in L^p[c, 0]$ we write

$$\begin{aligned} I^{\gamma}_{0+} R^{\gamma}_{c,0} f(y) &= \int_0^y \frac{(y-x)^{\gamma-1}}{\Gamma(\gamma)} \left[\frac{\sin(\pi\gamma)}{\pi} \int_c^0 \left(\frac{|t|}{x}\right)^{\gamma} \frac{f(t)}{x+|t|}\, dt \right] dx \\ &= \frac{\sin(\pi\gamma)}{\pi\Gamma(\gamma)} \int_c^0 f(t)|t|^{\gamma} \left[\int_0^y \frac{(y-x)^{\gamma-1} x^{-\gamma}}{x+|t|}\, dx \right] dt. \end{aligned}$$

The application of Fubini's Theorem is justified because the double integral is still absolutely convergent if $c = -\infty$ and fractional integrals of order γ on the whole real axis are defined for L^p-functions if $1 \leq p < 1/\gamma$. Finally the inner integral may be evaluated after the change of variable $z = \dfrac{-t(y-x)}{(x-t)y}$, and so we obtain (14). □

Theorem 3. *Let $a < c < b$, $\alpha, \beta \in R^+$. The following formulae hold*

$$(21)\qquad {}_{a,b}I_+^{\alpha+\beta} = I_{b+}^{\beta} \circ_{a,b} I_+^{\alpha} \quad + \quad {}_{a,b}I_+^{\beta} \circ I_{a+}^{\alpha}\,,$$

$$(22)\qquad {}_{a,b}E_+^{\alpha,\beta} = I_{b+}^{\beta} \circ {}_{a,b}I_+^{\alpha}\,,$$

$$(23)\qquad {}_{a,b}D_{c+}^{\alpha,\beta} = I_{b+}^{\beta} \circ {}_{a,c}I_+^{\alpha} \quad - \quad {}_{c,b}I_+^{\beta} \circ I_{c+}^{\alpha}\,,$$

$$(24)\qquad {}_{a,b}D_{a+}^{\alpha,\beta} = -{}_{a,b}I_+^{\beta} \circ I_{a+}^{\alpha}\,,$$

$$(25)\qquad {}_{a,b}D_{b+}^{\alpha,\beta} = I_{b+}^{\beta} \circ {}_{a,b}I_+^{\alpha}\,,$$

$$(26)\qquad {}_{a,b}D_{b+}^{\alpha,\beta} \quad - \quad {}_{a,b}D_{c+}^{\alpha,\beta} = {}_{c,b}\, I_+^{\alpha+\beta}.$$

Corollary 1. *If $a < c < b$, $\alpha, \beta \in R$ and $0 < \alpha < \alpha + \beta < 1$, then*

$$ {}_{a,b}D_{c+}^{\alpha,\beta} \; = \; I_{b+}^{\alpha+\beta} \left(R_{a,b}^{\alpha} - R_{c,b}^{\alpha+\beta} \right).$$

3. On some boundedness conditions.

Proposition 2. *Let $\alpha, \beta \in R^{\,+}$. There exist positive constants c_1, c_2 such that for every positive x we have the inequality*

$$(27)\qquad c_1(1+x)^{-\beta} \le B_{\alpha,\beta}(x) \le c_2(1+x)^{-\beta}.$$

Moreover, we may take $c_1 = \min\left\{\beta^{-1}, Be(\alpha, \beta)\right\}$, $c_1 = \max\left\{\beta^{-1}, Be(\alpha, \beta)\right\}$.

Proof. Given a positive x and assuming $\alpha \ge 1$ we have

$$(28)\qquad B_{\alpha,\beta}(x) \le \frac{(1+x)^{-\beta}}{\beta}.$$

In particular, if $x \to 0+$ in (28) there result $Be(\alpha, \beta) \le 1/\beta$ and on one hand (27) holds. If $0 < \alpha < 1$ we write

$$(29)\qquad B_{\alpha,\beta}(x) \ge \frac{(1+x)^{-\beta}}{\beta}$$

and making $x \to 0+$ in (29) we obtain $Be(\alpha, \beta) \ge 1/\beta$. We now introduce the functions of the non negative real variable x

$$f(x) = (1+x)^{\beta} B_{\alpha,\beta}(x)$$

and

$$g(x) = B_{\alpha,\beta}(x) - \frac{x^{\alpha-1}(1+x)^{1-\alpha-\beta}}{\beta}.$$

