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ON THE LIFTING PROBLEM IN CODIMENSION TWO

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In this note we prove a special case of the following conjecture of Mezzetti's [5]:

Let $X \subseteq \mathbb{P}^{n+2}$ be an integral, nondegenerate variety of dimension n. Suppose that its general hyperplane section lies on a hypersurface of degree s, while the variety itself does not. Then the degree of X is bounded by:

$$\deg X \leq s^2 - (n-1)s + \binom{n}{2} + 1.$$

Introduction.

Let $X \subseteq \mathbb{P}^{n+2}$ be a reduced irreducible projective variety of codimension 2, and let $Y = X \cap H$ be its general hyperplane section.
A nonliftable section of $\mathcal{I}_Y$ in degree s is a nonzero element

$$\alpha \in \operatorname{coker}(H^0(\mathcal{I}_X(s)) \to (H^0(\mathcal{I}_Y(s))) = \ker(H^1(\mathcal{I}_X(s-1)) \to H^1(\mathcal{I}_X(s)));$$

following [2], we call α a sporadic zero of X of degree s.
The order of an element $\beta \in H^1(\mathcal{I}_X(s))$ is the maximum integer p such that β is of form $\beta = H^p \cdot \gamma$, $\gamma \in H^1(\mathcal{I}_X(s-p))$. β is primitive if its order is zero.
Let C and Γ be the general $\mathbb{P}^3$- and $\mathbb{P}^2$-sections of X; it will be proved that, if X has a sporadic zero of degree s, then C has one of degree $\leq s$.
We can now state the main result of this paper:

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Theorem 0.1. *Let* $r = \dim I_{X,s}$ *and suppose that the following hold:*

(i) X *has a sporadic zero in degree* s*;*
(ii) $I_{\Gamma,s-1} = 0$*;*
(iii) a sporadic zero of C *in degree* s *is primitive.*

Then

$$\deg X \leq s^2 - (n + r - 1)s + \binom{n+r}{2} + 1. \tag{1}$$

In this paper we freely use results and terminology of initial ideal theory, as exposed in [2].

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1. Sporadic zeroes and differentiation.

Notation 1.1. $X \subseteq \mathbb{P}^{n+2}$ is a reduced irreducible nondegenerate subvariety of codimension 2;

^mH is a general linear subspace of codimension m, $m = 1, 2, \ldots, n$;
the ^mH's form general flag, i.e.

$$^nH \subseteq {}^{n-1}H \subseteq \cdots \subseteq {}^1H.$$

As special notations, we use the following:

$$H = {}^1H, \quad Y = H \cap X, \quad W = {}^2H \cap X, \quad C = {}^{n-1}H \cap X, \quad \Gamma = {}^nH \cap X.$$

We also use C and Γ to denote a reduced irreducible nondegenerate curve in $\mathbb{P}^3$ and its general plane section, and similarly we use Γ to denote a set of points of $\mathbb{P}^2$ in general position.

Definition 1.2. *A sporadic zero of degree* s *of* X *is an element of* $I_{Y,s}$ *that is not restriction of any element of* $I_{X,s}$*, i.e. a nonzero element of the cokernel of the restriction map* $I_{X,s} \to I_{Y,s}$*.*
Equivalently, it is a nonzero element of $\ker(H^1(\mathcal{I}_X(s-1)) \to H^1(\mathcal{I}_x(s)))$.

