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ON A BOOLEAN ALGEBRAS WHICH HAVE THE VITALI-HAHN-SAKS PROPERTY

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Given a boolean algebra A, a submeasure $\lambda : A \to \mathbb{R}_+$ is called a quasi σ-measure if for each disjoint sequence $(a_n)_{n\in\mathbb{N}} \subset A$ there is a subsequence $(b_n)_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ and a $b \in A$ such that $\lambda(b_n - b) = 0$ for each $n \in \mathbb{N}$ and $\lambda(b - \overset{n}{\underset{i=1}{\vee}} b_i) \overset{n}{\to} 0$. We say that a boolean algebra A verifies the Drewnowski condition if each exhaustive submeasure on A it is a quasi σ-measure. In the paper we prove that if a boolean algebra verifies the Drewnowski condition then A has the Vitali-Hahn-Saks property. Also other related questions are investigated.

In the sequel by A denote a boolean algebra.

Definition 1. A function $\lambda : A \to \mathbb{R}_+$ is called submeasure iff: $\lambda(0) = 0$; $\lambda(a) \le \lambda(b)$ for $a \le b$; $\lambda(a \vee b) \le \lambda(a) + \lambda(b)$, for $a, b \in A$.
λ is called exhaustive iff for each disjoint sequence $(a_n)_{n\in\mathbb{N}} \subset A$ we have $\lambda(a_n) \overset{n}{\to} 0$. By measure we mean an additive function on A.

For λ a submeasure we denote by $N_\lambda = \{a \in A \mid \lambda(a) = 0\}$, which evidently is an ideal of A and thus the quotient algebra A/N_λ is defined. For $a \in A$ we denote $\hat{a}$ the corresponding class of a in A/N_λ and by $\hat{\lambda} : A/N_\lambda \to \mathbb{R}_+$, $\hat{\lambda}(\hat{a}) = \lambda(a)$, $a \in A$. The following definition it is inspired from [4] Lemma and [3], Prop. 7.10, p. 155.

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Definition 2. A function $\lambda : A \to \mathbb{R}_+$ it is called a quasi σ-measure iff for each disjoint sequence $(a_n)_{n\in\mathbb{N}} \subset A$ there is a subsequence $(b_n)_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ and $b \in A$ such that:

$$\lambda(b_n - b) = 0 \text{ for each } n \in \mathbb{N}$$

and

$$\lambda(b - \bigvee_{i=1}^{n} b_i) \xrightarrow{n} 0.$$

Also we say that a boolean algebra A satisfies the Drewnowski condition iff each exhaustive submeasure on A it is a quasi σ-measure. Recall that A has the sequential completeness property (SCP) if each disjoint sequence $(a_n)_{n\in\mathbb{N}}$ from A has a subsequence $(a_{k_n})_{n\in\mathbb{N}}$ with $\bigvee_{n\in\mathbb{N}} a_{k_n} \in A$ (see [1]). We say that A has the subsequential interpolation property (SIP) if for any disjoint sequence $(a_n)_{n\in\mathbb{N}}$ in A and any infinite M of $\mathbb{N}$ there is an infinite subset I of M and an element $b \in A$ such that $a_n \le b$ for all $n \in I$ and $a_n \wedge b = 0$ for all $n \in \mathbb{N} \setminus I$ (see [5], [3]). We say that a boolean algebra A has the Vitali-Hahn-Saks property (VHS) with respect to an abelian topological group G if every sequence $(\mu_i)_{i\in\mathbb{N}}$ of exhaustive measures defined on A with values in G which is such that $\lim_i \mu_i(a)$ exists for each $a \in A$, is uniformly exhaustive (see [5], [3]). The following lemma has been proved in [1] for boolean algebras of sets. Our proof is modelled on the proof of Lemma from [4].

Lemma 3. *If A has the* (SCP)*, then A satisfies the Drewnowski condition.*

Proof. Let $\lambda : A \to \mathbb{R}_+$ be an exhaustive submeasure and $(a_n)_{n\in\mathbb{N}} \subset A$ be a disjoint sequence. Let $\mathbb{N} = \bigcup_{k\in\mathbb{N}} N_k$ be a partition on $\mathbb{N}$ with N_k infinite for each $k \in \mathbb{N}$.

