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STRONG SOLVABILITY FOR A CLASS OF NONLINEAR PARABOLIC EQUATIONS

LUBOMIRA SOFTOVA

Dedicated to the memory of Professor Filippo Chiarenza

Existence of strong solutions to Cauchy-Dirichlet problem for nonlinear parabolic equation is established. The nonlinear operator is prescribed by Carathéodory's function which satisfies an ellipticity condition due to S. Campanato. The main results are reached through Aleksandrov-Bakel'man-Pucci type maximum principle and topological fixed point theorem.

1. Introduction.

The general goal of this paper is to study strong solvability properties of the, Cauchy-Dirichlet problem for the nonlinear parabolic equation

$$a(x, t, u, u_x, u_{xx}) - u_t = f(x, t, u, u_x) \tag{$*$}$$

in the rectangle $Q = \{(x, t) \in (0, d) \times (0, T)\}$. The functions $a(x, t, z, p, \xi)$ and $f(x, t, z, p)$ are supposed to be measurable in $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ and continuous in the other variables $(z, p, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, i.e., a and f are Carathéodory's functions. Our main results are derived assuming that $a(x, t, z, p, \xi)$ satisfies an ellipticity condition due to S. Campanato which ensures the operator

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$a(x, t, u, u_x, u_{xx}) - u_t$ to be "near" to the heat operator $u_{xx} - u_t$ both considered as mappings in suitable Sobolev spaces. As it concerns the right-hand side $f(x, t, z, p)$ we allow quadratic growth with respect to the variable p.

Strong solvability result for $(*)$ has been obtained by Campanato in [1] in the case $a = a(x, t, \xi)$ and $f = f(x, t) \in L^2(Q)$. Strong solvability of the quasilinear equation $(*)$ (i.e., $a(x, t, z, p, \xi) = a'(x, t, z, p)\xi + a''(x, t, z, p)p$) with linear growth of $f(x, t, z, p)$ with respect to the gradient p has been proved by Maugeri in [6]. We should note that results similar to the ones prescribed here have been proved for nonlinear elliptic equations in [8] when the term f grows sub-quadratically in p, and in [10] if f has a quadratic growth with respect to p.

The existence of strong solution to the Cauchy-Dirichlet problem for the equation $(*)$ is established through Leray-Schauder's fixed point theorem by applying a standard procedure. Essential part of this procedure consists of deriving an $L^4(Q)$ a priori estimate for the gradient u_x of all eventual solutions to $(*)$. Making use of Aleksandrov-Bakel'man-Pucci type maximum principle, which is due to Krilov [4] and Tso [11], we establish also an $L^\infty(Q)$ a priori estimate in order to derive the strong solvability result.

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2. Setting of the problem and main results.

We shall study the Cauchy-Dirichlet problem

$$
\begin{cases} a(x, t, u, u_x, u_{xx}) - u_t = f(x, t, u, u_x) & \text{a.e. in } Q \\ u = 0 & \text{on } \partial Q \end{cases} \tag{1}
$$

in the rectangle $Q = \{(x, t) \in (0, d) \times (0, T)\}$ with parabolic boundary $\partial Q = \{(x, 0), x \in [0, d]\} \cup \{(0, t), t \in [0, T]\} \cup \{(d, t), t \in [0, T]\}$. Suppose that $a(x, t, z, p, \xi)$ and $f(x, t, z, p)$ are real-valued functions which fulfill Carathéodory's condition.

We shall consider strong solutions to the problem (1), i.e. twice weakly differentiable functions with respect to x and once in t which satisfy the equation in (1) a.e. in Q and achieve their boundary values in the sense of $H^1(0, d)$, (H^k means the Sobolev space of all functions having L^2-summable derivatives up to order k).