Fix coordinates $x_1, \dots, x_{n+3}$ in $\mathbb{P}^{n+2}$ and let $t_1, \dots, t_{n+3}$ be the dual coordinates in $\mathbb{P}^{n+2*}$, then H has equation $\sum_i t_i x_i$ (We sometimes write $H(t)$ when we want to emphasize its depending on $t \in \mathbb{P}^{n+2*}$.) It induces a map

$$H\cdot : H^1(\mathcal{I}_X(s-1)) \otimes \mathcal{O}_{\mathbb{P}^*}(-1) \to H^1(\mathcal{I}_X(s)) \otimes \mathcal{O}_{\mathbb{P}^*}.$$

Let $\mathcal{K}$ be the kernel of $H\cdot$, then the existence of a sporadic zero in degree s means that $\mathcal{K}$ has positive rank. So, for some $m \geq 0$, $\mathcal{K}(m)$ has sections. An element $\alpha \in H^0(\mathcal{K}(m))$ is a (varying) sporadic zero of X (in degree s). Since $\mathcal{K}(m)$ is a subsheaf of $H^1(\mathcal{I}_X(s-1)) \otimes \mathcal{O}_{\mathbb{P}^*}(-1)$, α can be viewed as an element of

$$H^0(H^1(\mathcal{I}_X(s-1)) \otimes \mathcal{O}_{\mathbb{P}^*}(m-1)) = H^1(\mathcal{I}_X(s-1)) \otimes \mathbb{C}[t]_{m-1}$$

i.e. $\alpha = \alpha(t)$ is a homogeneous polynomial of degree $m-1$ in the dual coordinates t, with coefficients in $H^1(\mathcal{I}_X(s-1))$. By definition, a sporadic zero α has the property that, for any $H \in \mathbb{P}^*$,

$$H \cdot \alpha(t) = 0. \tag{2}$$

Note that $\alpha(t)$ is defined only up to a constant factor, i.e. $\alpha(t) \in \mathbb{P}(H^1(\mathcal{I}_X(s-1)))$, but (2) holds for any choice of $\alpha(t)$, because $H\cdot : H^1(\mathcal{I}_X(s-1)) \to H^1(\mathcal{I}_X(s))$ is a linear map.
The set of (varying) elements $H^1(\mathcal{I}_X(s-1)) \otimes \mathbb{C}[t]$ can be extended to consider the (homogeneous) elements of $H^1(\mathcal{I}_X(s-1)) \otimes \mathbb{C}(t)$. Then $\alpha(t) \in H^1(\mathcal{I}_X(s-1)) \otimes \mathbb{C}(t)$ is a rational function on $\mathbb{P}^*$ with values in $\mathbb{P}(H^1(\mathcal{I}_X(s-1)))$; it is a sporadic zero if satisfies (2). Two elements $\alpha, \beta \in H^1(\mathcal{I}_X(s-1)) \otimes \mathbb{C}(t)$ represent the same sporadic zero iff $\alpha = \rho(t)\beta$, where $\rho(t) \in \mathbb{C}(t)$ is a homogeneous rational function.
$\mathbb{C}(t)$ is a field with derivations: the operators $\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_{n+3}}$ are derivations, i.e. linear maps of degree -1 satisfying Leibnitz rule. The differential operators $\frac{\partial}{\partial t_i}$ extend to $H^1(\mathcal{I}_X(s-1)) \otimes \mathbb{C}(t)$ by acting on the second factor.

The definitions above can be extended *verbatim* to the case of $H^i(\mathcal{O}_{\mathbb{P}}(s)) \otimes \mathbb{C}(t)$ and $H^i(\mathcal{O}_X(s)) \otimes \mathbb{C}(t)$ – indeed to any $U \otimes \mathbb{C}(t)$, where U is a $\mathbb{C}$-space – so we can define differential operators on all these cohomology spaces. The operators $\frac{\partial}{\partial t_i}$ satisfy the expected computation rules. In particular, most important for our purpose will be the following rule: let $\tilde{\alpha} \in H^0(\mathcal{O}_X(s)) \otimes \mathbb{C}(t)$ be homogeneous and let $\alpha \in H^1(\mathcal{I}_X(s)) \otimes \mathbb{C}(t)$ be its image under the natural cohomology map $\delta : H^0(\mathcal{O}_X(s)) \to H^1(\mathcal{I}_X(s))$, then $\frac{\partial \alpha}{\partial t_i}$ is the image of $\frac{\partial \tilde{\alpha}}{\partial t_i}$, i.e. $\delta \frac{\partial}{\partial t_i} = \frac{\partial}{\partial t_i}\delta$; furthermore, if $H = \sum_i t_i x_i$ denotes the general hyperplane, then

$$\frac{\partial}{\partial t_i}(H^q \cdot \tilde{\alpha}) = qH^{q-1}x_i \cdot \tilde{\alpha} + H^q \cdot \frac{\partial \tilde{\alpha}}{\partial t_i}$$

