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THE PICARD AND THE GAUSS-WEIERSTRASS SINGULAR INTEGRALS OF FUNCTIONS OF TWO VARIABLES

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We study some approximation properties of the Picard and the Gauss-Weierstrass singular integrals of functions of two variables belonging to some exponential weighted spaces. In Sec. 3.1 and 3.2 we give two theorems on the degree of approximation and two theorems of the Voronovskaya type. In Sec. 3.3 the Bernstein inequalities for these singular integrals are proved.

Some similar results for the Picard and the Gauss-Weierstrass singular integral of functions of one variable were given in [1] – [3].

The Picard and the Gauss-Weierstrass singular integral of functions f of one variable

$$P_r(f;x) := \frac{1}{2r}\int_{-\infty}^{+\infty} f(x+t)e^{-\frac{|t|}{r}}\,dt,$$

$$W_r(f;x) := \frac{1}{\sqrt{\pi r}}\int_{-\infty}^{+\infty} f(x+t)e^{-\frac{t^2}{r}}\,dt,$$

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$x \in \mathbb{R}$, $r > 0$, were examined in many papers, for example [1] – [3]. In these papers were proved some approximation properties of these integrals in various spaces.

In the present paper we shall consider the Picard and the Gauss-Weierstrass singular integrals for functions of two variables belonging to some exponential weighted spaces.

1. Preliminaries.

1.1. Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := N \cup \{0\}$, $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}_0 := [0, +\infty)$, $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ and let for a fixed $q \in \mathbb{R}_0$

$$\nu_{1,q}(x) := e^{-q|x|}, \quad x \in \mathbb{R}, \tag{1}$$

$$\nu_{2,q}(x) := e^{-qx^2}, \quad x \in \mathbb{R}. \tag{2}$$

For a fixed $q_1, q_2 \in \mathbb{R}_0$ let

$$\nu_{1;q_1,q_2}(x, y) := \nu_{1,q_1}(x)\nu_{1,q_2}(y), \quad (x, y) \in \mathbb{R}^2, \tag{3}$$

$$\nu_{2;q_1,q_2}(x, y) := \nu_{2,q_1}(x)\nu_{2,q_2}(y), \quad (x, y) \in \mathbb{R}^2. \tag{4}$$

Next, for fixed $1 \leq p \leq \infty$, $q_1, q_2 \in \mathbb{R}_0$ and $i = 1, 2$, we denote by $L_i^{p;q_1,q_2}$ the exponential weighted space of all real-valued function f defined on $\mathbb{R}^2$ for which $\nu_{i;q_1,q_2}(\cdot\,,\cdot)f(\cdot\,,\cdot)$ is a function Lebesgue-integrable with p-th power over $\mathbb{R}^2$ if $1 \leq p < \infty$, and $\nu_{i;q_1,q_2}(\cdot\,,\cdot)f(\cdot\,,\cdot)$ is uniformly continuous and bounded on $\mathbb{R}^2$ if $p = \infty$. The norm in $L_i^{p;q_1,q_2}$, $i = 1, 2$, is defined by the formula

$$\|f\|_{i;p;q_1,q_2} \equiv \|f(\cdot)\|_{i;p;q_1,q_2} := \tag{5}$$

$$= \begin{cases} \left(\iint_{\mathbb{R}^2} \left|\nu_{i;q_1q_2}(x, y)f(x, y)\right|^p dxdy \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \sup\limits_{(x,y)\in\mathbb{R}^2} \nu_{i;q_1,q_2}(x, y)|f(x, y)| & \text{if } p = \infty. \end{cases}$$

For $f \in L_i^{p;q_1,q_2}$, with some fixed p, q_1, q_2, i, and for $t, s \in \mathbb{R}_0$ we introduce the following moduli of smoothness

$$\omega^*(f, L_i^{p;q_1,q_2}; t, s) : = \sup_{\substack{|h|\leq t \\ |\delta|\leq s}} \left\|\Delta^*_{h,\delta} f(\cdot\,,\cdot)\right\|_{t;p;q_1q_2},$$

$$\omega_2(f, L_i^{p;q_1,q_2}; t, 0) : = \sup_{|h|\leq t} \left\|\Delta^2_{h,0} f(\cdot\,,\cdot)\right\|_{t;p;q_1q_2},$$

$$\omega_2(f, L_i^{p;q_1,q_2}; 0, s) : = \sup_{|\delta|\leq s} \left\|\Delta^2_{0,\delta} f(\cdot\,,\cdot)\right\|_{t;p;q_1q_2},$$

where

$$\Delta^*_{h,\delta} f(x, y) := f(x+h, y+\delta) + f(x+h, y-\delta) + \\ + f(x-h, y+\delta) + f(x-h, y-\delta) - 4f(x, y),$$

$$\Delta^2_{h,0} f(x, y) := f(x+h, y) + f(x-h, y) - 2f(x, y),$$

$$\Delta^2_{0,\delta} f(x, y) := f(x, y+\delta) + f(x, y-\delta) - 2f(x, y),$$

for all $(x, y) \in \mathbb{R}^2$ and $h, \delta \in \mathbb{R}$.
From this and by (1) – (5) we get for $f \in L_1^{p;q_1,q_2}$ and for all $t, s, \lambda \geq 0$