In particular, $f(x) \equiv 1/\beta$ if $\alpha = 1$ and our claim follows. On the other hand we may write

$$f'(x) = \beta(1+x)^{\beta-1}g(x),$$
$$g'(x) = \frac{1-\alpha}{\beta}x^{\alpha-2}(1+x)^{-\alpha-\beta},$$

i.e. g is monotone increasing or decreasing according as $0 < \alpha \leq 1$ or $\alpha \geq 1$ respectively. But $g(+\infty) = 0$, i.e. f becomes a monotone decreasing function in the first case and a monotone increasing one in the second. Moreover, $f(0) = Be(\alpha, \beta)$ and $f(+\infty) = 1/\beta$ and hence both estimates follow. □

Remark 1. We'll consider formally the following expressions

$$[a]\ \sup_{\kappa>0} \kappa^{1-\alpha} \left(\int_a^c \frac{w_1(x)^{-p'/p}\,dx}{(c-x+\kappa)^{(1-\alpha)p'}} \right)^{\frac{1}{p'}} \left(\int_b^{+\infty} \frac{(y-b)^{\beta q} w_2(y)\,dy}{(y-c+\kappa)^{(1-\alpha)q}} \right)^{\frac{1}{q}},$$

$$[b]\ \sup_{\kappa>0} \kappa^{1-\beta} \left(\int_c^b \frac{(b-x)^{\alpha p'} w_1(x)^{-p'/p}\,dx}{(b-x+\kappa)^{(1-\beta)p'}} \right)^{\frac{1}{p'}} \left(\int_b^{+\infty} \frac{w_2(y)dy}{(y-b+\kappa)^{(1-\beta)q}} \right)^{\frac{1}{q}},$$

$$[c]\ \sup_{\kappa>0} \left(\int_c^b \left(\frac{b-x+\kappa}{b-x} \right)^{(1-\beta)p'} (b-x)^{\alpha p'} w_1(x)^{-p'/p}\,dx \right)^{\frac{1}{p'}} \cdot$$
$$\cdot \left(\int_b^{+\infty} \frac{w_2(y)\,dy}{(y-b+\kappa)^{(1-\beta)q}} \right)^{\frac{1}{q}},$$

$$[d]\ \sup_{\kappa>0} \left(\int_a^c \left(\frac{c-x+\kappa}{c-x} \right)^{(1-\alpha)p'} w_1(x)^{-p'/p}\,dx \right)^{1/p'} \cdot$$
$$\cdot \left(\int_b^{+\infty} \frac{(y-b)^{\beta q} w_2(y)\,dy}{(y-c+\kappa)^{(1-\alpha)q}} \right)^{1/q},$$

$$[e]\ \left[\int_a^c \left(\int_b^{+\infty} \frac{(y-b)^{\beta q} w_2(y)\,dy}{(y-x)^{(1-\alpha)q}} \right)^{r/q} \cdot \right.$$
$$\left. \cdot \left(\int_a^c \frac{w_1(z)^{1-p'}\,dz}{\left(1+\frac{c-z}{c-x}\right)^{(1-\alpha)p'}} \right)^{r/q'} w_1(x)^{1-p'}\,dx \right]^{1/r},$$

$$[f]\ \left[\int_c^b \left(\int_b^{+\infty} \frac{w_2(y)\,dy}{(y-x)^{(1-\beta)q}} \right)^{r/q} \cdot \right.$$

$$\cdot\left(\int_c^b \frac{(b-z)^{\alpha p'} w_1(z)^{1-p'}\,dz}{\left(1+\frac{b-z}{b-x}\right)^{(1-\beta)p'}}\right)^{r/q'} (b-x)^{\alpha p'} w_1(x)^{1-p'}\,dx\Bigg]^{1/r}.$$

Remark 2. Here and throughout the paper, when any of p, q, or p' is ∞ integrals such as those $[a]$ – $[f]$ have the current interpretations and as usual $w_1(x)^{-p'/p} = (w_1(x)^{-1})^{p'/p}$. Thus for example

$$\left(\int_a^c \frac{w_1(x)^{-p'/p}\,dx}{(c-x+\kappa)^{(1-\alpha)p'}}\right)^{\frac{1}{p'}} = \operatorname*{ess\,sup}_{a\le x\le c} \frac{w_1(x)^{-1}dx}{(c-x+\kappa)^{1-\alpha}}\ ,\quad (p=1)$$

and

$$w_1(x)^{-p'/p} = \begin{cases} 1 & \text{if } 0 < w_1(x) < \infty \\ \infty & \text{if } w_1(x) = 0 \\ 0 & \text{if } w_1(x) = \infty \end{cases} \qquad (p=\infty).$$

Moreover, products of the form $0\cdot\infty$ are taken to be zero.