(H and x_i are viewed as linear maps between the appropriate $\mathbb{C}(t)$-vector spaces).

Definition 1.3. *The order of a (fixed) element $\alpha \in H^1(\mathcal{I}_X(s))$-with respect to a hyperplane H-is the maximum integer p such that*

$$\alpha \in \operatorname{im}(H^1(\mathcal{I}_X(s-p)) \xrightarrow{H^p\cdot} H^1(\mathcal{I}_X(s))).$$

α is primitive if its order is zero.

Remark. (i) Note that α is primitive iff $\alpha|_H \in H^1(\mathcal{I}_Y(s))$ is not zero

(ii) For a varying element $\alpha \in H^1(\mathcal{I}_X(s)) \otimes \mathbb{C}(t)$, its order is the order of the generic $\alpha(t)$ with respect to the hyperplane $H = \sum_i t_i x_i$, or, equivalently, the maximum p such that $\alpha \in \operatorname{im}(H^1(\mathcal{I}_X(s-p)) \otimes \mathbb{C}(t) \xrightarrow{H^p\cdot} H^1(\mathcal{I}_X(s)) \otimes \mathbb{C}(t))$.

Lemma 1.4. *If X has a sporadic zero of degree s, then Y has a sporadic zero in degree $\leq s$.*

Proof. As noted earlier, the general hyperplane H has equation $\sum t_i x_i$; furthermore, $x_1, \ldots, x_{n+2}$ are, in a natural way, coordinates on H. We denote by l a general hyperplane in H – i.e. l is a linear variety of dimension n.

Assume that X has a sporadic zero of degree s and order $p-1$, i.e. there exists $\beta = \beta(t) \in H^1(\mathcal{I}_X(s-p))$ such that $H^{p-1} \cdot \beta \neq 0$, $H^p \cdot \beta = 0$ and β is not of form $\beta = H \cdot \gamma$. Differentiating $H^p \cdot \beta = 0$ with respect to $\frac{\partial^p}{\partial x_{i_1} \ldots \partial x_{i_p}}$ we get $p!x_{i_1} \ldots x_{i_p} \cdot \beta + H \cdot \delta = 0$, where $\delta \in H^1(\mathcal{I}_X(s-1))$. Restricting to H, it becomes $x_{i_1} \ldots x_{i_p} \cdot \beta(H)|_H = 0$ in $H^1(\mathcal{I}_Y(s))$ – now x_i, $i = 1, \ldots, n+2$ are coordinates in H.

Now, $\hat{\beta} := \beta(H)|_H \neq 0$ because β is not of form $\beta = H \cdot \gamma$, and, for any monomial x^I of degree p, $x^I \cdot \hat{\beta} = 0$, so $l^p \cdot \hat{\beta} = 0$. Thus there exists $0 \leq r \leq p-1$ such that $l^r \cdot \hat{\beta} \neq 0$ and $l^{r+1} \cdot \hat{\beta} = 0$, for general l-note that $\hat{\beta}$ is constant, i.e. does not depend on l. In other terms, $l^r \cdot \hat{\beta}$ is a nonzero element of $\ker(l\cdot : H^1(\mathcal{I}_Y(s-p+r)) \to H^1(\mathcal{I}_Y(s-p+r+1)))$, i.e. it is a sporadic zero for Y of degree $s-p+r+1 \leq s$. □

Lemma 1.5. *Suppose that X has a sporadic zero in degree s;*

(i) if $I_{W,s-1} = 0$ then Y has a sporadic zero in degree s;

(ii) if $I_{X,s-1} = 0$, then $h^0(\mathcal{I}_Y(s)) > h^0(\mathcal{I}_X(s))$.