We shall denote by $W_0^{2,1;2}(Q)$ the real Sobolev space

$$W_0^{2,1;2}(Q) = \left\{ u \in L^2(0, T, H^2(0, d) \cap H_0^1(0, d)) : \frac{\partial u}{\partial t} \in L^2(Q), u(x, 0) = 0 \right\}$$

equipped with the norm

$$\|u\|_{(\beta)}^2 = \|u\|_{W_0^{2,1;2}(Q)}^2 = \int_Q (|u_{xx}|^2 + \beta^2 |u_t|^2) \, dx dt$$

where $\beta > 0$ is a constant (cf. [1]). Further, we impose the following requirements on the functions $a(x, t, z, p, \xi)$ and $f(x, t, z, p)$:

(A) (Campanato's ellipticity condition)
There exist positive constants α and K, $K < 1$, such that

$$|\xi - \alpha[a(x, t, z, p, \xi + \tau) - a(x, t, z, p, \tau)]| \leq K|\xi|$$

and $a(x, t, z, p, 0) = 0$;

(B) $|f(x, t, z, p)| \leq f_1(|z|)\Big[f_2(x, t) + |p|^2 \Big]$,
where the functions f_1 and f_2 are positive, $f_1 \in C^0(\mathbb{R}^+)$ is monotone nondecreasing function and $f_2 \in L^2(Q)$;

(C) $2zf(x, t, z, p) \geq -\mu_1(x, t)2zp - \mu_2(x, t)z^2 - \mu_3(x, t)$,
where $\mu_1(x, t)$ and $\mu_3(x, t)$ belong to the class $L^2(Q)$, and $\mu_2(x, t) \in L^\infty$.

Now we can formulate our main results.

Theorem 1 (Gradient estimate). *Assume conditions* (A) *and* (B) *to be fulfilled. Then there exists a constant* $C = C(\alpha, K, f_1, f_2, d, T, \|u\|_{L^\infty(Q)})$ *such that*

$$\|u_x\|_{L^4(Q)} \leq C \tag{2}$$

for every strong solution $u \in W_0^{2,1;2}(Q)$ *of the Cauchy-Dirichlet problem* (1).

Theorem 2 (Existence). *Let the conditions* (A), (B) *and* (C) *be satisfied. Then the Cauchy-Dirichlet problem* (1) *has a solution* $u \in W_0^{2,1;2}(Q)$.

Remark. Let us note that the condition (A) means ellipticity of the equation in (1). In fact, it follows from the Lemma in [8] that the function $\xi \to a(x,t,z,p,\xi)$ is differentiable almost everywhere with respect to ξ and $0 < \lambda \le \frac{\partial a}{\partial \xi}(x,t,z,p,\xi) \le \Lambda$ with constants λ and Λ depending on α and K. Vice versa, if the derivative $\frac{\partial a}{\partial \xi}$ exists almost everywhere and $0 < \lambda \le \frac{\partial a}{\partial \xi}(x,t,z,p,\xi) \le \Lambda$, then the condition (A) is fulfilled with suitable constants α and $K < 1$.

Here the ellipticity condition in the form (A) is more convenient because of the use of Campanato's theory of near mappings (cf. [1]).

3. Proofs of the results.

Proof of Theorem 1. Let $u \in W_0^{2,1;2}(Q)$ be a bounded solution of the problem (1). We shall rewrite the equation in (1) as follows:

$$a(x,t,u,u_x,u_{xx}) - \frac{f(x,t,u,u_x)\left[f_2(x,t)+u_x^2\right]}{f_2(x,t)+u_x^2} - u_t = 0,$$

$$\begin{aligned} a(x,t,u,u_x,u_{xx}) - &\frac{f(x,t,u,u_x)}{f_2(x,t)+u_x^2}u_x^2 - f_2(x,t)u - u_t \\ &= -f_2(x,t)u + \frac{f(x,t,u,u_x)}{f_2(x,t)+u_x^2} f_2(x,t). \end{aligned}$$

Now, defining the functions

$$b(x,t) = -\frac{f(x,t,u,u_x)}{f_2(x,t)+u_x^2}$$

and

$$F(x,t) = \frac{f(x,t,u,u_x)}{f_2(x,t)+u_x^2} f_2(x,t) - f_2(x,t)u,$$

we get the problem

$$(3) \quad \begin{cases} a(x,t,u,u_x,u_{xx}) + b(x,t)u_x^2 - f_2(x,t)u - u_t = F(x,t) & \text{a.e. in } Q \\ u = 0 & \text{on } \partial Q. \end{cases}$$

According to the assumption (B) one has $|b(x,t)| \leq f_1(\|u\|_{L^\infty}) < \infty$, i.e. $b(x,t) \in L^\infty(Q)$, and $F(x,t) \in L^2(Q)$.