Theorem 4. *Let $-\infty < a < c < b < +\infty$, α, $\beta \in R^+$, w_1, w_2 two weights defined on $[a,b]$ and $[b,+\infty]$ respectively and $1 \le p, q \le +\infty$. Let:*

(i) $0 < \alpha, \beta \le 1$, $1 \le p \le q \le +\infty$;
(ii) $0 < \alpha \le 1 < \beta$, $1 \le p \le q \le +\infty$, $1 < q$;
(iii) $0 < \beta \le 1 < \alpha$, $1 \le p \le q \le +\infty$, $1 < q$;
(iv) $1 < \alpha$, $1 < \beta$, $1 \le p \le +\infty$, $1 < q$;
(v) $0 < \alpha, \beta < 1$, $1 < q < p < +\infty$;
(vi) $0 < \beta < 1 \le \alpha$, $1 < q < p < +\infty$;
(vii) $0 < \alpha < 1 \le \beta$, $1 < q < p < +\infty$;

the general linear deviation operator ${}_{a,b}D^{\alpha,\beta}_{c+}$ will be bounded among the weighted Lebesgue spaces $L^p_{w_1(x)dx}[a,b]$ and $L^q_{w_2(y)dy}[b,+\infty]$ (2) if and only if, with the notation of the above remark, the following respective conditions hold:

In (i), the expressions $[a]$ and $[b]$ are finite.
In (ii), the expressions $[a]$ and $[c]$ are finite.
In (iii), the expressions $[b]$ and $[d]$ are finite.
In (iv), the expressions $[c]$ and $[d]$ are finite.
In (v), with $r = 1/q - 1/p$, $[e]$ and $[f]$ are finite.
In (vi), with $r = 1/q - 1/p$, $[d]$ and $[f]$ are finite.
In (vii), with $r = 1/q - 1/p$, $[c]$ and $[e]$ are finite.

(2) As usual, in this case we write ${}_{a,b}D^{\alpha,\beta}_{c+} \in \left[L^p_{w_1(x)dx}[a,b], L^q_{w_2(y)dy}[b,+\infty]\right]$.

Proof. Given $f \in L^1_{loc}[a,+\infty]$, from (2) and (6) we have

$$ {}_{a,b}D^{\alpha,\beta}_{c+} f(y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left[\int_a^c (y-x)^{\alpha+\beta-1} B_{\alpha,\beta}\left(\frac{b-x}{y-b}\right) f(x)\,dx - \right. $$

$$ \left. - \int_c^b (y-x)^{\alpha+\beta-1} B_{\beta,\alpha}\left(\frac{y-b}{b-x}\right) f(x)\,dx \right] $$

and hence we write

$$ {}_{a,b}D^{\alpha,\beta}_{c+} f = {}_{a,b}D^{\alpha,\beta}_{1;c+} f + {}_{a,b}D^{\alpha,\beta}_{2;c+} f, \tag{30} $$

with

$$ {}_{a,b}D^{\alpha,\beta}_{1;c+} f(y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^c (y-x)^{\alpha+\beta-1} B_{\alpha,\beta}\left(\frac{b-x}{y-b}\right) f(x)\,dx $$

and

$$ {}_{a,b}D^{\alpha,\beta}_{2;c+} f(y) = -\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_c^b (y-x)^{\alpha+\beta-1} B_{\beta,\alpha}\left(\frac{y-b}{b-x}\right) f(x)\,dx. $$

Since

$$ {}_{a,b}D^{\alpha,\beta}_{1;c+} = {}_{a,b}D^{\alpha,\beta}_{c+}\chi_{a,c]} \quad \text{and} \quad {}_{a,b}D^{\alpha,\beta}_{2;c+} = -{}_{a,b}D^{\alpha,\beta}_{c+}\chi_{[c,b]} \tag{31} $$

it is immediate that ${}_{a,b}D^{\alpha,\beta}_{c+} \in \left[L^p_{w_1(x)dx}[a,b], L^q_{w_2(y)dy}[b,+\infty]\right]$ iff

$$ \begin{aligned} &{}_{a,b}D^{\alpha,\beta}_{1;c+} \in \left[L^p_{w_1(x)dx}[a,c], L^q_{w_2(y)dy}[b,+\infty]\right] \quad \text{and} \\ &{}_{a,b}D^{\alpha,\beta}_{2;c+} \in \left[L^p_{w_1(x)dx}[c,b], L^q_{w_2(y)dy}[b,+\infty]\right]. \end{aligned} \tag{32} $$

