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0-DIMENSIONAL SUBSCHEMES OF CURVES LYING ON A SMOOTH QUADRIC SURFACE

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We characterize all the possible Hilbert functions for 0-dimensional subschemes of an irreducible curve C lying on a smooth quadric surface $Q \subset \mathbb{P}^3_k$.

Introduction.

The three-codimensional projective schemes and, in particular, the 0-dimensional subschemes of $\mathbb{P}^3$, represent a subject of the algebraic geometry where very little is still known. Specifically we consider the following general problem: how do some properties of the irreducible curves of $\mathbb{P}^3$ affect the Hilbert function of their 0-dimensional subschemes? So one can consider curves whose minimal surface has fixed degree, complete intersection curves, arithmetically Cohen-Macaulay curves, arithmetically Buchsbaum curves etc..

In this paper we answer one of these questions: we characterize all the possible Hilbert function of the 0-dimensional subschemes of an irreducible curve lying on a smooth quadric, just using the type (a, b) of the curve. The postulation of 0-dimensional schemes on a smooth quadric have been extensively studied in [3] and in [5].

More precisely let Q be a smooth quadric. In [7] the author proves that if $X \subset Q$ is a 0-dimensional scheme contained in a complete intersection

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$(2, b, s)$, $2 \leq b \leq s$, for all $n \in \mathbb{N}$ such that $n \geq s$ and $\Delta H(X, n) > 1$ then $\Delta^2 H(X, n+1) \leq -2$.

Here we prove that if C is an irreducible curve of type (a, b) with $a \leq b$, lying on a smooth quadric surface and $\{v_n\}$ is a O-sequence, definitively zero, such that $v_n \leq \Delta H(C, n)$, $s = \min\{n \in \mathbb{N} \mid v_n < \Delta H(C, n)\} \geq b$ and $\Delta v_{n+1} \leq -2$ for all $n \geq s$ such that $v_n > 1$, then on C there is a 0-dimensional scheme X, such that $\Delta H(X, n) = v_n$.

In Section 1 we fix notation and preliminaries, in the second section we give the general form of the Hilbert function for a 0-dimensional subscheme of a curve of type (a, b). The third section forms the technical heart of this paper. In particular we explain how one can construct a 0-dimensional scheme X, with assigned Hilbert function, on some lines of a smooth quadric surface Q, in a way such that it is possible to embed X in a suitable curve of Q. This result will be very useful to prove the theorem in Section 4.

1. Notation and preliminaries.

Throughout this paper Q will denote a smooth quadric surface in $\mathbb{P}^3_k$, k algebraically closed field. Let $C \subset Q$ be a curve of type (a, b), $a \leq b$. We set $c = b - a$ and $d = a + b = \deg C$. Let $R = k[x, y, z, w]$ be the polynomial ring over k in four indeterminates and I_C the saturated homogeneous ideal of C in R.

It is known that a minimal free resolution of R/I_C as R-module is:

if $c = 0$

$$0 \to R(-b-2) \to R(-b) \oplus R(-2) \to R \to R/I_C \to 0;$$

if $c = 1$

$$0 \to R(-b-1)^2 \to R(-b)^2 \oplus R(-2) \to R \to R/I_C \to 0;$$

if $c \geq 2$

$$0 \to R(-b-2)^{c-1} \to R(-b-1)^{2c} \to R(-b)^{c+1} \oplus R(-2) \to R \to R/I_C \to 0;$$

(see [4]).

If $\phi : \mathbb{N}_0 \to \mathbb{Z}$ is a function, the *first difference* of ϕ is $\Delta\phi(0) = \phi(0)$, $\Delta\phi(n) = \phi(n) - \phi(n-1)$ if $n \geq 1$, and recursively the *n-th difference* is $\Delta^n\phi(0) = \Delta^{n-1}\phi(0)$, $\Delta^n\phi(n) = \Delta^{n-1}\phi(n) - \Delta^{n-1}\phi(n-1)$ if $n \geq 1$.

The above resolutions allows us to compute the Hilbert function of C. In particular we get

n	0	1	2	$\cdots$	$b-1$	b	0	$\cdots$
$\Delta^2 H(C,n)$	1	2	2	$\cdots$	2	$1-c$	0	$\cdots$

If M is a graded R-module, we write M_n for the degree n piece of M and if M is a finite length graded R-module we define *diameter* of M (write diam M) the number of the nonzero graded pieces of M.