Let $\rho \in [0,1]$ be a parameter and for the unknown $v(x,t) \in W_0^{2,1;2}(Q)$ consider the problem

$$(4)\quad \begin{cases} a(x,t,u,u_x,v_{xx}) + b(x,t)v_x^2 - f_2(x,t)v - v_t = \rho F(x,t) & \text{a.e. in } Q \\ v = 0 & \text{on } \partial Q. \end{cases}$$

If $\rho = 0$, the problem (4) has a solution $v \equiv 0$. In the case $\rho = 1$, $v \equiv u$ is one of the solutions to this problem. Thus, if we know in addition uniqueness result for the problem (4), then the solution $v(x,t)$ of (4) with $\rho = 1$ coincides with the fixed solution $u(x,t)$ of the problem (3).

Proposition 3. *Let $v', v'' \in W_0^{2,1;2}(Q)$ be two solutions of the problem* (4) *corresponding to the parameters* $0 \leq \rho' < \rho'' \leq 1$. *Then*

$$(5)\qquad \|v' - v''\|_{L^\infty(Q)} \leq (\rho'' - \rho')[f_1(\|u\|_{L^\infty(Q)}) + \|u\|_{L^\infty(Q)}].$$

Proof. Subtracting the equations satisfied by v' and v'' we get

$$(6)\qquad \begin{cases} a(x,t,u,u_x,v'_{xx}) + b(x,t){v'_x}^2 - f_2(x,t)v' - \\ \quad v'_t - a(x,t,u,u_x,v''_{xx}) - b(x,t){v''_x}^2 + \\ \quad + f_2(x,t)v'' + v''_t = (\rho' - \rho'')F(x,t) & \text{a.e. in } Q \\ v' - v'' = 0 & \text{on } \partial Q. \end{cases}$$

According to the Lemma in [8], the function $\xi \to a(x,t,z,p,\xi)$ is differentiable almost everywhere with respect to ξ, the derivative $a_\xi(x,t,z,p,\xi)$ is strictly positive and belongs to $L^\infty(Q \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$. This allows us to linearize the problem (6). Thus, imposing the new notations

$$w = v' - v'',$$

$$A(x,t) = \int_0^1 a_\xi(x,t,u,u_x,sw_{xx} + v''_{xx})\,ds,$$

$$B(x,t) = 2b(x,t)\int_0^1 (sw_x + v''_x)\,ds,$$

the problem (6) reads

$$(7)\qquad \begin{cases} A(x,t)w_{xx} + B(x,t)w_x - f_2(x,t)w - w_t = \\ \qquad = (\rho' - \rho'')F(x,t) & \text{a.e. in } Q \\ w = 0 & \text{on } \partial Q. \end{cases}$$

We can estimate the function $F(x,t)$ in view of the condition (B), namely

$$F(x,t) \leq |F(x,t)| \leq \frac{|f(x,t,u,u_x)| f_2(x,t)}{f_2(x,t)+u_x^2} + f_2(x,t)|u|$$
$$\leq f_2(x,t)\big[f_1(|u|)+|u|\big] \leq f_2(x,t)\big[f_1(\|u\|_{L^\infty(Q)}) + \|u\|_{L^\infty(Q)}\big].$$

If we denote by P the linear parabolic operator in (7) and apply it to the constant $M = (\rho'' - \rho')[f_1(\|u\|_{L^\infty(Q)}) + \|u\|_{L^\infty(Q)}]$ we get $PM = -f_2(x,t)(\rho'' - \rho')[f_1(\|u\|_{L^\infty(Q)}) + \|u\|_{L^\infty(Q)}]$. Hence

$$\begin{cases} P(w-M) \geq 0 & \text{a.e. in } Q \\ w - M \leq 0 & \text{on } \partial Q. \end{cases}$$