Proof. (i) By Lemma 1.4, Y has a sporadic zero α of degree $\leq s$. If $\deg \alpha < s$, then $I_{W,s-1} \neq 0$, contradiction. Hence Y has a sporadic zero of degree s.

(ii) Since a sporadic zero β of X gives rise to an element of $I_{Y,s}$ that is not restriction of an element of $I_{X,s}$, it is enough to prove that no nonzero element of $I_{X,s}$ maps to $0 \in I_{Y,s}$ under the restriction map. But, for a general hyperplane $H \subseteq \mathbb{P}^{n+2}$, the exact sequence $0 \to \mathcal{I}_X(-1) \to \mathcal{I}_X \to \mathcal{I}_Y \to 0$ gives in cohomology $0 \to H^0(\mathcal{I}_X(s-1)) \to H^0(\mathcal{I}_X(s)) \to H^0(\mathcal{I}_Y(s))$. Since $I_{X,s-1} = 0$, then $I_{X,s} \to I_{Y,s}$ is injective. □

Proposition 1.6. *If X has a sporadic zero in degree s and $I_{{}^mH\cap X,s-1} = 0$, then $h^0(\mathcal{I}_{{}^mH\cap X}(s)) \geq m + h^0(\mathcal{I}_X(s))$.*

Proof. By induction, $h^0(\mathcal{I}_{{}^{m-1}H\cap X}(s)) \geq m - 1 + h^0(\mathcal{I}_X(s))$; by Lemma 1.5 (i) – with ${}^{m-2}H \cap X$ playing the rôle of X – we have that ${}^{m-1}H \cap X$ has a sporadic zero in degree s, so we can apply Lemma 1.5 (ii) to ${}^{m-1}H \cap X$ and get

$$h^0(\mathcal{I}_{{}^mH\cap X}(s)) > h^0(\mathcal{I}_{{}^{m-1}H\cap X}(s)),$$

i.e. $h^0(\mathcal{I}_{{}^mH\cap X}(s)) \geq m + h^0(\mathcal{I}_X(s))$. □

Corollary 1.7. *If $X \subseteq \mathbb{P}^{n+2}$ has a sporadic zero of degree s and $I_{\Gamma,s-1} = 0$, then* $\dim I_{\Gamma,s} \geq n + \dim I_{X,s}$.

The ideas underlying the results in the remaining of this section are due to Strano ([7]); the methods of proof, using differentiation of sporadic zeroes, are due to Green ([2]).

Proposition 1.8 ([2]). *Let $\alpha \in H^1(\mathcal{I}_X(s-p)) \otimes \mathbb{C}(t)$ be an element of* $\ker H^p$, *then $\alpha_Y (= \alpha(H)|_H)$ belongs to* $(0 : \mathfrak{m}_H^p)$, *i.e. $\alpha_Y \in H^1(\mathcal{I}_Y(s-p))$ is annihilated by all polynomials of degree p in H.*

Proof. Since $\alpha \in \ker H^p\cdot$, then $H^p \cdot \alpha = 0$ in $H^1(\mathcal{I}_X(s)) \otimes \mathbb{C}(t)$. Differentiating this relation with respect to $\frac{\partial^p}{\partial t_{i_1} \ldots \partial t_{i_p}}$ we get $p! x_{i_1} \ldots x_{i_p} \cdot \alpha + H \cdot \beta = 0$, with $\beta \in H^1(\mathcal{I}_X(s-1)) \otimes \mathbb{C}(t)$. Restricting to H, we have $x_{i_1} \ldots x_{i_p} \cdot \alpha_Y = 0$ in $H^1(\mathcal{I}_Y(s))$. But $x_{i_1} \ldots x_{i_p}$ restricted to H, for all $i_1, \ldots, i_p$, generate the set of all polynomials of degree p, so the proposition is proved. □