If $\mathscr{I}_C$ is the ideal sheaf of C, we recall that the module

$$M_C = \bigoplus_{n\in\mathbb{Z}} H^1(\mathscr{I}_C(n))$$

is called the Hartshorne-Rao module (or simply the Rao module) of C. M_C is a finite length R-module and $M_C = 0$ if and only if C is an arithmetically Cohen-Macaulay curve if and only if $c \leq 1$. Moreover if $c \geq 2$ and $\rho(n) = \dim_k M_{C,n}$ we have that

$$\rho(n) = \begin{cases} (b-n-1)(n-a+1) & \text{for } a \leq n \leq b-2 \\ 0 & \text{for } n \leq a-1 \text{ or } n \geq b-1 \end{cases}$$

(see [2]), consequently diam $M_C = c - 1$ and we can obtain the first and the second difference of $\rho(n)$:

n	0	$\cdots$	$a-1$	a	$a+1$	$\cdots$	$b-1$	b	$b+1$	$\cdots$
$\Delta\rho(n)$	0	$\cdots$	0	$c-1$	$c-3$	$\cdots$	$1-c$	0	0	$\cdots$
$\Delta^2\rho(n)$	0	$\cdots$	0	$c-1$	-2	$\cdots$	-2	$c-1$	0	$\cdots$

2. Hilbert function of the subschemes of C.

In this section we found the general form of the first difference of the Hilbert function of a 0-dimensional scheme of a curve of type (a, b) on a smooth quadric, such that

$$s = \min\left\{n \in \mathbb{N} \mid \Delta H(X,n) < \Delta H(C,n)\right\} \geq b.$$

Let $C \subset \mathbb{P}^3_k$ be a curve, $\mathscr{I}_C$ its ideal sheaf, $S \subset \mathbb{P}^3_k$ a surface such that no irreducible component of C is contained in S. Then $X = C \cap S$ is a 0-dimensional scheme.

Let $F = 0$, $F \in R_s$, be the equation of S; we have for any n the following short exact sequence of sheaves:

$$0 \to \mathscr{I}_C(n-s) \xrightarrow{F} \mathscr{I}_C(n) \to \mathscr{J}_X(n) \to 0,$$

where the first map is the multiplication by F and $\mathscr{J}_X$ is its cokernel: of course it is the ideal sheaf of X as a subscheme of S. It induces a long exact sequence on the cohomological groups

$$0 \to H^0(\mathscr{I}_C(n-s)) \to H^0(\mathscr{I}_C(n)) \to H^0(\mathscr{J}_X(n)) \to$$
$$\to H^1(\mathscr{I}_C(n-s)) \to H^1(\mathscr{I}_C(n)) \to H^1(\mathscr{J}_X(n)) \to \cdots$$

or, if $J_X = \bigoplus_n H^0(\mathscr{J}_X(n))$, the exact sequence of graded R-modules:

$$0 \to I_C(-s) \xrightarrow{F} I_C \to J_X \to M_C(-s) \xrightarrow{F} M_C \to \cdots .$$

and if K is the kernel of the map:

$$M_C(-s) \xrightarrow{F} M_C$$

we have the exact sequence of graded R-modules:

$$(*) \qquad 0 \to I_C(-s) \xrightarrow{F} I_C \to J_X \to K \to 0.$$

From the following exact sequence:

$$0 \to \mathscr{I}_S(n) \to \mathscr{I}_X(n) \to \mathscr{J}_X(n) \to 0,$$

since $H^1(\mathscr{I}_S(n)) = 0$ for any n, we obtain the exact sequence of graded R-modules:

$$0 \to I_S \to I_X \to J_X \to 0,$$

hence

$$\dim_k J_{X,n} = \dim_k I_{X,n} - \dim_k I_{S,n} = \dim_k I_{X,n} - \dim_k R_{n-s}$$

and from $(*)$ we get the following relation between the Hilbert function of C and the Hilbert function of X:

$$H(X,n) = H(C,n) - H(C,n-s) - \dim_k K_n.$$

Let us consider now a curve $C \subset Q$ of type (a, b), $a \leq b$ and a surface S, such that $s \geq b$. Since $s \geq b > c - 1 = \operatorname{diam} M_C$ we have