But by Prop. 1 we deduce that (32) will hold iff

$$ \begin{aligned} &{}_{a,c}I^{\alpha}_{+} \in \left[L^p_{w_1(x)dx}[a,c], L^q_{(y-b)^{\beta q}w_2(y)dy}[b,+\infty]\right] \quad \text{and} \\ &{}_{c,b}I^{\beta}_{+} \in \left[L^p_{(b-x)^{-\alpha p}w_1(x)dx}[c,b], L^q_{w_2(y)dy}[b,+\infty]\right]. \end{aligned} \tag{33} $$

Moreover the following inequalities hold

$$c_1 \frac{(y-b)^\beta}{\Gamma(\beta)} {}_{a,c}I_+^\alpha f(y) \le_{a,b} D_{1;c+}^{\alpha,\beta} f(y) \le c_2 \frac{(y-b)^\beta}{\Gamma(\beta)} {}_{a,c}I_+^\alpha f(y), \tag{34}$$

$$\left|{}_{a,b}D_{2;c+}^{\alpha,\beta} f(y)\right| \le c_2 \; {}_{c,b}I_+^\beta \left[\frac{(b-y)^\alpha}{\Gamma(\alpha)} |f(y)|\right] \tag{35}$$

and

$$\left|{}_{c,b}I_+^\beta \left[\frac{(b-y)^\alpha}{\Gamma(\alpha)} f(y)\right]\right| \le_{c,b} I_+^\beta \left[\frac{(b-y)^\alpha}{\Gamma(\alpha)} |f(y)|\right] \le \frac{1}{c_1} \left|{}_{a,b}D_{2;c+}^{\alpha,\beta} f(y)\right| \tag{36}$$

where by c_1 and c_2 we denote the constants determined in Prop. 2.

In particular we may write

$${}_{a,c}I_+^\alpha = \frac{1}{\Gamma(\alpha)} \tau_{-c} \; S_{1-\alpha} (\tau_c)^\sim, \tag{37}$$

where $S_{1-\alpha}$ denote the usual Stieltjes transform. From the corresponding representations of ${}_{a,c}I_+^\alpha$ and ${}_{c,b}I_+^\beta$ as composition of maps

$$\begin{array}{rcr} L^p_{w_1(x)dx}[a,c] & & L^p_{(b-x)^{-\alpha p} w_1(x)dx}[c,b] \\ \downarrow (\tau_c)^\sim & & \downarrow (\tau_b)^\sim \\ L^p_{w_1(c-x)dx}[0,c-a] & & L^p_{x^{-\alpha p} w_1(b-x)dx}[0,b-c] \\ \downarrow S_{1-\alpha} & \text{and} & \downarrow S_{1-\beta} \\ L^q_{(y-b+c)^{\beta q} w_2(y+c)dy}[b-c,+\infty] & & L^q_{w_2(y+b)dy}[0,+\infty] \\ \downarrow \tau_{-c} & & \downarrow \tau_{-b} \\ L^q_{(y-b)^{\beta q} w_2(y)dy}[b,+\infty] & & L^q_{w_2(y)dy}[b,+\infty] \end{array} \tag{38}$$

it is clear that (33) will hold iff

$$S_{1-\alpha} \in \left[L^p_{w_1(c-x)dx}[0,c-a], L^q_{(y-b+c)^{\beta q} w_2(y+c)dy}[b-c,+\infty]\right] \tag{39}$$

and

$$S_{1-\beta} \in \left[L^p_{x^{-\alpha p} w_1(b-x)dx}[0,b-c], L^q_{w_2(y+b)dy}[0,+\infty]\right]. \tag{40}$$

We've therefore reduced our matter to the study of boundedness conditions of the general Stieltjes transform. Therefore known results of K. Andersen [1] and G. Sinnamon [8] in this direction go on.