$$\omega^*(f, L_1^{p;q_1,q_2}; t, s) \leq 2\{e^{q_2 s}\omega_2(f, L_1^{p;q_1,q_2}; t, 0) + \\ + e^{q_1 t}\omega_2(f, L_1^{p;q_1,q_2}, 0, s)\}, \tag{6}$$

$$\omega_2(f, L_1^{p;q_1,q_2}; \lambda t, 0) \leq (\lambda+1)^2\{e^{\lambda q_1 t}\omega_2(f, L_1^{p;q_1,q_2}; t, 0), \tag{7}$$

$$\omega_2(f, L_1^{p;q_1,q_2}; 0, \lambda s) \leq (\lambda+1)^2\{e^{\lambda q_2 s}\omega_2(f, L_1^{p;q_1,q_2}; 0, s). \tag{8}$$

If $f \in L_2^{p;q_1,q_2}$ with some fixed p, q_1, q_2, then for all $t, s, \lambda \geq 0$ we have

$$\omega^*(f, L_2^{p;2q_1,2q_2}; t, s) \leq 2\{e^{2q_2 s^2}\omega_2(f, L_2^{p;2q_1,q_2}; t, 0) + \\ + e^{2q_1 t^2}\omega_2(f, L_2^{p;q_1,2q_2}; 0, s)\}, \tag{9}$$

$$\omega_2(f, L_2^{p;2q_1,q_2}; \lambda t, 0) \leq (\lambda+1)^2\{e^{2q_1\lambda^2 t^2}\omega_2(f, L_2^{p;q_1,q_2}; t, 0), \tag{10}$$

$$\omega_2(f, L_2^{p;q_1,2q_2}; 0, \lambda s) \leq (\lambda+1)^2\{e^{2q_2\lambda^2 s^2}\omega_2(f, L_2^{p;q_1,q_2}; 0, s). \tag{11}$$

1.2. The Picard singular integral of function f belonging to a fixed space $L_1^{p;q_1 q_2}$ we define by the formula

$$P_{r,\rho}(f; x, y) := \frac{1}{4r\rho} \iint_{\mathbb{R}^2} f(x+u, y+v) e^{-\frac{|u|}{r} - \frac{|v|}{\rho}}\, du dv, \\ (x, y) \in \mathbb{R}^2,\ r, \rho > 0. \tag{12}$$

The Gauss-Weierstrass singular integral for function f belonging to a given space $L_2^{p;q_1,q_2}$ we define by

$$W_{r,\rho}(f; x, y) := \frac{1}{\pi\sqrt{r\rho}} \iint_{\mathbb{R}^2} f(x+u, y+v) e^{-\frac{u^2}{r} - \frac{v^2}{\rho}}\, du dv, \\ (x, y) \in \mathbb{R}^2,\ r, \rho > 0. \tag{13}$$

It is easily verified that the integral $P_{r,\rho}(f)$ is well-defined for every $f \in L_1^{p;q_1,q_2}$ with $1 \le p \le \infty$ and $q_1, q_2 \in \mathbb{R}_0$, provided that $0 < r < 1/q_1$ and $0 < \rho < 1/q_2$. (In this paper we set $1/q = +\infty$ if $q = 0$).
Similarly, the integral $W_{r,\rho}(f)$ is well-defined for every $f \in L_2^{p;q_1,q_2}$, $1 \le p \le \infty$ and $q_1, q_2 \in \mathbb{R}_0$, provided that $0 < r < 1/(2q_1)$ and $0 < \rho < 1/(2q_2)$.

In this paper we shall denote by $M_{a,b}$ the suitable positive constants depending only on indicate parameters.

2. Auxiliary results.

By elementary calculations we can prove the following four lemmas:

Lemma 1. *For all $n \in \mathbb{N}_0$ and $y > 0$ we have*

$$\int_0^{+\infty} t^n e^{-yt}\, dt = n!\, y^{-n-1}, \tag{14}$$

$$\int_0^{\infty} t^n e^{-yt^2}\, dt = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{\pi}{y}} & \text{if} \quad n = 0, \\ \dfrac{(2k-1)!!}{2(2y)^k}\sqrt{\dfrac{\pi}{y}} & \text{if} \quad n = 2k \ge 2, \\ \dfrac{k!}{2y^{k+1}} & \text{if} \quad n = 2k+1, \end{cases} \tag{15}$$

where $(2k-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2k-1)$ for $k \in \mathbb{N}$.