According to the Aleksandrov-Bakel'man-Pucci type maximum principle ([4], [11]) one has

$$\max_Q (w-M) \leq \max_{\partial Q} (w-M)^+ = 0,$$

i.e. $w \leq M$. Considering the same problem but for $-w$, we get an estimate from bellow $w \geq -M$. Hence $\|w\|_{L^\infty(Q)} \leq M$, that is what we needed to prove. □

Corollary 4. *If the problem* (4) *has a solution for some* $\rho \in [0,1)$ *then it is a unique solution.*

Proof. It follows immediately from (5) putting $\rho' = \rho''$. Then $v' \equiv v''$. Moreover, if $\rho' = 0$ and $v' \equiv 0$, then we get L^∞ estimate for the solution v''. Since ρ' and ρ'' are arbitrary, that estimate is true for any solution v and $\rho \in [0,1)$. □

For our further considerations we need Campanato's definition of "nearness" between operators. Let $\mathcal{A}$ and $\mathcal{B}$ be two operators acting from a Hilbert space $\mathcal{H}$ into $\mathcal{H}'$:

$$\mathcal{A},\ \mathcal{B} : \mathcal{H} \longrightarrow \mathcal{H}'.$$

Definition 1. *We shall say that* $\mathcal{A}$ *is "near"* $\mathcal{B}$ *if there exist two positive constants* α *and* K, $K \in (0,1)$, *such that for each* $u, v \in \mathcal{H}$ *we have*

$$\|\mathcal{B}u - \mathcal{B}v - \alpha[\mathcal{A}u - \mathcal{A}v]\|_{\mathcal{H}'} \leq K\|\mathcal{B}u - \mathcal{B}v\|_{\mathcal{H}'}.$$

Let us recall the following definition of monotonicity.

Definition 2. *The operator $\mathcal{A}$ is said to be monotone with respect to the operator $\mathcal{B}$ if for each $u, v \in \mathcal{H}$ we have*

$$(\mathcal{A}u - \mathcal{A}v, \mathcal{B}u - \mathcal{B}v)_{\mathcal{H}'} \geq 0.$$

Here $(\cdot, \cdot)_{\mathcal{H}'}$ is the inner product in the space $\mathcal{H}'$.

We are in a position now to derive the gradient estimate (2) for each solution $u(x, t)$ of (1). For this goal consider the solutions v' and v'' of (4) which correspond to parameters $\rho' < \rho''$. Taking the difference between the corresponding equations we have

$$(8)\quad \begin{cases} a(x, t, u, u_x, v'_{xx}) - a(x, t, u, u_x, v''_{xx}) - v'_t + v''_t = G(x, t) & \text{a.e. in } Q \\ v' - v'' = 0 & \text{on } \partial Q, \end{cases}$$

where

$$G(x, t) = F(x, t)(\rho' - \rho'') - b(x, t)({v'_x}^2 - {v''_x}^2) + f_2(x, t)(v' - v'').$$

Having in mind condition (A) and Young's inequality we obtain

$$|w_{xx} - \alpha w_t|^2 \leq K^2(1 + \varepsilon)|w_{xx}|^2 + (\alpha^2 + \frac{\alpha^2}{\varepsilon})|G(x, t)|^2,$$

where $w = v' - v''$. On the other hand, Lemma 2.3 in [2] yields

$$\begin{aligned} \|w\|^2_{(\alpha)} &= \int_Q [|w_{xx}|^2 + \alpha^2 |w_t|^2]\, dxdt \\ &\leq \int_Q |w_{xx} - \alpha w_t|^2\, dxdt \leq \int_Q K^2(1 + \varepsilon)|w_{xx}|^2\, dxdt \\ &+ \int_Q \Big(\alpha^2 + \frac{\alpha^2}{\varepsilon}\Big)|G(x, t)|^2\, dxdt. \end{aligned}$$