$$H(X, n) = H(C, n) - H(C, n - s) - \rho(n - s)$$

(we use the convention that $H(C, n) = 0$ for $n \leq -1$), and consequently

$$\Delta H(X, n) = \Delta H(C, n) - \Delta H(C, n - s) - \Delta\rho(n - s);$$

so for $n \leq s + a - 1$

$$\Delta H(X, n) = \Delta H(C, n) - \Delta H(C, n - s).$$

Moreover

$$\begin{aligned}\Delta H(X, s + a) &= \Delta H(C, s + a) - \Delta H(C, a) - c + 1\\ &= a + b - (2a + 1) - c + 1 = 0,\end{aligned}$$

and, since $\Delta H(X, n)$ is a O-sequence, $\Delta H(X, n) = 0$ for $n \geq s + a$. Finally we obtain:

$$\Delta H(X, n) = \begin{cases} \Delta H(C, n) - \Delta H(C, n - s) & \text{for } n \leq s + a - 1 \\ 0 & \text{for } n \geq s + a. \end{cases}$$

Let $X \subset C$ be a 0-dimensional scheme. We define

$$s = \min\left\{n \in \mathbb{N} \mid \Delta H(X, n) < \Delta H(C, n)\right\}$$

and we assume that $s \geq b$. Let S be a surface, $\deg S = s$, such that $X \subset S$ but no irreducible component of C is contained in S. Since $X \subset S \cap C$, $\Delta H(X, n) \leq \Delta H(S \cap C, n)$, i.e. the last part of the first difference of the Hilbert function of X is

n	$b-1$	b	$\cdots$	$s-1$	s	$\cdots$	$s+e-1$	$s+e$	$\cdots$
$\Delta H(X, n)$	$2b - 1$	d	$\cdots$	d	h_1	$\cdots$	h_e	0	$\cdots$

$d - 1 \geq h_1$, $h_e \geq 1$, $0 \leq e \leq a$ and ([7] Theorem 2.2) $h_n - h_{n+1} \geq 2$ for $1 \leq n \leq e - 1$.

3. Technical lemmas.

In this section we prove three lemmas, which will be useful in the proof of the main theorem.

Lemma 3.1. *Let E be a scheme of t skew lines $L_1, \dots, L_t$ lying on Q. Let $X \subset E$ be a finite set of points such that $|X \cap L_i| = l_i \geq t$ for $i = 1, \dots, t$. Then the Hilbert function of X depends only on the number of the points arranged on each line.*

Proof. Of course, we can assume that

$$l_1 \geq l_2 \geq \dots \geq l_t.$$

We want to compute $H(X, n)$ for every integer n. We distinguish three cases: $1 \leq n \leq l_t - 1$ or $l_t \leq n \leq l_1 - 1$ or $n \geq l_1$.

If $1 \leq n \leq l_t - 1$ we have that $I_{X,n} = I_{E,n}$, so $H(X, n) = H(E, n)$.

If $l_t \leq n \leq l_1 - 1$ we set $\beta = \max\{i \mid 1 \leq i \leq t, l_i \geq n+1\}$. We want to prove that $H(X, n) = H(L_1 \cup \dots \cup L_\beta, n) + \sum_{i=\beta+1}^{t} l_i$. We call $P_{\beta+1,1}, \dots, P_{\beta+1,l_{\beta+1}}, \dots, P_{t,1}, \dots, P_{t,l_t}$ the points of X such that $P_{i,j} \in L_i$ for $\beta + 1 \leq i \leq t$ and $1 \leq j \leq l_i$. We have to prove that these points impose independent conditions to the linear system of the surfaces of degree n containing the lines $L_1, \dots, L_\beta$. To do this it is enough to prove that there is a surface S of degree $\leq n$ containing the lines $L_1, \dots, L_\beta$, passing through the points $P_{\beta+1,1}, \dots, P_{\beta+1,l_{\beta+1}}, \dots, P_{\gamma,1}, \dots, P_{\gamma,k}$, $\beta + 1 \leq \gamma \leq t$, $1 \leq k \leq l_\gamma$ but not through $P_{\gamma,k+1}$ (we use the convention that if $\gamma \leq t-1$ then $P_{\gamma,l_\gamma+1} = P_{\gamma+1,1}$).

If $\gamma - 1 \geq k$ we take the plane Π_i identified by the line L_i and by the point $P_{\gamma,i}$ for $1 \leq i \leq k$, and we take a plane Π_i passing through the line L_i but not through $P_{\gamma,k+1}$ for $k+1 \leq i \leq \gamma - 1$. We set $S = \Pi_1 \cup \dots \cup \Pi_{\gamma-1}$.