Lemma 2. *Suppose that $f \in L_1^{p;q_1,q_2}$ with some $1 \le p \le \infty$, $q_1, q_2 \in \mathbb{R}_0$ and let r_1, ρ_1 be fixed numbers such that*

$$0 < r_1 < 1/q_1, \quad 0 < \rho_1 < 1/q_2\,. \tag{16}$$

The for all $r \in (0, r_1]$ and $\rho \in (0, \rho_1]$ we have

$$\left\| P_{r,\rho}(f, \cdot, \cdot) \right\|_{1;p;q_1,q_2} \le \left\| f \right\|_{1;p;q_1,q_2} A(r, \rho, q_1, q_2), \tag{17}$$

where

$$A(r, \rho, q_1, q_2) := \frac{1}{4r\rho} \iint_{\mathbb{R}^2} e^{|t|\left(q_1 - \frac{1}{r}\right) + |z|\left(q_2 - \frac{1}{\rho}\right)}\, dt dz \le \tag{18}$$

$$\le \frac{1}{(1 - q_1 r_1)(1 - q_2 \rho_1)}\,.$$

The formulas (12), (17) *and* (18) *show that $P_{r,\rho}$, $r \in (0, r_1]$ and $\rho \in (0, \rho_1]$, is a linear positive operator from the space $L_1^{p;q_1,q_2}$ into $L_1^{p;q_1,q_2}$.*

Lemma 3. *Suppose that $f \in L_2^{p;q_1,q_2}$ with some $1 \le p \le \infty$, $q_1, q_2 \in \mathbb{R}_0$, and let r_2, ρ_2 be fixed numbers such that*

$$0 < r_2 < 1/(2q_1), \quad 0 < \rho_2 < 1/(2q_2). \tag{19}$$

The for all $r \in (0, r_2]$ and $\rho \in (0, \rho_2]$ we have

$$\| W_{r,\rho}(f; \cdot, \cdot) \|_{2;p;2q_1,2q_2} \le \| f \|_{2;p;q_1,q_2} B(r, \rho, q_1, q_2),$$

where

$$B(r, \rho, q_1, q_2) := \frac{1}{\pi \sqrt{r\rho}} \iint_{\mathbb{R}^2} e^{t^2(2q_1 - \frac{1}{r}) + z^2(2q_2 - \frac{1}{\rho})} \, dt dz \le$$
$$\le \frac{1}{\sqrt{(1 - 2q_1 r_2)(1 - 2q_2 \rho_2)}} .$$

The above inequalities and (13) *prove that $W_{r,\rho}$, $r \in (0, r_2]$ and $\rho \in (0, \rho_2]$, is a linear positive operator from the space $L_2^{p;q_1,q_2}$ into $L_2^{p;2q_1,2q_2}$.*

Lemma 4. *Suppose that the assumptions of Lemma 2 are satisfied and $m, n \in \mathbb{N}_0$. Then for all $(x, y) \in \mathbb{R}^2$ and $r \in (0, r_1]$, $\rho \in (0, \rho_1]$ we have*

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} P_{r,\rho}(f; x, y) = (-1)^{m+n} r^{-m} \rho^{-n} P_{r,\rho}(f; x, y).$$

Using these results, we shall prove the next properties of $P_{r,\rho}$ and $W_{r,\rho}$.

Lemma 5. *Suppose that the assumptions of Lemma 3 are satisfied. Then for every $m, n \in \mathbb{N}_0$ there exist real numbers $A_{m,n;j,k}$, $0 \le j \le [m/2]$, $0 \le k \le [n/2]$, depending only on m, n, j, k such that*

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} W_{r,\rho}(f; x, y) = \tag{20}$$

$$= \sum_{j=0}^{[\frac{m}{2}]} \sum_{k=0}^{[\frac{n}{2}]} A_{m,n,j,k} r^{-m+j} \rho^{-n+k} W_{r,\rho}\big(f(t, z)(t - x)^{m-2j} (z - y)^{n-2k}; x, y \big),$$

for all $(x, y) \in \mathbb{R}_0^2$ and $r \in (0, r_2]$, $\rho \in (0, \rho_2]$, where $[x]$ denotes the integral part of $x \in \mathbb{R}$.

Proof. From (13) follows

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} W_{r,\rho}(f; x, y) =$$
$$= \frac{1}{\pi\sqrt{r\rho}} \iint_{\mathbb{R}^2} f(u, v) \left(\frac{\partial^m}{\partial x^m} e^{\frac{-(x-u)^2}{r}}\right) \left(\frac{\partial^n}{\partial y^n} e^{\frac{-(y-v)^2}{\rho}}\right) dudv.$$

But, using the mathematical induction, we can prove that for every $m \in \mathbb{N}_0$ there exist real numbers $a_{m,j}$, $0 \le j \le [m/2]$, depending only on m, j such that for all $x \in \mathbb{R}$ and $r = \text{const} > 0$ holds

$$\frac{d^m}{dx^m} e^{\frac{-x^2}{r}} = e^{\frac{-x^2}{r}} \sum_{j=0}^{[\frac{m}{2}]} a_{m,j} \left(-\frac{2}{r}\right)^j \left(-\frac{2x}{r}\right)^{m-2j}.$$

From the above we obtain

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} W_{r,\rho}(f; x, y) =$$
$$= \sum_{j=0}^{[\frac{m}{2}]} \sum_{k=0}^{[\frac{n}{2}]} A_{m,n,j,k} r^{-m+j} \rho^{-n+k} \frac{1}{\pi\sqrt{r\rho}} \iint_{\mathbb{R}^2} f(u, v)(x-u)^{m-2j}(y-v)^{n-2k} \cdot$$
$$\cdot e^{-\frac{(x-u)^2}{r} - \frac{(y-v)^2}{\rho}} dudv =$$
$$= \sum_{j=0}^{[\frac{m}{2}]} \sum_{k=0}^{[\frac{n}{2}]} A_{m,n,j,k} r^{-m+j} \rho^{-n+k} W_{r,\rho}\big(f(t, z)(t-x)^{m-2j}(z-y)^{n-2k}; x, y\big),$$

for $m, n \in \mathbb{N}_0$, $(x, y) \in \mathbb{R}^2$, $r \in (0, r_2]$ and $\rho \in (0, \rho_2]$.
Thus the proof of (20) is completed. □