Now, if $\varepsilon > 0$ is chosen so small that $K^2(1 + \varepsilon) < 1$, we get

$$\|w\|^2_{(\beta)} \leq C_1(\alpha, K, \varepsilon) \int_Q |G(x, t)|^2\, dxdt$$

where $\beta^2 = \dfrac{\alpha^2}{1 - K^2(1+\varepsilon)}$, i.e.

$$\begin{aligned}
\|w\|_{(\beta)} &\le C_1 \|G(x,t)\|_{L^2(Q)} \\
&\le C_1 \Big[\|F\|_{L^2(Q)} + \|b\|_{L^\infty(Q)} (\|v'_x\|^2_{L^4(Q)} + \|v''_x\|^2_{L^4(Q)}) + \\
&\quad + \|f_2\|_{L^2(Q)} \|w\|_{L^\infty(Q)} \Big] \\
&\le C_2 \Big[1 + \|v'_x\|^2_{L^4(Q)} + \|v''_x\|^2_{L^4(Q)} \Big] \\
&\le C_3 \Big[1 + \|v'_x\|^2_{L^4(Q)} + \|w_x\|^2_{L^4(Q)} \Big].
\end{aligned}$$

Hence, the Gagliardo-Nirenberg interpolation inequality (see [7])

$$\|w_x\|^2_{L^4(Q)} \le N \|w_{xx}\|_{L^2(Q)} \|w\|_{L^\infty(Q)}$$

implies

$$\begin{aligned}
\|w\|_{(\beta)} \le C_3 \Big\{ & 1 + \|v'_x\|^2_{L^4(Q)} + \\
& + N(\rho'' - \rho') \big[f_1(\|u\|_{L^\infty(Q)} + \|u\|_{L^\infty(Q)}) \big] \|w_{xx}\|_{L^2(Q)} \Big\}
\end{aligned}$$

with N being a constant which depends only on Q. (Indeed, to derive that version of Gagliardo-Nirenberg's inequality, one should apply at first the classical result with respect to x for a fixed t and then integrate with respect to t. We refer the reader to the monograph [5] for details).

If $\rho'' - \rho' = \tau$ is so small that $C_3 N \tau [f_1(\|u\|_{L^\infty(Q)}) + \|u\|_{L^\infty(Q)}] < 1$ we may move $\|w_{xx}\|_{L^\infty(Q)}$ to the left-hand side of the last inequality. Moreover, having in mind

$$\|w_{xx}\|_{L^2(Q)} \le \|w\|_{(\beta)}$$

we get

$$\|w_{xx}\|_{L^2(Q)} \le C_4 (1 + \|v'_x\|^2_{L^4(Q)}).$$

That is why

$$\begin{aligned}
\|v''_x\|^2_{L^4(Q)} &\le \|w_x\|^2_{L^4(Q)} + \|v'_x\|^2_{L^4(Q)} \\
&\le N \|w_{xx}\|_{L^2(Q)} \|w\|_{L^\infty(Q)} + \|v'_x\|^2_{L^4(Q)} \\
&\le C_5 + C_6 \|v'_x\|^2_{L^4(Q)}.
\end{aligned}$$

We put $\rho' = 0$, $v' = 0$, $\rho'' = \tau$, $v'' = v^\tau$ above and get immediately an estimate for the $L^4(Q)$ norm of the gradient of v^τ

$$\|v_x^\tau\|^2_{L^4(Q)} \leq C_5(\alpha, K, \varepsilon, \|f_1\|_{L^\infty}, \|f_2\|_{L^2}, \|u\|_{L^\infty}) \tag{9}$$

whenever there exists a solution v^τ of the problem (4) with $\rho = \tau$.

Now we shall prove strong solvability of the problem (4) with $\rho = \tau$. For this goal Leray-Schauder's fixed point theorem will be used.