If $\gamma - 1 < k$ we take the plane Π_i identified by the line L_i and by the point $P_{\gamma,i}$ for $1 \leq i \leq \gamma - 1$ and a plane Π_i passing through $P_{\gamma,i}$ but not through $P_{\gamma,k+1}$ for $\gamma \leq i \leq k$. We set $S = \Pi_1 \cup \dots \cup \Pi_k$.

In both cases $\deg S = \max\{\gamma - 1, k\} \leq n$.

If $n \geq l_1$ then similar arguments show that $H(X, n) = \sum_{i=1}^{t} l_i$. □

We call Σ_1 and Σ_2 the two rulings of lines lying on Q.

Lemma 3.2. *Let E be a scheme of t lines $L_1, \dots, L_t$ of Σ_1 and $v \leq t$ lines $M_1, \dots, M_v$ of Σ_2. Let $X \subset E$ be a finite set of points such that $|X \cap L_i \cap M_j| = \emptyset$, $\forall i, j$, and $l_i \geq t + m$, for $i = 1, \dots, t$, where $l_i = |X \cap L_i|$, for $i = 1, \dots, t$, $m_i = |X \cap M_i|$ for $i = 1, \dots, v$ and $m = \max\{m_i \mid 1 \leq i \leq v\}$, $m_i \geq 1$. Then the Hilbert function of X depends only on the numbers of the points arranged on each line.*

Proof. We can assume that

$$l_1 \geq \ldots \geq l_t$$

and

$$m_1 \geq \ldots \geq m_v.$$

If $1 \leq n \leq t + m_v - 1$ then $H(X, n) = H(E, n)$. In fact let f be $\in I_{X,n}$; since on L_i there are $l_i \geq t + m_1 \geq t + m_v \geq n + 1$ points, $f \in I_{L_1 \cup \ldots \cup L_t, n}$ and since the points are different from the intersection points between the lines L_i and M_j, f is zero on $t + m_j \geq n + 1$ points on M_j, so f is zero on all M_j, i.e. $f \in I_{E,n}$.

If $t + m_v \leq n \leq t + m_1 - 1$, $(m_1 > m_v)$, denote $\gamma = \max\{j \mid 1 \leq j \leq v,\, m_j + t \geq n + 1\}$, then $H(X, n) = H(L_1 \cup \ldots \cup L_t \cup M_1 \cup \ldots \cup M_\gamma, n) + \sum_{i=\gamma+1}^{v} m_i$. In fact, as in the previous case, $I_{L_1 \cup \ldots \cup L_t \cup M_1 \cup \ldots \cup M_\gamma, n} \subset I_{X,n}$, so it is enough to prove that the points lying on $M_{\gamma+1}, \ldots, M_v$ impose independent conditions to the linear system of the surfaces of degree n containing the lines $L_1, \ldots, L_t, M_1, \ldots, M_\gamma$. We denote these points $Q_{\gamma+1,1}, \ldots, Q_{\gamma+1,m_{\gamma+1}}, \ldots, Q_{v,1}, \ldots, Q_{v,m_v}$. To do this we prove that there is a surface of degree $\leq n$, containing the lines $L_1, \ldots, L_t, M_1, \ldots, M_\gamma$, passing through

$$Q_{\gamma+1,1}, \ldots, Q_{\gamma+1,m_{\gamma+1}}, \ldots, Q_{\sigma,1}, \ldots, Q_{\sigma,k}$$

but not through $Q_{\sigma,k+1}$, $\gamma + 1 \leq \sigma \leq v$, $1 \leq k \leq m_\sigma$ (we use again the convention that if $\sigma \leq v-1$ then $Q_{\sigma,m_\sigma+1} = Q_{\sigma+1,1}$). Let us consider the planes Π_i containing the lines L_i and M_i, $1 \leq i \leq \sigma - 1$; Π_i passing through L_i but not containing $Q_{\sigma,k+1}$, $\sigma \leq i \leq t$; Π_{t+i} passing through $Q_{\sigma,i}$ but not through $Q_{\sigma,k+1}$, $1 \leq i \leq k$. We have that $\deg(\Pi_1 \cup \ldots \cup \Pi_{t+k}) = t + k \leq t + m_\sigma \leq n$.