Lemma 6. *Let* (x_0, y_0) *be a fixed point in* $\mathbb{R}^2$ *and let* $g(\cdot, \cdot; x_0, y_0)$ *be a given function belonging to some space* $L_1^{\infty; q_1, q_2}$, $q_1, q_2 \in \mathbb{N}_0$, *and*

$$\lim_{(t,z)\to(x_0,y_0)} g(t, z; x_0, y_0) = 0. \tag{21}$$

Then

$$\lim_{\substack{r\to 0^+\\ \rho\to 0^+}} P_{r,\rho}(g(t, z; x_0, y_0); x_0, y_0) = 0, \quad r \in (0, r_1],\ \rho \in (0, \rho_1], \tag{22}$$

where r_1 *and* ρ_1 *are given in Lemma 2.*

Proof. Choose $\varepsilon > 0$. By the properties of $g(\cdot\,,\cdot; x_0, y_0)$ there exist positive constants $\delta \equiv \delta(\varepsilon)$ and M such that

$$(23)\qquad \nu_{1,q_1,q_2}(t,z)\Big|g(t,z;x_0,y_0)\Big| < \\ < \frac{\varepsilon}{4}(1-q_1r_1)(1-q_2\rho_1) \text{ for } |t-x_0|<\delta, |z-y_0|<\delta,$$

$$(24)\qquad \nu_{1;q_1,q_2}(t,z)\Big|g(t,z;x_0,y_0)\Big| \le M \quad \text{for all } (t,z)\in\mathbb{R}^2.$$

From (12) we get

$$\nu_{1;q_1,q_2}(x_0,y_0)\big|P_{r,\rho}(g(t,z;x_0,y_0);x_0,y_0)\big| \le \\ \le \frac{\nu_{1;q_1,q_2}(x_0,y_0)}{4r\rho}\iint_{\mathbb{R}^2}\big|g(t,z;x_0,y_0)\big|e^{\frac{-|t-x_0|}{r}-\frac{|z-y_0|}{\rho}}\,dtdz = \\ = \frac{\nu_{1;q_1,q_2}(x_0,y_0)}{4r\rho}\Big(\iint_{|t-x_0|<\delta\ |z-y_0|<\delta}+\iint_{|t-x_0|<\delta\ |z-y_0|\ge\delta}+ \\ +\iint_{|t-x_0|\ge\delta\ |z-y_0|<\delta}+\iint_{|t-x_0|\ge\delta\ |z-y_0|\ge\delta}\Big)\big|g(t,z;x_0,y_0)\big|e^{\frac{-|t-x_0|}{r}-\frac{|z-y_0|}{\rho}}\,dtdz := \\ = I_1+I_2+I_3+I_4$$

for all $r\in(0,r_1]$, $\rho\in(0,\rho_1]$, where r_1,ρ_1 are given by (16). By (23) and Lemma 2 we have

$$I_1 < \frac{\varepsilon}{4}(1-q_1r_1)(1-q_2\rho_1)\,A(r,\rho,q_1,q_2) \le \frac{\varepsilon}{4}$$

and, by (1), (3) and (24), we have

$$I_2 \le \frac{Me^{-q_1|x_0|-q_2|y_0|}}{4r\rho}\left(\int_{-\infty}^{+\infty} e^{q_1|t|-\frac{|t-x_0|}{r}}\,dt\right)\left(\int_{|z-y_0|\ge\delta} e^{q_2|z|-\frac{|z-y_0|}{\rho}}\,dz\right).$$

But from $|z-y_0|\ge\delta$ follows $1\le\delta^{-2}(z-y_0)^2$ and further

$$I_2 \le \frac{M}{\delta^2 r\rho}\left(\int_0^{+\infty} e^{-t\left(\frac{1}{r}-q_1\right)}\,dt\right)\left(\int_0^{+\infty} z^2e^{-z\left(\frac{1}{\rho}-q_2\right)}\,dz\right),$$

which by (14) yields

$$I_2 \le \frac{2M}{\delta^2 r\rho} \frac{r}{1-rq_1} \frac{\rho^3}{(1-\rho q_2)^3} \le \frac{2M\rho^2}{\delta^2(1-r_1q_1)(1-\rho_1q_2)^3}, \qquad r \in (0, r_1],\ \rho \in (0, \rho_1].$$

Analogously,

$$I_3 \le \frac{2Mr^2}{\delta^2(1-r_1q_1)^3(1-\rho_1q_2)} .$$

Arguing similarly as in the case of the integral I_2 and using the inequality $1 \le \delta^{-4}(t-x_0)^2(z-y_0)^2$ for $|t-x_0| \ge \delta$, $|z-y_0| \ge \delta$, we get

$$I_4 \le \frac{M}{\delta^4 r\rho} \left(\int_0^{+\infty} t^2 e^{-t\left(\frac{1}{r}-q_1\right)}\, dt \right) \left(\int_0^{+\infty} z^2 e^{-z\left(\frac{1}{\rho}-q_2\right)}\, dz \right),$$

and further by (14) we obtain

$$I_4 \le \frac{4Mr^2\rho^2}{\delta^4(1-r_1q_1)^3(1-\rho_1q_2)^3} \quad \text{for} \quad r \in (0, r_1],\ \rho \in (0, \rho_1].$$