Define the space

$$\mathcal{S} = \left\{ y : \quad y \in L^\infty(Q), \quad y_x \in L^4(Q) \right\}$$

equipped with the norm

$$\|y\|_{\mathcal{S}} = \|y\|_\infty + \|y_x\|_{L^4(Q)},$$

as it was done in [6]. Now, define the operator $\mathcal{M} : [0, 1] \times \mathcal{S} \to W_0^{2,1;2}(Q)$ to act as follows. For each $\sigma \in [0, 1]$ and for each $y \in \mathcal{S}$ we consider the problem

$$\begin{cases} a(x, t, u, u_x, z_{xx}) - z_t = \\ \quad = \sigma[\tau F(x, t) - b(x, t)|y_x|^2 + f_2(x, t)y] & \text{a.e. in } Q \\ z = 0 & \text{on } \partial Q. \end{cases} \tag{10}$$

Let us note that the right-hand side above belongs to L^2 since $F(x, t) \in L^2(Q)$, $b(x, t) \in L^\infty(Q)$, $y \in L^\infty(Q)$, $y_x \in L^4(Q)$ and

$$\int_Q f_2^2(x, t) y^2\, dx dt \leq \|y\|^2_\infty \|f_2\|^2_{L^2(Q)} < \infty.$$

Since the condition (A) means "nearness" between $a(x, t, u, u_x, z_{xx})$ and the Laplace operator z_{xx}, and the operator z_t is monotone with respect to z_{xx} ([2]), it follows by Theorem 9 in [1] that the parabolic operator $a(x, t, u, u_x, z_{xx}) - z_t$ is "near" to the heat operator $z_{xx} - \alpha z_t$. On the other hand, the right-hand side in the Cauchy-Dirichlet problem (10) belongs to L^2. Since the problem

$$\begin{cases} z_{xx} - \alpha z_t = \sigma[\tau F(x, t) - b(x, t)|y_x|^2 + f_2(x, t)y] & \text{a.e. in } Q \\ z = 0 & \text{on } \partial Q. \end{cases} \tag{11}$$

has a unique solution $z \in W_0^{2,1;2}(Q)$, according to [2] our problem (10) admits a unique solution lying at the same space. In such a way, the operator $\mathcal{M}$ is well defined. Further on, it follows by Lemmas (2-1) and (2-2) in [6] that $W_0^{2,1;2}(Q) \subset \mathcal{S}$ and thus we may consider $\mathcal{M}$ to act from $[0, 1] \times \mathcal{S}$ into $\mathcal{S}$.

Now, the condition $a(x, t, u, u_x, 0) = 0$, as required in (A), shows that $\mathcal{M}(0, y) = 0$ for each $y \in \mathcal{S}$. Similar arguments as these used in [8] imply continuity of $\mathcal{M}$. Finally, it is proved in [6] (pp. 387–388) that $\mathcal{M}$ is a compact operator considered as a mapping from $[0, 1] \times \mathcal{S}$ into $\mathcal{S}$. The a priori estimate (9) provides a uniform with respect to σ and y bound for each solution to the equation $\mathcal{M}(\sigma, y) = y$ which is equivalent to the problem

$$\begin{cases} a(x, t, u, u_x, y_{xx}) - y_t = \sigma[\tau F(x, t) - b(x, t)|y_x|^2 + f_2(x, t)y] & \text{a.e. in } Q \\ y = 0 & \text{on } \partial Q. \end{cases}$$

Therefore, the Leray-Schauder theorem implies existence of a fixed point of the mapping $\mathcal{M}(1, \cdot)$ which, in view of the definition of $\mathcal{M}$, becomes solution to the problem (4) with $\rho = \tau$.

Finally, separating the interval $[0, 1]$ of m subintervals of length less than or equal to τ and repeating the above procedure m times, we get the desired estimate for the gradient of the solution

$$\|u_x\|^2_{L^4(Q)} \leq C_7. \tag{12}$$

This completes the proof of Theorem 1. □

Returning to the problem (1), we need an L^∞ a priori estimate for u in order to derive its strong solvability. A variant of this estimate can be found in [6], p. 393 but, having in mind the ellipticity condition (A), we prefer to propose a direct proof.