Proposition 1.9 ([2]). *Let*

$$0 \to \oplus_i S(-a_{n,i}) \xrightarrow{\phi} \oplus_i S(-a_{n-1,i}) \to \cdots \to \oplus_i S(-a_{0,i}) \to I_Y \to 0$$

be a minimal free resolution of I_Y. Then there exists a nonzero element of $H^1(\mathcal{I}_Y(s-p)) \cap (0 : \mathfrak{m}_H^p)$ – i.e. annihilated by all polynomials of degree p–iff there is a nonzero element of $\oplus_{a_{n,i} \leq s+n+1} H^{n+1}(\mathcal{O}_{\mathbb{P}^{n+1}}(s-p-a_{n,i}))$ mapping to zero under the natural map induced by the resolution.

Proof. The sheafification of the resolution of I_Y is

$$0 \to \oplus_i \mathcal{O}_{\mathbb{P}^{n+1}}(-a_{n,i}) \xrightarrow{\phi} \oplus_i \mathcal{O}_{\mathbb{P}^{n+1}}(-a_{n-1,i}) \to \cdots$$
$$\cdots \to \oplus_i \mathcal{O}_{\mathbb{P}^{n+1}}(-a_{0,i}) \to \mathcal{I}_Y \to 0.$$

Twisting by $s - p$ and taking hypercohomology, we see that

$$H^1(\mathcal{I}_Y(s-p)) \simeq \ker(\oplus_i H^{n+1}(\mathcal{O}_{\mathbb{P}^{n+1}}(s-p-a_{n,i})) \xrightarrow{\phi}$$
$$\oplus_i H^{n+1}(\mathcal{O}_{\mathbb{P}^{n+1}}(s-p-a_{n-1,i}))).$$

Now, by Serre duality, an element of $H^{n+1}(\mathcal{O}_{\mathbb{P}^{n+1}}(q))$ is annihilated by all polynomials of degree p iff $q \geq -p-n-1$. Hence an element $\alpha \in H^1(\mathcal{I}_Y(s-p)) \cap (0 : \mathfrak{m}_H^p)$ corresponds to an element $\hat{\alpha} \in \oplus_{a_{n,i} \leq s+n+1} H^{n+1}(\mathcal{O}_{\mathbb{P}^{n+1}}(s-p-a_{n,i})) \cap \ker\phi$. □

Theorem 1.10 (Re [6])**.** *If X has a sporadic zero of degree s, then Y has a syzygy of order n and degree $\leq s+n+1$.*

Proof. A sporadic zero of X in degree s in a nonzero homogeneous element α of $\ker(H\cdot : H^1(\mathcal{I}_X(s-1)) \otimes \mathbb{C}(t) \to H^1(\mathcal{I}_X(s)) \otimes \mathbb{C}(t))$. Arguing inductively on whether $\alpha \in \operatorname{im}(H\cdot : H^1(\mathcal{I}_X(s-2)) \otimes \mathbb{C}(t) \to H^1(\mathcal{I}_X(s-1)) \otimes \mathbb{C}(t))$, we can assume that, for some $p \geq 1$, there exists a primitive $\beta \in H^1(\mathcal{I}_X(s-p)) \otimes \mathbb{C}(t)$ such that $H^p \cdot \beta = 0$. By Proposition 1.8, β_Y is annihilated by all polynomials of degree p and furthermore $\beta_Y \neq 0$, because β is primitive. So, by Proposition 1.9, there exists a nonzero element in

$$\oplus_{a_{n,i} \leq s+n+1} H^{n+1}(\mathcal{O}_{\mathbb{P}^{n+1}}(s-p-a_{n,i})) \cap \ker\phi. \tag{3}$$

In particular, $a_{n,j} \leq s+n+1$ for some j, i.e. there exists a n-th syzygy of degree $a_{n,j} \leq s+n+1$. □

An immediate consequence of Theorem 1.10 is the following proposition.