It is obvious that for fixed $\varepsilon, \delta, M > 0$ and $r_1\rho_1$ satisfying the conditions (16) there exist $r^* \in (0, r_1]$ and $\rho^* \in (0, \rho_1]$ such that

$$I_k < \frac{\varepsilon}{4} \quad \text{for all} \quad r \in (0, r^*),\ \rho \in (0, \rho^*) \text{ and } k = 1, 2, 3, 4.$$

Hence,

$$\nu_{1;q_1,q_2}(x_0, y_0)\left|P_{r,\rho}\big(g(t, z; x_0, y_0); x_0, y_0\big)\right| < \varepsilon$$

for all $r \in (0, r^*)$ and $\rho \in (0, \rho^*)$, which shows that

$$\lim_{\substack{r\to 0^+\\ \rho\to 0^+}} \nu_{1;q_1,q_2}(x_0, y_0) P_{r,\rho}\big(g(t, z; x_0, y_0); x_0, y_0\big) = 0.$$

From this and (3) follows the desired assertion (22) and we complete the proof. □

Similarly we can prove the following

Lemma 7. *Let* $(x_0, y_0) \in \mathbb{R}^2$ *be a fixed point and let* $g(\cdot\,,\cdot\,; x_0, y_0)$ *be a given function belonging to a given space* $L_2^{\infty;q_1,q_2}$, $q_1, q_2 \in \mathbb{R}_0$, *and satisfying the condition* (21). *Then*

$$\lim_{\substack{r\to 0^+\\ \rho\to 0^+}} W_{r,\rho}(g; x_0, y_0) = 0.$$

Applying Lemma 1, we easily derive from (12) and (13)

Lemma 8. *For all* $(x, y) \in \mathbb{R}^2$, $r, \rho > 0$ *and* $m, n \in \mathbb{N}_0$ *we have*

$$P_{r,\rho}(1; x, y) = 1 = W_{r,\rho}(1; x, y),$$

$$P_{r,\rho}\Big((t-x)^{2m}; x, y\Big) = (2m)!r^{2m}, \ W_{r,\rho}\Big((t-x)^{2m}; x, y\Big) = = (2m-1)!!\Big(\frac{r}{2}\Big)^m,$$

$$P_{r,\rho}\Big((z-y)^{2m}; x, y\Big) = (2m)!\rho^{2m}, \ W_{r,\rho}\Big((z-y)^{2m}; x, y\Big) = = (2m-1)!!\Big(\frac{\rho}{2}\Big)^m.$$

Moreover, if $m = 2k+1$ *and* $k, n \in \mathbb{N}_0$ *or* $n = 2k+1$ *and* $k, m \in \mathbb{N}_0$, *then*

$$P_{r,\rho}\Big((t-x)^m(z-y)^n; x, y\Big) = 0 = W_{r,\rho}\Big((t-x)^m(z-y)^n; x, y\Big).$$

3. Main theorems.

3.1. First we shall give two theorems on the order of approximation of function f by $P_{r,\rho}(f)$ and $W_{r,\rho}(f)$.

Theorem 1. *Suppose that* $f \in L_1^{p;q_1,q_2}$ *with some* $1 \le p \le \infty$, $q_1, q_2 \in \mathbb{R}_0$ *and let* r_1, ρ_1 *be fixed numbers satisfying the condition* (16). *Then there exists a positive constant* $M^* \le \frac{5}{2}(1-r_1q_1)^{-3} \cdot (1-\rho_1 q_2)^{-3}$ *such that*

$$\big\| P_{r,\rho}(f; \cdot, \cdot) - f(\cdot, \cdot)\big\|_{1;p;q_1,q_2} \le \le M^*\big\{\omega_2(f, L_1^{p;q_1,q_2}; r, 0) + \omega(f, L_1^{p;q_1,q_2}; 0, \rho)\big\} \tag{25}$$

for all $r \in (0, r_1]$ *and* $\rho \in (0, \rho_1]$.

Proof. From (12) and by (14) follows

$$P_{r,\rho}(f; x, y) - f(x, y) = \frac{1}{16r\rho}\iint_{\mathbb{R}^2} \Delta^*_{u,v} f(x, y) e^{\frac{-|u|}{r} - \frac{|v|}{\rho}}\, dudv,$$

for all $(x, y) \in \mathbb{R}^2$ and $r \in (0, r_1]$, $\rho \in (0, \rho_1]$, where $\Delta^*_{u,v} f(x, y)$ is defined in Sec. 1.1. From this and by Lemma 2 we get

$$\big\| P_{r,\rho}(f; \cdot, \cdot) - f(\cdot, \cdot)\big\| \le \frac{1}{16r\rho}\iint_{\mathbb{R}^2} \big\|\Delta^*_{u,v} f(\cdot, \cdot)\big\|_{1;p;q_1,q_2} e^{\frac{-|u|}{r} - \frac{|v|}{\rho}}\, dudv$$