If $t + m_1 \leq n \leq l_1 - 1$, $(l_1 > t + m_1)$, called $\beta = \max\{i \mid 1 \leq i \leq t,\, l_i \geq n + 1$, then $H(X, n) = H(L_1 \cup \ldots \cup L_\beta, n) + \sum_{i=\beta+1}^{t} l_i + \sum_{j=1}^{v} m_j$. In fact as in the previous cases we have only to prove that the points

$$P_{\beta+1,1}, \ldots, P_{\beta+1,l_{\beta+1}}, \ldots, P_{t,1}, \ldots, P_{t,l_t},$$

$$Q_{1,1}, \ldots, Q_{1,m_1}, \ldots, Q_{v,1}, \ldots, Q_{v,m_v}$$

$P_{i,j} \in L_i$ and $Q_{i,j} \in M_i$, impose independent conditions to the linear system of the surfaces of degree n containing the lines $L_1, \ldots, L_\beta$. For the points $P_{i,j}$ it is the same proof of the Lemma 1. So it is enough to prove that there is a surface of degree $\leq n$ containing the lines $L_1, \ldots, L_\beta$, passing through the points

$$P_{\beta+1,1}, \ldots, P_{\beta+1,l_{\beta+1}}, \ldots, P_{t,1}, \ldots, P_{t,l_t},$$

$$Q_{1,1}, \ldots, Q_{1,m_1}, \ldots, Q_{\sigma,1}, \ldots, Q_{\sigma,k}$$

$1 \le \sigma \le v$, $1 \le k \le m_\sigma$ and not through $Q_{\sigma,k+1}$ (with the usual convention). Let us consider the planes Π_i passing through L_i and M_i, $1 \le i \le \sigma - 1$; Π_i passing through the lines L_i but not containing $Q_{\sigma,k+1}$, $\sigma \le i \le t$; Π_{t+i} passing through $Q_{\sigma,1}, \dots, Q_{\sigma,k}$, but not through $Q_{\sigma,k+1}$, $1 \le i \le k$. We have that $\deg(\Pi_1 \cup \dots \cup \Pi_{t+k}) = t + k \le t + m_\sigma \le n$.

If $n \ge l_1$ then similar arguments show that $H(X, n) = \deg X$. $\square$

In [8] it is proved that for any O-sequence satisfying a "decreasing type" condition there is a 0-dimensional subscheme of a smooth quadric with this O-sequence as its Hilbert function. Unfortunately the construction given in [8] cannot be used for our purpose, so we need a construction "ad hoc". The following technical lemma gives this construction.

Lemma 3.3. *Let* $\{v_n\}$ *be a O-sequence of the type*

n	0	1	$\cdots$	$t-1$	t	$t+1$	$\cdots$	$t+u-1$	$t+u$	$\cdots$
v_n	1	3	$\cdots$	$2t-1$	k_1	k_2	$\cdots$	k_u	0	$\cdots$

$2t \ge k_1 \ge k_2 \ge \dots \ge k_u > 0$, $u \ge 1$.

If $\delta = \max\{t, k_1\}$, *then there is a* 0*-dimensional subscheme* X, *lying on* δ *lines of* Q, *such that* $\Delta H(X, n) = v_n$, *for all* $n \in \mathbb{N}_0$ *and* $\Delta H(X, n)$ *depends only on the number of the points arranged on each line.*

Proof. We rename the integers $k_1, \dots, k_u$ and settle them in the following way:

$$k_1 = h_{q,\alpha_q} = \dots = h_{q,1} > h_{q-1,\alpha_{q-1}} = \dots = h_{q-1,1} > \dots$$

$$\dots > h_{1,\alpha_1} = \dots = h_{1,1} = k_u.$$

We set $\beta_i = h_{i,1}$; $s = t + u$; $r = \sum_{i=0}^{\infty} v_i = t^2 + \sum_{i=1}^{q} \alpha_i \beta_i$.

We have that $\beta_1 = k_u$, $\beta_q = k_1$ and $u = \sum_{i=1}^{q} \alpha_i$.

Let $Q \subset \mathbb{P}^3_k$ be a smooth quadric and Σ_1 and Σ_2 the two rulings of lines on Q. We have two cases.