Proposition 1.11. *If X has a primitive sporadic zero of degree s, then Y has a n-th syzygy of degree (exactly) $s+n+1$.*

Proof. The hypothesis of primitivity implies that $p = 1$ in (3). So, for some $a_{n,j} \leq s+n+1$, we have $H^{n+1}(\mathcal{O}_{\mathbb{P}^{n+1}}(s-1-a_{n,j})) \neq 0$, hence, by Serre duality, $s-1-a_{n,j} \leq -n-2$, then $a_{n,j} \geq s+n+1$.
It follows that $a_{n,j} = s+n+1$, for some j, i.e. Y has a n-th syzygy of degree $s+n+1$. □

Corollary 1.12. *If C has a primitive sporadic zero of degree s, then Γ has a syzygy of degree (exactly) $s+2$.*

Remark. (i) Both Theorem 1.10 and Proposition 1.11 hold for any (integral, nondegenerate, projective) $X \subseteq \mathbb{P}^{n+2}$, regardless of its codimension, as a straightforward check of their proofs shows.

(ii) Corollary 1.12 is a particular case, of a more general Theorem of Strano (see [7], Theorem 2).

2. A bound on the degree.

Let $f(x) \in \mathbb{C}[x]$ be a homogeneous polynomial, in multiindex notation $f(x) = \sum_K a_K x^K$, $x^K = x_1^{k_1} \dots x_n^{k_n}$. Define the *initial monomial* of $f(x)$ as

$$\text{in}(f(x)) := \max\{x^K \mid a_K \neq 0\},$$

where max is with respect to the reverse lexicographic order on the monomials of $\mathbb{C}[x]$.
Let $I \subseteq \mathbb{C}[x]$ be a homogeneous ideal, define in(I) to be the ideal generated by the monomials in$(f(x))$, for all $f(x) \in I$, $f(x)$ homogeneous.
Let $Z \subseteq \mathbb{P}$ be a (nondegenerate, integral projective) variety; it is a fact that, for general coordinates in $\mathbb{P}$, $in(I_Z)$ stays constant, i.e. it does not depend on the (general) coordinates chosen. This is the *generic initial ideal* of Z, denoted by gin(I_Z); it is of course a monomial ideal.
The relationship between the generators of I and the generators of in(I) is essentially the same as between a basis (i.e. a minimal system of generators) and a Gröbner basis of I. It is well known that any Gröbner basis contains a basis of the ideal, so we can assume that the generators of gin(I_Z) be the initial monomials of such a Gröbner basis, containing a basis of I_Z; some (but not necessarily all) generators of gin(I_Z) are initial monomials of generators of I_Z.
Especially, for a system of points Γ in $\mathbb{P}^2$, gin(I_Γ) is generated by monomials not involving x_3, where x_1, x_2, x_3 are the variables in $\mathbb{P}^2$.
For more details, see [2].

Definition 2.1. *Let* gin(I_Γ) *be minimally generated by*

$$x_1^k, x_1^{k-1}x_2^{\lambda_{k-1}}, \dots, x_2^{\lambda_0},$$

then $\lambda_0, \dots, \lambda_{k-1}$ *are called the invariants of* Γ.

The difference sequence of Γ, $(d_k, d_{k+1}, \dots)$, *is defined by*

$$d_m := h(m) - h(m-1),$$

where h *is the Hilbert function of* Γ.

We denote by g_m *and* σ_m *the number of generators and syzygies in degree* m *for a minimal free resolution of* I_Γ.