and by (5) – (8),

$$\begin{aligned}\left\|\Delta^*_{u,v}f(\cdot\,,\cdot)\right\|_{1;q;q_1,q_2} &\le \omega^*(f, L_1^{p;q_1,q_2}; |u|, |v|) \le \\ &\le 2\left\{e^{q_2|v|}\omega_2(f, L_1^{p;q_1,q_2}; |u|, 0) + e^{q_1|u|}\omega_2(f, L_1^{p;q_1,q_2}; 0, |v|)\right\} \le \\ \le 2e^{q_1|u|+q_2|v|}&\left\{\left(\frac{|u|}{r}+1\right)^2\omega_2(f, L_1^{p;q_1,q_2}; r, 0) + \left(\frac{|v|}{\rho}+1\right)^2\omega_2(f, L_1^{p;q_1,q_2}; 0, \rho)\right\}\end{aligned}$$

for $r \in (0, r_1]$ and $\rho \in (0, \rho_1]$. Consequently,

$$\begin{aligned}&\left\|P_{r,\rho}(f; \cdot\,,\cdot) - f(\cdot\,,\cdot)\right\|_{1;p;q_1,q_2} \le \\ &\le \omega_2(f, L_1^{p;q_1,q_2}; r, 0)\frac{1}{2r\rho}\left(\int_0^\infty e^{-v\left(\frac{1}{\rho}-q_2\right)}\,dv\right)\left(\int_0^\infty \left(\frac{u}{r}+1\right)^2 e^{-u\left(\frac{1}{r}-q_1\right)}\,du\right)+ \\ &+\omega_2(f, L_1^{p;q_1,q_2}; 0, \rho)\frac{1}{2r\rho}\left(\int_0^\infty e^{-u\left(\frac{1}{r}-q_1\right)}du\right)\left(\int_0^\infty \left(\frac{v}{\rho}+1\right)^2 e^{-v\left(\frac{1}{\rho}-q_2\right)}dv\right).\end{aligned}$$

Now, using (14) and the inequalities $0 < 1/(1 - rq_1) \le 1/(1 - r_1q_1)$, $0 < 1/(1 - \rho q_2) \le 1/(1 - \rho_1 q_2)$ for $r \in (0, r_1]$, $\rho \in (0, \rho_1]$, we obtain

$$\begin{aligned}&\frac{1}{\rho}\int_0^\infty e^{-v\left(\frac{1}{\rho}-q_2\right)}\,dv \le \frac{1}{1-\rho_1 q_2}\,, \\ &\frac{1}{2r}\int_0^\infty \left(\frac{u}{r}+1\right)^2 e^{-u\left(\frac{1}{r}-q_1\right)^2}\,du \le \frac{5}{2}\frac{1}{(1-r_1q_1)^3}\,, \\ &\frac{1}{r}\int_0^\infty e^{-u\left(\frac{1}{r}-q_1\right)}\,du \le \frac{1}{1-r_1q_1}\,, \\ &\frac{1}{2\rho}\int_0^\infty \left(\frac{v}{\rho}+1\right)^2 e^{-v\left(\frac{1}{\rho}-q_2\right)^2}du \le \frac{5}{2}\frac{1}{(1-\rho_1 q_2)^3}\,,\end{aligned}$$

for all $0 < r \le r_1$ and $0 < \rho \le \rho_1$. Combining these, we obtain the desider assertion (25). □

Arguing as in the proof of Theorem 1 and using (2), (4), (5), (8) – (11), (13), (15) and Lemma 3, we can prove the following

Theorem 2. *Suppose that $f \in L_2^{p;q_1,q_2}$ with some $1 \le p \le \infty$, $q_1, q_2 \in \mathbb{R}_0$, and let r_2, ρ_2 be fixed numbers satisfying the conditions* (18). *Then there exists a* *positive constant* $M^* \equiv M_{p,q_1,q_2,r_2,\rho_2}$ *such that*

$$\left\| W_{r,\rho}(f; \cdot, \cdot) - f(\cdot, \cdot) \right\|_{2;p;2q_1-2q_2} \leq$$
$$\leq M^* \Big\{ \omega_2(f, L_2^{p;q_1,q_2}; \sqrt{r}, 0) + \omega_2(f, L_2^{p;q_1,q_2}; 0, \sqrt{\rho}) \Big\}$$

for all $r \in (0, r_2]$ *and* $\rho \in (0, \rho_2]$.

3.2. In this part we shall give the Voronovskaya theorem ([4]) for the singular integrals $P_{r,\rho}(f)$ and $W_{r,\rho}(f)$.