Proposition 5. *Let conditions* (*A*) *and* (*C*) *hold. Then each solution* $u \in W_0^{2,1;2}(Q)$ *of the problem* (1) *satisfies the estimate*

$$\|u\|_{L^\infty(Q)} \leq \Big(c_1\sqrt{d}\exp\{c_2 d^{-1}\|\mu_1\|^2_{L^2(Q)}\}\|(-\mu_3 e^{-Mt})^-\|_{L^2(Q)}\Big)^{1/2} \cdot \cdot \exp\left\{\frac{MT}{2}\right\}. \tag{13}$$

where $\|\mu_2(x, t)\|_{L^\infty(Q)} \leq M$.

Proof. The problem (1) is equivalent to the next one

$$\begin{cases} A(x, t)u_{xx} - u_t = f(x, t, u, u_x) & \text{a.e. in } Q \\ u = 0 & \text{on } \partial Q, \end{cases}$$

where

$$A(x,t) = \int_0^1 a_\xi(x,t,u,u_x,su_{xx})\,ds.$$

According to the ellipticity condition (A), $A(x,t) \in L^\infty(Q)$ and $A(x,t) \geq 0$, as it is proved in [8]. Multiplying the equation by $2u$ and using condition (C) we get

$$A(x,t)(u^2)_{xx} + \mu_1(x,t)(u^2)_x + \mu_2(x,t)u^2 - (u^2)_t \geq -\mu_3(x,t).$$

Putting $u^2 = we^{Mt}$ we get

$$Aw_{xx} + \mu_1 w_x + (\mu_2 - M)w - w_t \geq -\mu_3 e^{-Mt}.$$

Since $\mu_2 - M$ is nonpositive we can apply the Aleksandrov-Bakel'man-Pucci maximum principle [11] that yields the estimate

$$\max_Q w \leq c_1\sqrt{d}\exp\{c_2 d^{-1}\|\mu_1\|^2_{L^2(Q)}\}\|(-\mu_3 e^{-Mt})^-\|_{L^2(Q)}$$

where c_1 and c_2 are constants depending on Q. Since $\max_Q u^2 \leq \max_Q we^{MT}$ we get what we needed to prove. □

Proof of Theorem 2. It follows by the Leray-Schauder fixed point theorem and the estimates (13) and (2) in a similar way as that already used in the proof of Theorem 1 (cf. [8] also). We omit the details. □

REFERENCES

[1] S. Campanato, *On the condition of nearness between operators,* Ann. Mat. Pura Appl., (4) 167 (1994), pp. 243–256.

[2] S. Campanato, *Non variational basic parabolic systems of second order,* Rend. Mat. Acc. Lincei, 3 (1992), pp. 129–136.

[3] D. Gilbarg - N. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer-Verlag, Berlin, Heidelberg, 1983.

[4] N. Krilov, *Sequences of convex functions and estimates of the maximum of the solution of a parabolic equation,* Siberian Math. J., 17 (1976), pp. 226–236.

[5] O. Ladyzhenskaya - V. Solonnikov - N. Uralt'seva, *Linear and Quasilinear Equations of Parabolic Type,* M. Nauka, 1967, (in Russian).

[6] A. Maugeri, *Sul problema di Cauchy-Dirichlet per le equazioni paraboliche non lineari del secondo ordine in due variabili,* Le Matematiche, 22 (1967), pp. 383–396.

[7] L. Nirenberg, *On elliptic partial differential equations,* Ann. Sc. Norm. Sup. Pisa, (3) 13 (1959), pp. 115–162.

[8] D. Palagachev, *Global strong solvability of Dirichlet problem for a class of nonlinear elliptic equations in the plane,* Le Matematiche, 48 (1993), pp. 311–321.

[9] S. Sobolev, *Selected Questions of the Theory of Functional Spaces and Generalized Functions,* M. Nauka, 1989.

[10] L. Softova, *Gradient a priori estimate for a class of nonlinear elliptic equations in the plane,* (to appear).

[11] K. Tso, *On an Aleksandrov-Bakel'man Type maximum principle for second-order parabolic equations,* Comm. Part. Diff. Eq., 10 (1985), pp. 543–553.

Department of Mathematics,
Higher Military Transport School,
158 "Geo Milev" Street,
1574 Sofia (BULGARIA),
e-mail: luba@vvtu.bg