Since A has the (SCP), there exists M_k infinite $\subset N_k$ so that $x_k = \bigvee_{n\in M_k} a_n \in A$.

Evidently $(x_k)_{k\in\mathbb{N}}$ are disjoint and thus λ being exhaustive, $\lambda(x_k) \xrightarrow{k} 0$. Thus there exist $k_1 \ge 1$ such that $\lambda(x_{k_1}) \le 1$ i.e. there exist $P_1 = M_{k_1}$ infinite $\subset \mathbb{N}$, $c_1 = x_{k_1} \in A$ such that $\lambda(c_1) \le 1$.

In this way we can construct P_k infinite $\subset \mathbb{N}$ with the properties: $c_k = \bigvee_{n\in P_k} a_n \in A$

$$P_{k+1} \subset P_k \text{ , } \min P_k < \min P_{k+1}$$

$$\lambda(c_k) \le \frac{1}{k}$$

for each $k \in \mathbb{N}$.

If $p_k = \min P_k$, then for $(a_{p_k})_{k\in\mathbb{N}} \subset A$ using the (SCP) for A, there exist a subsequence $(k_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ such that $\bigvee_{l\in\mathbb{N}} a_{p_{k_l}} \overset{\text{not}}{=} a \in A$. If we denote $b_l = a_{p_{k_l}} \in A$ then we have: $a - \bigvee_{l=1}^{n} b_l = \bigvee_{l\geq n+1} b_1$. As $a_n \leq c_k$ for each $n \in P_k$ and for $l \geq n+1$, $p_{k_l} \in P_{k_l} \subset P_{k_{n+1}}$ we obtain $b_l \leq c_{k_{n+1}}$ for each $l \geq n+1$ hence: $\bigvee_{l\geq n+1} b_1 \leq c_{k_{n+1}}$, from where λ being increasing $\lambda(a - \bigvee_{l=1}^{n} b_l) \leq \lambda(C_{k_{n+1}}) \leq \frac{1}{k_{n+1}} \overset{n}{\to} 0$ i.e. A satisfies the Drewnowski condition. □

Proposition 4. *Let $\lambda : A \to \mathbb{R}_+$ be a submeasure. Then:*

a) *If λ is a quasi σ-measure, then A/N_λ has the* (SCP).

b) *If λ is a exhaustive and A/N_λ has the* (SCP)*, then λ is a quasi σ-measure.*

Hence A satisfies the Drewnowski condition if and only if A/N_λ has the (SCP) *for each $\lambda : A \to \mathbb{R}_+$ exhaustive submeasure.*

Proof. a) 1. Let $(a_n)_{n\in\mathbb{N}} \subset A$ be a disjoint sequence; since λ is quasi σ-measure there exist a $b \in A$ and a subsequence $(b_n)_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that: $\lambda(b_n - b) = 0$, for each $n \in \mathbb{N}$ and $\lambda(b - \bigvee_{i=1}^{n} b_i) \overset{n}{\to} 0$. Then $\hat{b}_n \leq \hat{b}$ for each $n \in \mathbb{N}$ and if $\hat{c} \geq \hat{b}_n$, for each $n \in \mathbb{N}$ then $\hat{c} \geq \bigvee_{i=1}^{n} \hat{b}_i$, $\hat{b} - \hat{c} \leq \hat{b} - \bigvee_{i=1}^{n} \hat{b}_i$, from where $\lambda(b - c) \leq \lambda(b - \bigvee_{n=1}^{n} b_i) \overset{n}{\to} 0$; thus $\lambda(b - c) = 0$ i.e. $\hat{b} \leq \hat{c}$ i.e. $\hat{b} = \bigvee_{i=1} \hat{b}_n$ in A/N_λ.