Theorem 3. *Let* f *be a function belonging to a given space* $L_1^{\infty;q_1,q_2}$, $q_1, q_2 \in \mathbb{R}_0$, *which the partial derivatives of the order* ≤ 2 *belong also to* $L_1^{\infty;q_1,q_2}$. *Then for every* $(x, y) \in \mathbb{R}^2$ *we have*

$$\lim_{r\to 0^+} r^{-2}\big\{P_{r,r}(f; x, y) - f(x, y)\big\} = f''_{xx}(x, y) + f''_{yy}(x, y). \tag{26}$$

Proof. Let (x_0, y_0) be a fixed point in $\mathbb{R}^2$. By the Taylor formula we have for every $(t, z) \in \mathbb{R}^2$

$$f(t, z) = f(x_0, y_0) + f'_x(x_0, y_0)(t - x_0) + f'_y(x_0, y_0)(z - y_0) + \tag{27}$$
$$+ \frac{1}{2}\big\{f''_{xx}(x_0, y_0)(t - x_0)^2 + 2f''_{xy}(x_0, y_0)(t - x_0)(z - y_0) +$$
$$+ f''_{yy}(x_0, y_0)(z - y_0)^2\big\} + \varphi(t, z; x_0, y_0)\sqrt{(t - x_0)^4 + (z - y_0)^4}\,,$$

where $\varphi(\cdot, \cdot; x_0, y_0) \in L_1^{\infty;q_1,q_2}$ and $\lim_{(t,z)\to(x_0,y_0)} \varphi(t, z; x_0, y_0) = 0$. From (27) we get

$$P_{r,r}(f(t, z); x_0, y_0) - f(x_0, y_0) = f'_x(x_0, y_0)P_{r,r}(t - x_0; x_0, y_0) +$$
$$+ f'y(x_0, y_0)P_{r,r}(z - y_0; x_0, y_0) + \frac{1}{2}\big\{f''_{xx}(x_0, y_0)P_{r,r}((t - x_0)^2; x_0, y_0) +$$
$$+2f''_{xy}(x_0, y_0)P_{r,r}((t-x_0)(z-y_0); x_0, y_0)+f''_{yy}(x_0, y_0)P_{r,r}((z-y_0)^2; x_0, y_0)\big\}+$$
$$+ P_{r,r}(\varphi(t, z; x_0, y_0)\sqrt{(t - x_0)^4 + (z - y_0)^4}; x_0, y_0)$$

and further by Lemma 8 follows

$$P_{r,r}(f(t, z); x_0, y_0) - f(x_0, y_0) = r^2\big\{f''_{xx}(x_0, y_0) + \tag{28}$$
$$+ f''_{yy}(x_0, y_0)\big\} + P_{r,r}(\varphi(t, z; x_0, y_0)\sqrt{(t - x_0)^4 + (z - y_0)^4}; x_0, y_0),$$

for $0 < r < \min(1/q_1, 1/q_2)$. Using the Hölder inequality, we can write

$$\left|P_{r,r}(\varphi(t,z;x_0,y_0)\sqrt{(t-x_0)^4+(z-y_0)^4};x_0,y_0)\right| \le$$
$$\le \left\{P_{r,r}(\varphi^2(t,z;x_0,y_0);x_0,y_0)\right\}^{1/2}\left\{P_{r,r}((t-x_0)^4+(z-y_0)^4;x_0,y_0)\right\}^{1/2}$$

for $0 < r < \min(1/(2q_1), 1/(2q_2))$. Moreover $\varphi^2(\cdot,\cdot;x_0,y_0) \in L_1^{\infty;2q_1,2q_2}$ and $\lim\limits_{(t,z)\to(x_0,y_0)} \varphi^2(t,z;x_0,y_0) = 0$. Hence, using Lemma 6, we get,

$$\lim_{r\to 0^+} P_{r,r}\left(\varphi^2(t,z;x_0,y_0);x_0,y_0\right) = 0.$$

By (12) and Lemma 8 follows

$$P_{r,r}\left((t-x_0)^4+(z-y_0)^4;x_0,y_0\right) = 2\cdot 4!r^4 \quad \text{for } r > 0.$$

Consequently,

$$\lim_{r\to 0^+} r^{-2}P_{r,r}\left(\varphi(t,z;x_0,y_0)\sqrt{(t-x_0)^4+(z-y_0)^4};x_0,y_0\right) = 0. \tag{29}$$

Next, using (29) to (28), we obtain (26) in the point (x_0, y_0). Thus the proof is completed. □

Analogously, using (27), Lemma 7 and Lemma 8, we can prove the following

Theorem 4. *Suppose that f is a function belonging to a given space $L_2^{\infty;q_1,q_2}$, $q_1, q_2 \in \mathbb{R}_0$, which the partial derivatives of the order ≤ 2 belong also to $L_2^{\infty;q_1,q_2}$. Then for every $(x,y) \in \mathbb{R}^2$ we have*

$$\lim_{r\to 0^+} r^{-1}\left\{W_{r,r}(f;x,y) - f(x,y)\right\} = \frac{1}{4}\left\{f''_{xx}(x,y) + f''_{yy}(x,y)\right\}.$$

3.3. Finaly we shall prove the Bernstein inequality for the singular integrals $P_{r,\rho}$ and $W_{r,\rho}$.

Theorem 5. *Suppose that the assumptions of Theorem 1 are satisfied. Then, for all $m, n \in \mathbb{N}_0$, $r \in (0, r_1)]$ and $\rho \in (0, \rho_1]$, we have*

$$\left\|\frac{\partial^{m+n}}{\partial x^m \partial y^n} P_{r,\rho}(f;x,y)\right\|_{1;p;q_1,q_2} \le \frac{r^{-m}\rho^{-n}}{(1-q_1r_1)(1-q_2\rho_1)}\|f\|_{1;q;q_1,q_2}. \tag{30}$$