First case: $0 < k_1 \le t$. We arrange r points on any t lines of Σ_1, in the following way:

on	β_1	lines we take any	s	points on each
"	$\beta_2 - \beta_1$	"	$s - \alpha_1$	"
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
"	$\beta_q - \beta_{q-1}$	"	$s - \sum_{i=1}^{q-1} \alpha_i$	"
"	$t - \beta_q$	"	$s - \sum_{i=1}^{q} \alpha_i$	"

So the number of points is

$$\beta_1 s + \sum_{i=2}^{q}(\beta_i - \beta_{i-1})(s - \sum_{j=1}^{i-1}\alpha_j) + t(t - \beta_q)$$

$$= \beta_1 s + \sum_{i=2}^{q}\beta_i(s - \sum_{j=1}^{i-1}\alpha_j) - \sum_{i=1}^{q-1}\beta_i(s - \sum_{j=1}^{i}\alpha_j) + t^2 - t\beta_q$$

$$= \beta_1 s + \beta_q s - \beta_1 s + \beta_1\alpha_1 - \beta_q\sum_{j=1}^{q-1}\alpha_j + \sum_{i=2}^{q-1}\beta_i\alpha_i + t^2 - t\beta_q$$

$$= \beta_q u + t^2 + \sum_{i=1}^{q-1}\beta_i\alpha_i - \beta_q\sum_{j=1}^{q-1}\alpha_j$$

$$= \beta_q\alpha_q + t^2 + \sum_{i=1}^{q-1}\beta_i\alpha_i = r.$$

We have to verify that the first difference of the Hilbert function of this set of points X is equal to v_n for all $n \in \mathbb{N}$. We write E_i for a scheme made up by i skew lines on Q. The hypotheses of the Lemma 3.1 are satisfied. So for computing $H(X, n)$ we have to calculate the number γ_n of the lines on which we put more than n points and the number ζ_n of the points that lie on the other $t - \gamma_n$ lines. If $1 \le n \le t - 1$, $\gamma_n = t$, so $\Delta H(X, n) = \Delta H(E_t, n) = v_n$.

If $t \le n \le s - 1$ then we can write $n = t + \sum_{i=m+1}^{q}\alpha_i + k$, $1 \le m \le q$, $0 \le k \le \alpha_m - 1$, so $\gamma_n = \beta_m$. If $k \ge 1$ then also $\gamma_{n-1} = \beta_m$, so, by the Lemma 3.1, $\Delta H(X, n) = H(X, n) - H(X, n-1) = H(E_{\beta_m}, n) + \zeta_n - H(E_{\beta_m}, n-1) - \zeta_n = \Delta H(E_{\beta_m}, n) = \beta_m = v_n$; if $k = 0$, $\gamma_{n-1} = \beta_{m+1}$, so $\Delta H(X, n) = H(X, n) - H(X, n-1) = H(E_{\beta_m}, n) + \zeta_n - H(E_{\beta_{m+1}}, n-1) - \zeta_{n-1} = \beta_m^2 + (n - \beta_m + 1)\beta_m + \sum_{i=m}^{q}(\beta_{i+1} - \beta_i)(s - \sum_{j=1}^{i}\alpha_j) - \beta_{m+1}^2 + (n - 1 - \beta_{m+1} + 1)\beta_{m+1} - \sum_{i=m+1}^{q}(\beta_{i+1} - \beta_i)(s - \sum_{j=1}^{i}\alpha_j) = n\beta_m + \beta_m - n\beta_{m+1} + (\beta_{m+1} - \beta_m)(t + \sum_{j=m+1}^{q}\alpha_j) = n(\beta_m - \beta_{m+1}) + \beta_m + (\beta_{m+1} - \beta_m)n = \beta_m = v_n$.

If $n \ge s$ then $\gamma_n = 0$ and $\zeta_n = r$, so $\Delta H(X, n) = 0 = v_n$.

Second case: $t + 1 \le k_1 \le 2t$. We set $p = \min\{1 \le i \le q \mid \beta_i \ge t\}$ and $\bar{b} = \beta_p - t$. We have that $0 \le \bar{b} < \beta_p - \beta_{p-1}$. We choose our points X on t

lines $L_1, \ldots, L_t$ of Σ_1 in the following way:

on	β_1	lines we take any	s	points on each
”	$\beta_2 - \beta_1$	”	$s - \alpha_1$	”
...	...	...	...	...
”	$\beta_{p-1} - \beta_{p-2}$	”	$s - \sum_{i=1}^{p-2} \alpha_i$	”
”	$\beta_p - \beta_{p-1} - \bar{b}$	”	$s - \sum_{i=1}^{p-1} \alpha_i$	”

and on $k_1 - t$ lines $M_1, \ldots, M_{k_1-t}$ of Σ_2 with the only restriction $X \cap L_i \cap M_j = \emptyset$, $\forall i, j$, in the following way:

on	$\bar{b}$	lines we take	$u - \sum_{i=1}^{p-1} \alpha_i$	points on each
”	$\beta_{p+1} - \beta_p$	”	$u - \sum_{i=1}^{p} \alpha_i$	”
...	...	...	...	...
”	$\beta_q - \beta_{q-1}$	”	$u - \sum_{i=1}^{q-1} \alpha_i$	”