2. If $(\hat{a}_n)_{n\in\mathbb{N}} \subset A/N_\lambda$ is a disjoint sequence and $b_n = a_n - \bigvee_{i<n} a_i$, then $\hat{b}_n = \hat{a}_n$, since $(\hat{a}_n)_{n\in\mathbb{N}}$ is a disjoint sequence and by 1 there exists a subsequence $(b_{k_n})_{n\in\mathbb{N}}$ such that $\bigvee_{n\in\mathbb{N}} \hat{b}_{k_n} \in A/N_\lambda$ i.e. $\bigvee_{n\in\mathbb{N}} \hat{a}_{k_n} \in A/N_\lambda$.

b) Let $\lambda : A \to \mathbb{R}_+$ be an exhaustive submeasure such that A/N_λ has the (SCP). If $\hat{\lambda} : A/N_\lambda \to \mathbb{R}_+$, $\hat{\lambda}(\hat{a}) = \lambda(a)$, $a \in A$ then $\hat{\lambda}$ is well defined and remains exhaustive (use case 2 from a). From Lemma 3 it follows that $\hat{\lambda}$ is a quasi σ-measure, hence if $(a_n)_{n\in\mathbb{N}} \subset A$ are disjoint the for $(\hat{a}_n)_{n\in\mathbb{N}}$ there exist a $b \in A$ and a sequence $(b_n)_{n\in\mathbb{N}}$ such that $\hat{\lambda}(\hat{b}_n - \hat{b}) = 0$, $n \in \mathbb{N}$, and $\hat{\lambda}(\hat{b} - \bigvee_{i=1}^{n} \hat{b}_i) \overset{n}{\to} 0$ i.e. $\lambda(b_n - b) = 0$, $n \in \mathbb{N}$ and $\lambda(b - \bigvee_{i=1}^{n} b_i) \overset{n}{\to} 0$ i.e. λ is a quasi σ-measure. □

Proposition 5. *If A has the* (SIP)*, then A satisfies the Drewnowski condition.*

Proof. This proposition has been suggested by Lemma from [4] and Proposition 7.10 from [3]. Also we make the remark that this implication gives to use the matriceal technics from [2] and this is the point of the beginning of this paper. Let $\lambda : A \to \mathbb{R}_+$ be an exhaustive submeasure. Since A has the (SIP), exactly as in [5] (beginning of the proof of the Theorem 4) for $k \in \mathbb{N}$ we can construct $b_k \in A$, $N_k \subset \mathbb{N}$, N_k infinite, so that:

$$a_n \leq b_k, \forall n \in N_k, a_n \wedge b_k = 0, \forall n \in \mathbb{N} \setminus N_k$$

$$b_k \geq b_{k+1}, \lambda(b_k) \leq \frac{1}{2^k}$$

$$n_k \in N_{k-1}, n_k < \min N_k, N_{k+1} \subset N_k$$

and $a \in A$, $M \subset \mathbb{N}$, M is infinite so that

$$a_{n_k} \leq a, \forall k \in M; a_{n_k} \wedge a = 0, \forall k \in \mathbb{N} \setminus M$$

$$a \wedge [b_k - (a_{n_{k+1}} \wedge b_{k+1})] = 0, \forall k \in N, k \geq 0$$

$$b_0 = 1 \text{ (the unit element of A)}, \mathbb{N}_0 = \mathbb{N}.$$

Then:

$$a \leq a_{n_1} \vee \ldots \vee a_{n_k} \vee b_k \text{ and } a - \vee\{a_{n_i} \mid i \leq k, i \in M\} = a - \bigvee_{i=1}^{k} a_{n_i} \leq b_k$$

from which using the fact that λ is increasing we obtain:

$$\lambda(a - \vee\{a_{n_i} \mid i \leq k, i \in M\}) \leq \lambda(b_k) \leq \frac{1}{2^k} \xrightarrow{k} 0$$

and $a_{n_k} \leq a$, for each $k \in M$; moreover $\lambda(a_{n_k} - a) = 0$ for each $k \in M$. Hence the subsequence $(a_{n_k})_{k \in M}$ verifies the condition of Definition 2. □

Theorem 6. *If A satisfies the Drewnowski condition, then A has the (VHS) with respect to each abelian topological group.*

Proof. Let G be an abelian topological group, $\mu_i : a \to G$ exhaustive measure for each $i \in \mathbb{N}$ so that: $\lim_i \mu_i(a)$ exist for each $a \in A$. We can do suppose that G is a quasinormed group.

Let $\lambda = \sum_{i=1}^{\infty} \frac{1}{2^i} \min(1, \bar{\mu}_i)$ be, where $\bar{\mu}_i$ is the submeasure majorant of μ_i, which remains exhaustive ([4]): hence λ is an exhaustive submeasure. Since A satisfies the Drewnowski condition it follows that λ is a quasi σ-measure.