Proof. Using Lemma 4 and Lemma 2, we immediately obtain (30). □

Theorem 6. *Suppose that the assumptions of Theorem 2 are satisfied. Then, for every fixed $m, n \in \mathbb{N}_0$, there exists a positive constant M^* depending only on $m, n, q_1, q_2, r_2, \rho_2$ such that*

$$\left\| \frac{\partial^{m+n}}{\partial x^m \partial y^n} W_{r,\rho}(f; x, y) \right\|_{2;p;2q_1,2q_2} \leq M^* r^{\frac{-m}{2}} \rho^{\frac{-n}{2}} \|f\|_{2;p;q_1,q_2}, \tag{31}$$

for all $r \in (0, r_2]$ and $\rho \in (0, \rho_2]$.

Proof. The inequality (31) for $m = n = 0$ is given in Lemma 3. If $m^2+n^2 > 0$, then by (20) we get

$$\left\| \frac{\partial^{m+n}}{\partial x^m \partial y^n} W_{r,\rho}(f; x, y) \right\|_{2;p;2q_1,2q_2} \leq$$

$$\leq M_{m,n} \sum_{j=0}^{[\frac{m}{2}]} \sum_{k=0}^{[\frac{n}{2}]} r^{j-m} \rho^{k-n} \left\| W_{r;\rho}\big(f(t, z)(t - \cdot)^{m-2j}(z - \cdot)^{n-2k}; \cdot, \cdot\big) \right\|_{2;p;2q_1,2q_2}$$

and by (5), (2) and (4),

$$\|W_{r,\rho}(f(t, z)(t - \cdot)^{m-2j}(z - \cdot)^{n-2k}; \cdot, \cdot)\|_{2;p;2q_1,2q_2} \leq$$

$$\leq \|f\|_{2;p;q_1,q_2} \frac{1}{\sqrt{\pi r}\sqrt{\pi \rho}} \iint_{\mathbb{R}^2} |t|^{m-2j} |z|^{n-2k} e^{-t^2(\frac{1}{r}-2q_1)-z^2(\frac{1}{\rho}-2q_2)} \, dt dz :=$$

$$= \|f\|_{2;p;q_1,q_2} \frac{4}{\sqrt{\pi r}\sqrt{\pi \rho}} I_{m-2j}(r, q_1) I_{n-2k}(\rho, q_2),$$

where

$$I_{m-2j}(r, q_1) = \int_0^{\infty} t^{m-2j} e^{-t^2\left(\frac{1}{r}-2q_1\right)} \, dt,$$

$$I_{n-2k}(\rho, q_2) = \int_0^{\infty} z^{n-2k} e^{-z^2\left(\frac{1}{\rho}-2q_2\right)} \, dz.$$

Consequently,

$$\left\| \frac{\partial^{m+n}}{\partial x^m \partial y^n} W_{r,\rho}(f; x, y) \right\|_{2;p;2q_1,2q_2} \leq$$

$$\leq M_{m,n} \|f\|_{2;p;q_1,q_2} \left\{ \sum_{j=0}^{[\frac{m}{2}]} r^{j-m-\frac{1}{2}} I_{m-2j}(r, q_1) \right\} \left\{ \sum_{k=0}^{[\frac{n}{2}]} r^{k-n-\frac{1}{2}} I_{n-2k}(\rho, q_2) \right\}$$

for $r \in (0, r_2]$ and $\rho \in (0, \rho_2]$. Using (15) we get for $r \in (0, r_2]$

$$\sum_{j=0}^{[\frac{m}{2}]} r^{j-m-\frac{1}{2}} I_{m-2j}(r, q_1) \leq M_s \sum_{j=0}^{s} r^{j-2s-\frac{1}{2}} \frac{r^{s-j+\frac{1}{2}}}{(1-2r_2q_1)^{s-j+\frac{1}{2}}} \leq$$

$$\leq M_{s,q_1,r_2} r^{-s} = M_{m,q_1,r_2} r^{-\frac{m}{2}} \text{ if } m = 2s,$$

$$\sum_{j=0}^{[\frac{m}{2}]} r^{j-m-\frac{1}{2}} I_{m-2j}(r, \rho_1) \leq M_m \sum_{j=0}^{s} r^{j-2s-\frac{3}{2}} \frac{r^{s-j+1}}{(1-2r_2q_1)^{s-j+1}} \leq$$

$$\leq M_{m,q_1,r_2} r^{-s-\frac{1}{2}} = M_{m,q_1,r_2} r^{-\frac{m}{2}} \text{ if } m = 2s+1.$$

Analogously we obtain

$$\sum_{k=0}^{[\frac{n}{2}]} \rho^{k-n-\frac{1}{2}} I_{n-2k}(\rho, q_2) \leq M_{n,q_2,\rho_2} \rho^{-\frac{n}{2}} \text{ for } n \in \mathbb{N}_0,\ \rho \in (0, \rho_2].$$

Summing up, we obtain the Bernstein inequality (31). □

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