The number of these points is

$$\beta_1 s + \sum_{i=2}^{p-1} (\beta_i - \beta_{i-1})(s - \sum_{j=1}^{i-1} \alpha_j) + (\beta_p - \beta_{p-1} - \bar{b})(s - \sum_{j=1}^{p-1} \alpha_j) +$$

$$+ \bar{b}(u - \sum_{j=1}^{p-1} \alpha_j) + \sum_{i=p+1}^{q} (\beta_i - \beta_{i-1})(u - \sum_{j=1}^{i-1} \alpha_j) =$$

$$= t^2 + \sum_{i=1}^{q-1} \beta_i \alpha_i = r.$$

Again we have to prove that $\Delta H(X, n) = v_n$. We write E_{ij} for a scheme made up by i lines of Σ_1 and by j lines of Σ_2 and we set $E = E_{tk_1-t}$. Because of our choice the hypotheses of the Lemma 3.2 are satisfied.

If $1 \leq n \leq t - 1$ then $\Delta H(X, n) = \Delta H(E, n) = 2n + 1 = v_n$.

If $t \leq n \leq t + \alpha_q - 1$ then $\Delta H(X, n) = \Delta H(E, n) = \beta_q = v_n$.

If $t + \alpha_q \leq n \leq t + \sum_{i=p}^{q} \alpha_i - 1$ then every lines of E in Σ_1 have more than n points of X. Let γ_n be the number of the lines of Σ_2 on which we have more than n points and ζ_n the number of the points lying on the other $\beta_q - t - \gamma_n$ lines of Σ_2. Then $H(X, n) = H(E_{t\gamma_n}, n) + \zeta_n$. We write $n = t + \sum_{i=m+1}^{q} \alpha_i + k$, $p - 1 \leq m \leq q$, $0 \leq k \leq \alpha_m - 1$. Then

$\gamma_n = \bar{b}+\beta_m-\beta_p = \beta_m-t$. If $k \geq 1$ then also $\gamma_{n-1} = \beta_m-t$ and $\zeta_{n-1} = \zeta_n$, so, by the Lemma 3.2, $\Delta H(X,n) = \Delta H(E_{t\gamma_n}, n) = t+\gamma_n = t+\beta_m-t = \beta_m = v_n$. If $k = 0$, $\gamma_{n-1} = \beta_{m+1}-t$, so $\Delta H(X,n) = H(E_{t\gamma_n}, n)+\zeta_n - H(E_{t\gamma_{n-1}}, n) - \zeta_{n-1} = t^2 + (1+\sum_{i=m+1}^{q}\alpha_i)(t+\gamma_n) + \sum_{i=m}^{q}(\beta_{i+1}-\beta_i)\sum_{j=i+1}^{q}\alpha_j - t^2 - \sum_{i=m+1}^{q}\alpha_i(t+\gamma_{n-1}) - \sum_{i=m+1}^{q}(\beta_{i+1}-\beta_i)\sum_{j=i+1}^{q}\alpha_j = t+\gamma_n+\sum_{i=m+1}^{q}\alpha_i\gamma_n - \sum_{i=m+1}^{q}\alpha_i\gamma_{n-1} + (\beta_{m+1}-\beta_m)\sum_{j=m+1}^{q}\alpha_j = \beta_m = v_n$.

Similar arguments work for $t+\sum_{i=p}^{q}\alpha_i \leq n \leq s-1$ and $n \geq s$. □

Example 3.4. Let us consider the following O-sequence:

n	0	$\cdots$	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
v_n	1	$\cdots$	11	12	10	10	10	8	8	8	8	4	4	0	$\cdots$

We have that $\beta_1 = 4$, $\beta_2 = 8$, $\beta_3 = 10$, $\beta_4 = 12$; $\alpha_1 = 2$, $\alpha_2 = 4$, $\alpha_3 = 3$, $\alpha_4 = 1$; $t = 6$, $u = 10$, $s = 16$; $p = 2$, $\bar{b} = 2$.

We put our points in the following way:

on	4	lines of Σ_1 we take any	16	points on each
”	2	”	14	”

and, distinct from the intersection points:

on	2	lines of Σ_2 we take	8	points on each
”	2	”	4	”
”	2	”	1	point on each.

4. The main result.

In this section we prove the main result of this paper.