Theorem 2.2 (Gruson-Peskine [3]). *If every generator of* $\mathrm{gin}(I_\Gamma)$ *in degree* d *is the initial monomial of a generator of* I_Γ, *for some* $d \geq k + \lambda_{k-1}$, *then the points of* Γ *are not in uniform position.*

Proof. [2], Theorem 4.4 and Remark afterward, Corollary 4.8. □

We need the following relations among the invariants of Γ defined above.

Proposition 2.3. *If* Γ *are* d *points in uniform position, then:*

(i) $d_{m+1} \geq d_m + 2$ *for all* $\lambda_{k-1} + k - 1 \leq m < \lambda_0$;

(ii) if $d_{m+1} = d_m + 2$ *for some* $\lambda_{k-1} + k - 1 \leq m < \lambda_0$, *then* I_Γ *has no generators in degree* $m + 1$;

(iii) $d = \sum_{m=0}^{\lambda_0}(m + 1 - d_m)$;

(iv) $-d_{m-1} + 2d_m - d_{m+1} = \sigma_{m+1} - g_{m+1}$.

Proof. [2], Propositions 4.12 and 4.14. □

Proof of Theorem 0.1. Since X has a sporadic zero in degree s and $I_{\Gamma,s-1} = 0$, by Corollary 1.7, $I_{\Gamma,s}$ has dimension $\geq n+r$, so the element d_s in the difference sequence of Γ is at least $n + r$, say $d_s = \delta$.
By Lemma 1.5 (i), C has a sporadic zero in degree s; if it is primitive – as stated in (iii) – then Γ has a syzygy in degree $s + 2$, by Corollary 1.12. Now, by Proposition 2.3, the ideal I_Γ satisfies the relation

$$-d_s + 2d_{s+1} - d_{s+2} = -g_{s+2} + \sigma_{s+2}, \tag{4}$$

where g and σ are respectively the number of generators and syzygies in a given degree.
By uniform position, $d_{s+1} \geq \delta + 2$, $d_{s+2} \geq \delta + 4$; furthermore, as noted earlier, $\sigma_{s+2} \geq 1$.
If $d_{s+1} = \delta + 2$, then, from (4), we get

$$g_{s+2} = d_{s+2} - d_{s+1} + \sigma_{s+2} - 2 \geq d_{s+2} - d_{s+1} - 1.$$

It follows that every generator of gin(I_Γ) in degree $s+2$ is the initial monomial of a generator of I_Γ in the same degree. Indeed, $d_{s+2} - d_{s+1} - 1$ is the number of generators of gin(I_Γ) in degree $s + 2$; on the other hand, it is a general fact that, for any given degree, the number of generators of I is less or equal to the number of generators of in(I). (The last statement expresses the fact that any Gröbner basis contains a basis of I).
By Theorem 2.2, this is a contradiction to the uniform position of Γ, as soon as $n + r > 1$. Thus $d_{s+1} > \delta + 2$, and the difference sequence of Γ has form

$$d_s \geq n + r, d_{s+m} \geq n + r + 2m + 1, \text{ for } 0 < m \leq s - n - r.$$

It follows:

$$\begin{aligned}\deg X = \deg \Gamma = & \sum_{m=0}^{\infty}(m+1-d_m)\\ \leq & \ 1+2+\cdots+s+\\ & (s-n-r+1)+\\ & (s-n-r-1)+(s-n-r-2)+\cdots+1\\ = & \binom{s+1}{2}+(s-n-r+1)+\binom{s-n-r}{2}\\ = & \ s^2-(n+r-1)s+\binom{n+r}{2}+1. \qquad \square\end{aligned}$$

Remark. (i) The case $n+r=1$, i.e. $n=1, r=0$, is Laudal's Lemma [4]:

$$\deg C \leq s^2+1.$$

(ii) Mezzetti's bound is, of course, the case $r=0$, so theorem 0.1 proves her conjecture under the additional hypotheses that $I_{\Gamma,s-1}=0$ and (one of) the sporadic zero(es) of C be primitive.