Let now $(a_j)_{j \in \mathbb{N}} \subset A$ be a disjoint sequence. Then there exist a subsequence $(b_j)_{j \in \mathbb{N}}$ and a $b \in A$ so that $\lambda(b_j - b) = 0$ for $j \in \mathbb{N}$ and $\lambda(b - \bigvee_{j=1}^{n} b_j) \to^n 0$. Since $\lambda \geq \frac{1}{2^i} \min(1, \bar{\mu}_i)$, for each i, we have $\mu_i(b - \bigvee_{j=1}^{n} b_j) \xrightarrow{n} 0$ for each $i \in \mathbb{N}$, and $\mu_i(b_j - b) = 0$ for each $i \in \mathbb{N}$ and $j \in \mathbb{N}$. From this using the additivity of μ_i it follows that: $\mu_i(b) = \sum_{j=1}^{\infty} \mu_i(b_j)$ for each $i \in \mathbb{N}$. This shows that the matrix $(\mu_i(a_j))$ satisfies the hypothesis of (BMT) from [1], p. 7.

Hence $\lim_j \mu_i(a_j) = 0$ uniformly for $i \in \mathbb{N}$ i.e. the family $(\mu_i)_{i\in\mathbb{N}}$ is uniformly exhaustive i.e. A has the (VHS) with respect to abelian topological group G. $\square$

Proposition 7. *Let G be an abelian topological group. Then the following assertions are equivalent:*

i) A has the (VHS) *property with respect to G.*
ii) A/N_λ has the (VHS) *property with respect to G, for each exhaustive submeasure $\lambda : A \to \mathbb{R}_+$.*

Proof. i) $\Rightarrow$ ii) Let I be an ideal of A, $\hat{\mu}_n : A/I \to G$ exhaustive measures for each $n \in \mathbb{N}$ so that $\lim_n \hat{\mu}_n(\hat{a})$ exist for each $\hat{a} \in A/I$. We define $\mu_n : A \to G$, $\mu_n(a) = \hat{\mu}_n(\hat{a})$, $a \in A$. Then μ_n remain exhaustive measures for each $n \in \mathbb{N}$ and $\lim_n \mu_n(a)$ exist for each $a \in A$. As A has (VHS) property with respect to G it follows that the family $(\mu_n)_{n\in\mathbb{N}}$ is uniformly exhaustive. Thus $(\hat{\mu}_n)_{n\in\mathbb{N}}$ will be uniformly exhaustive (use case 2 from Proposition 4) i.e. A/I has the (VHS) property with respect to G.
ii) $\Rightarrow$ i) Let $\mu_n : A \to G$ be exhaustive measures so that $\lim_n \mu_n(a)$ exist for each $a \in A$. We may suppose that G is a quasinormed group. If $\lambda = \sum_{n=1}^{\infty} \frac{1}{2^n} \min(1, \bar{\mu}_n)$, then λ is an exhaustive submeasures in A. We consider $\hat{\mu}_n : A/N_\lambda \to G$, $\hat{\mu}_n(\hat{a}) = \mu_n(a)$, $a \in A$ which are will defined since $N_\lambda = \bigcap_{n\in\mathbb{N}} N_{\mu_n}$ and evidently $\hat{\mu}_n$ is exhaustive for each $n \in \mathbb{N}$. By ii) the family $(\hat{\mu}_n)_{n\in\mathbb{N}}$ will be uniformly exhaustive on A/N_λ, and thus $(\mu_n)_{n\in\mathbb{N}}$ will be uniformly exhaustive on A i.e. i). $\square$

The following theorem resumes the above results.

Theorem 8. *For a boolean algebra A we consider the following assertions:*

a) A has the (SIP).
b) A satisfies the Drewnowski condition.
c) A/N_λ has the (SCP) *for each $\lambda : A \to \mathbb{R}_+$ exhaustive submeasure.*
d) A/N_λ has the (VHS) *property with respect to each abelian topological group, for each $\lambda : A \to \mathbb{R}_+$ exhaustive submeasure.*
e) A has the (VHS) *property with respect to each abelian topological group.*

Then we have: a) $\Rightarrow$ b) $\Leftrightarrow$ c) $\Rightarrow$ d) $\Leftrightarrow$ e).

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