Theorem 4.1. *Let C be an irreducible curve, lying on a smooth quadric, of type (a,b), $1 \leq a \leq b$. Let $\{v_n\}$ be a O-sequence of the kind*

n	0	$\cdots$	$b-1$	b	$\cdots$	$s-1$	s	$\cdots$	$s+e-1$	$s+e$	$\cdots$
v_n	1	$\cdots$	$2b-1$	d	$\cdots$	d	h_1	$\cdots$	h_e	0	$\cdots$

$d = a+b$, $d-1 \geq h_1$, $h_e \geq 1$, $0 \leq e \leq a$ and $h_n - h_{n+1} \geq 2$ for $1 \leq n \leq e-1$.

Then there is a 0-dimensional scheme $X \subset C$, such that $\Delta H(X,n) = v_n$, for all $n \in \mathbb{N}_0$.

Proof. Let $Q \subset \mathbb{P}^3_k$ be a smooth quadric and let Σ_1 and Σ_2 be the rulings of lines on Q of type $(1, 0)$ and $(0, 1)$ respectively. We set $c = b - a$.

Let S be a surface such that $\deg S = s$ and $C \not\subset S$ and let $R_1, \dots, R_c \in \Sigma_1$, $R_i \neq R_j$, $\forall i \neq j$, such that $R_i \not\subset S$ for $1 \leq i \leq c$. We call D the curve $C \cup R_1 \cup \ldots \cup R_c$. Then D is a complete intersection curve on Q of type (b, b) and the scheme $Z = S \cap D$ is a 0-dimensional complete intersection.

If $\vartheta = s + b - 1$ then we consider the following O-sequence

$$\bar{v}_n = \Delta H(Z, \vartheta - n) - v_{\vartheta - n}$$

i.e.

n	0	$\cdots$	-1	t	$\cdots$	$b-1$	b	$\cdots$	$s-1$	s	$\cdots$
$\bar{v}_n$	1	$\cdots$	$2t-1$	k_1	$\cdots$	k_e	c	$\cdots$	c	0	$\cdots$

$t = b - e$, $2t \geq k_1 \geq \ldots \geq k_e \geq c$.

Let Y be a 0-dimensional scheme of cs points, arranged s for any line R_i, such that they are distinct from the b points where R_i meets C. If we are able to build a zero dimensional scheme $\bar{X} \subset C$, such that $\Delta H(\bar{X} \cup Y, n) = \bar{v}_n$ and a surface S, $\deg S = s$, such that $C \not\subset S$, $R_i \not\subset S$ for $1 \leq i \leq c$, $\bar{X} \cup Y \subset S$, then for the *liaison theory*, (see [6] and [1]), the theorem is proved.

To build the scheme $\bar{X}$ we want obviously to use the Lemma 3.3. Since each line of Σ_1 meets C in b points and each line of Σ_2 meets C in a points it is enough to show that the construction of the Lemma 3.3, in this situation, uses a number of points $\leq b$ for the lines of Σ_1 (except the points that we put on the lines $R_1, \dots, R_c$) and a number $\leq a$ for the lines of Σ_2.

With the same notation of the Lemma 3.3 if $s \geq b + 1$ then $\beta_1 = c$ and $\alpha_1 \geq s - b$. Our construction uses first of all c lines of Σ_1 where we put s points. We take as such lines just the $R_1, \dots, R_c$. Instead, on the other lines of Σ_1 we must take a number of points $\leq s - \alpha_1 \leq s - s + b = b$. If $\beta_q > t$ then our construction uses lines of Σ_2 too. We can have $p = 1$ or $p \geq 2$. If $p = 1$ then

$$c = \beta_1 \geq t \Rightarrow b - a \geq b - e \Rightarrow a \leq e \Rightarrow a = e,$$

hence $\bar{b} = 0$ so we must put on these lines a number of points $\leq s - t - \alpha_1 \leq s - b + e - \alpha_1 \leq s - b + e - s + b = e = a$. If $p \geq 2$ then the number of points is $\leq s - t - \alpha_1 - \ldots - \alpha_{p-1} \leq s - b - e - \alpha_1 \leq s - b + e - s + b = e \leq a$.

If $s = b$ then $\beta_1 = k_e = 2b - 1 - h_1 \geq 2b - 1 - a - b + 1 = c$, so we can repeat the same arguments.

So far we have built the 0-dimensional scheme $\bar{X} \cup Y$. The existence of the surface S is trivial. $\square$

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