As in the case of the original Mezzetti's conjecture, the bound (1) is sharp. To see this, we need the following construction of Chang [1].
Chang proves that all varieties $X \subseteq \mathbb{P}^{n+2}$ having a (special type of) Ω-resolution are arithmetically Buchsbaum of codimension two.
In particular, we are interested in the varieties having an Ω-resolution of form

$$(5) \qquad 0 \to \begin{matrix}(n+r)\mathcal{O}_{\mathbb{P}^{n+2}}(-1)\\ \oplus\\ \mathcal{O}_{\mathbb{P}^{n+2}}(n+r-s-1)\end{matrix} \to \begin{matrix}\Omega^1_{\mathbb{P}^{n+2}}(1)\\ \oplus\\ r\mathcal{O}_{\mathbb{P}^{n+2}}\end{matrix} \to \mathcal{I}_X(s) \to 0.$$

The following argument shows that these varieties satisfy the bound (1) as an equality if $s \geq n+r$.
Since $H^0(\mathcal{O}_{\mathbb{P}^{n+2}}(n+r-s-1))=0$ for $s \geq n+r$, taking H^0 in (5), we have

$$0 \to rH^0(\mathcal{O}_{\mathbb{P}^{n+2}}) \to H^0(\mathcal{I}_X(s)) \to 0,$$

so $\dim I_{X,s}=r$.
Restricting (5) to $H=\mathbb{P}^{n+1}$, we obtain (recall that $\Omega^1_{\mathbb{P}^{n+2}}(1)|_H = \Omega^1_H(1) \oplus \mathcal{O}_H$)

$$0 \to \begin{matrix}(n+r)\mathcal{O}_{\mathbb{P}^{n+1}}(-1)\\ \oplus\\ \mathcal{O}_{\mathbb{P}^{n+1}}(n+r-s-1)\end{matrix} \to \begin{matrix}\Omega^1_{\mathbb{P}^{n+1}}(1)\\ \oplus\\ (r+1)\mathcal{O}_{\mathbb{P}^{n+1}}\end{matrix} \to \mathcal{I}_Y(s) \to 0.$$

so, taking H^0, we similarly have that $h^0(\mathcal{I}_Y(s)) = r+1$, hence X has a sporadic zero of degree s.
Restricting (5) to the general $\mathbb{P}^2$ and twisting by -1, it becomes

$$0 \to \begin{matrix} (n+r)\mathcal{O}_{\mathbb{P}^2}(-2) \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}(n+r-s-2) \end{matrix} \to \begin{matrix} \Omega^1_{\mathbb{P}^2} \\ \oplus \\ (n+r)\mathcal{O}_{\mathbb{P}^2}(-1) \end{matrix} \to \mathcal{I}_\Gamma(s-1) \to 0.$$

which shows that $I_{\Gamma,s-1} = 0$.
Finally, the unique sporadic zero of C is primitive, because, twisting (5) by -2 and restricting to the general $\mathbb{P}^3$, we see that $H^1(\mathcal{I}_C(s-2)) = 0$, so the sporadic zero $\alpha \in H^1(\mathcal{I}_C(s-1))$ cannot be in the image of $H^1(\mathcal{I}_C(s-2))$, i.e. α is primitive.
Now, (5) also yields the exact sequence

$$0 \to \begin{matrix} (n+r)\mathcal{O}_{\mathbb{P}^2}(-1) \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}(n+r-s-1) \end{matrix} \to \begin{matrix} \Omega^1_{\mathbb{P}^2}(1) \\ \oplus \\ (n+r)\mathcal{O}_{\mathbb{P}^2} \end{matrix} \to \mathcal{O}_{\mathbb{P}^2}(s) \to \mathcal{O}_\Gamma(s) \to 0.$$

and a computation of Chern classes shows that

$$\deg X = \deg \Gamma = -c_2(\mathcal{O}_\Gamma(s)) = s^2 - (n+r-1)s + \binom{n+r}{2} + 1.$$

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