Remark 3. V. D. Stepanov [9] obtained necessary and sufficient conditions for the boundedness from weighted Lebesgue real spaces of Volterra convolution operators of the form

$$Kf(x) = \int_0^x k(x-y)\, f(y)\, dy,$$

where $k(x)$ is a non negative non decreasing kernel satisfying the inequality $k(x+y) \leq D\,(k(x)+k(y))$ for all $x, y \in R^+$. In particular, Stepanov research will allow us to relate ${}_{a,b}D_{c+}^{\alpha,\beta}$ with I_{c+}^{γ} when $\gamma \geq 1$, $\gamma = \alpha+\beta$, i.e. precisely when the corresponding Riemann - Liouville kernel $k_\gamma(x) = x^{\gamma-1}/\Gamma(\gamma)$ satisfies Stepanov conditions. With our notation we have

$${}_{a,b}I_{c+}^{\gamma} = \tau_{-a}\ o\ I_{0+}^{\gamma}\ o\ \tau_a\ o\ \chi_{[a,b]}$$

and it is of interest for us to consider

$$\begin{array}{ccc} L^p_{w_1(x)dx}[a,b] & \overset{{}_{a,b}I^{\gamma}_{c+}}{\rightarrow} & L^q_{w_2(y)dy}[b,+\infty]. \\ \downarrow \tau_a & & \uparrow \tau_{-a} \\ L^p_{\tau_a w_1(x)dx}[0,b-a] & \overset{I^{\gamma}_{0+}}{\rightarrow} & L^q_{\tau_a w_2(y)dy}[b-a,+\infty]. \end{array}$$

Following Stepanov research, for $t>0$, we must consider the numbers

$$A_0(t) = \left[\int_a^{\min\{t+a,b\}} w_1(x)^{-p'/p} dx\right]^{1/p'} \cdot \left[\int_{\max\{t+a,b\}}^{\infty} (y-a-t)^{q(\gamma-1)}\, w_2(y)\, dy\right]^{1/q}, \tag{41}$$

$$A_1(t) = \left[\int_a^{\min\{t+a,b\}} (t+a-x)^{p'(\gamma-1)}\, w_1(x)^{-p'/p} dx\right]^{1/p'} \cdot \left[\int_{\max\{t+a,b\}}^{\infty} w_2(y)\, dy\right]^{1/q}, \tag{42}$$

and

$$A_0 = \sup_{t>0} A_0(t), \qquad A_1 = \sup_{t>0} A_1(t), \qquad A = \max\{A_0, A_1\}. \tag{43}$$

Now, for the boundedness of the operator

$$L^p_{w_1(x)dx}[a,b] \overset{{}_{a,b}I^{\gamma}_{c+}}{\longrightarrow} L^q_{w_2(y)dy}[b,+\infty]$$

it is necessary and sufficient that the number A be finite (see [9],Th. 1).

Remark 4. Following Remark 3, we may prove that, in general, our deviation operators are small compared to the operators whose deviation they measure. For instance, let us consider the case $0 \le \alpha, \beta \le 1$, $1 \le \gamma$, $1 < p \le q < \infty$. We assume that ${}_{a,b}I_{+}^{\alpha,\beta} \in \left[L^p_{w_1(x)dx}[a,b], L^q_{w_2(y)dy}[b,+\infty]\right]$, and for a fixed $\kappa > 0$ we write

$$\begin{aligned}
A &\ge A_0 \\
&\ge A_0(b-a) \\
&= \left[\int_a^b w_1(x)^{-p'/p}\,dx\right]^{1/p'} \left[\int_b^\infty (y-b)^{q(\gamma-1)} w_2(y)dy\right]^{1/q} \\
&\ge \kappa^{1-\alpha}\left[\int_a^c \frac{w_1(x)^{-p'/p}\,dx}{(c-x+\kappa)^{(1-\alpha)p'}}\right]^{1/p'} \left[\int_b^{+\infty} \frac{(y-b)^{\beta q} w_2(y)\,dy}{(y-c+\kappa)^{(1-\alpha)q}}\right]^{1/q}
\end{aligned}$$

and we deduce that

$$\sup_{\kappa>0} \kappa^{1-\alpha}\left(\int_a^c \frac{w_1(x)^{-p'/p}\,dx}{(c-x+\kappa)^{(1-\alpha)p'}}\right)^{\frac{1}{p'}} \left(\int_b^{+\infty} \frac{(y-b)^{\beta q} w_2(y)\,dy}{(y-c+\kappa)^{(1-\alpha)q}}\right)^{\frac{1}{q}} < \infty.$$

Analogously

$$\begin{aligned}
A &\ge A_1 \\
&\ge A_1(b-a) \\
&= \left[\int_a^b (b-x)^{p'(\gamma-1)} w_1(x)^{-p'/p}\,dx\right]^{1/p'} \left[\int_b^\infty w_2(y)\,dy\right]^{1/q} \\
&\ge \kappa^{1-\beta}\left[\int_b^{+\infty} \frac{w_2(y)\,dy}{(y-b+\kappa)^{(1-\beta)q}}\right]^{1/q} \cdot \\
&\qquad \cdot \left[\int_c^b \frac{(b-x)^{\alpha p'}}{(b-x+\kappa)^{(1-\beta)p'}} w_1(x)^{-p'/p}\,dx\right]^{1/p'}
\end{aligned}$$

and now

$$\sup_{\kappa>0} \kappa^{1-\beta}\left(\int_c^b \frac{(b-x)^{\alpha p'} w_1(x)^{-p'/p}\,dx}{(b-x+\kappa)^{(1-\beta)p'}}\right)^{\frac{1}{p'}} \cdot \left(\int_b^{+\infty} \frac{w_2(y)\,dy}{(y-b+\kappa)^{(1-\beta)q}}\right)^{\frac{1}{q}} < \infty.$$

From Theorem 4 we deduce that ${}_{a,b}D_{c+}^{\alpha,\beta} \in \left[L^p_{w_1(x)dx}[a,b],\, L^q_{w_2(y)dy}[b,+\infty]\right]$ if ${}_{a,b}I_{+}^{\alpha,\beta} \in \left[L^p_{w_1(x)dx}[a,b],\, L^q_{w_2(y)dy}[b,+\infty]\right]$.

Nevertheless we may find weighted functions w_1, w_2 for which ${}_{a,b}D_{c+}^{\alpha,\beta} \in \left[L^p_{w_1(x)dx}[a,b],\, L^q_{w_2(y)dy}[b,+\infty]\right]$ but ${}_{a,b}I_{+}^{\alpha,\beta} \notin \left[L^p_{w_1(x)dx}[a,b],\, L^q_{w_2(y)dy}[b,+\infty]\right]$. For instance, for $a \le x \le b \le y$ we'll write

$$w_1(x) = (b-x)^{\sigma},\ p/p' \le \sigma < (\alpha + 1/p')p,$$
$$w_2(y) = (y-b)^{\lambda} e^{-y},\ \lambda > -1 + (1-\beta)q.$$

We observe that the number $A_0(b-a) = +\infty$ in (41), because $1 - \sigma p'/p \le 0$. Hence $A = +\infty$ in (43) and by the Stepanov condition ${}_{a,b}I_{+}^{\alpha,\beta}$ is not bounded. On the other hand,

$$\sup_{\kappa>0} \kappa^{1-\alpha} \left(\int_a^c \frac{w_1(x)^{-p'/p}\,dx}{(c-x+\kappa)^{(1-\alpha)p'}}\right)^{\frac{1}{p'}} \left(\int_b^{+\infty} \frac{(y-b)^{\beta q} w_2(y)\,dy}{(y-c+\kappa)^{(1-\alpha)q}}\right)^{\frac{1}{q}} \le$$

$$\le (b-c)^{\alpha-1} \left(\int_a^c (b-x)^{-\sigma p'/p}\,dx\right)^{\frac{1}{p'}} \left(\int_b^{+\infty} (y-b)^{\beta q+\lambda} e^{-y}\,dy\right)^{\frac{1}{q}}$$

and

$$\sup_{\kappa>0} \kappa^{1-\beta} \left(\int_c^b \frac{(b-x)^{\alpha p'} w_1(x)^{-p'/p}\,dx}{(b-x+\kappa)^{(1-\beta)p'}}\right)^{\frac{1}{p'}} \left(\int_b^{+\infty} \frac{w_2(y)\,dy}{(y-b+\kappa)^{(1-\beta)q}}\right)^{\frac{1}{q}} \le$$

$$\le \left(\int_c^b (b-x)^{(\alpha-\sigma/p)p'}\,dx\right)^{\frac{1}{p'}} \left(\int_b^{+\infty} (y-b)^{\lambda+(\beta-1)q} e^{-y}\,dy\right)^{\frac{1}{q}}.$$

On using Theorem 4 (i) we obtain ${}_{a,b}D_{c+}^{\alpha,\beta} \in \left[L^p_{w_1(x)dx}[a,b],\, L^q_{w_2(y)dy}[b,+\infty]\right]$.

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