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PARABOLIC TRANSMISSION PROBLEMS ACROSS IRREGULAR LAYERS

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We study second order transmission problems across either a fractal surface or across the corresponding pre-fractal surface. Existence uniqueness and regularity results for the strict solution in both cases are proved. The asymptotic behaviour of the solutions of the approximating problems is also investigated.

1. Introduction

There is a huge literature on transmission problems across interfaces which are usually assumed to be regular or Lipschitz surfaces; a good reference is the book of Dautray and Lions [4]. Here, we focus our attention on the so-called second order transmission problems which, in electrostatics and magnetostatics, model the heat transfer through an infinitely conductive layer (in this regard see the paper by Pham Huy and Sanchez Palencia [31] and the references listed in). It's to be pointed out that also in "hydraulic fracturing" – a technique used in order to increase the flow of oil from a reservoir into a producing oil well – the model problem is a transmission problem (see [2]). In all these cases the model problem is a parabolic boundary value problem involving a transmission condition on the layer of order two.

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In many applications, such as boundary value problems of absorption or irrigation type (see [3]), the surface effects are enhanced with respect to the surrounding volume; in this context, fractal boundaries and fractal layers may provide new interesting settings as we describe in the present paper.

As far as we know, the rigorous study of transmission problems–in the elliptic case– across irregular layers of fractal type is recent and has been treated firstly in [16], [17] and [27].

In this paper we describe a constructive approach to second order fractal transmission problems for parabolic operators (details and proofs are contained in a joint paper with P. Vernole [19]). In these problems the fractal layer S of Koch type is obtained in the limit of polyhedral (prefractal) surfaces S_h ,by self-similar iteration at small scales. From an analytic point of view, the main problem to be studied is the convergence of the solutions obtained with the prefractal layers when the geometry becomes fractal. The main difficulty in this study is the jump of the geometric dimension of the layer, which is two for all the prefractal approximations and intermediate between two and three for the limit layer. In other words, while the approximating layers have finite two-dimensional area, the limit layer is non rectifiable with infinite area. We first consider the elliptic case. We describe the variational approach consisting in proving the convergence of suitable energy forms in the Mosco's sense and we also describe some extensions to the heat equation by relying on the convergence of the related semigroups. In order to obtain these convergence properties, we have to introduce some renormalization factors.

More precisely, we consider a "cylindrical" fractal surface S of the type $S = F \times I$ where F is the Koch snowflake and I is the unit interval $[0,1]$. Fractal surfaces are usually non self-similar sets hence, for this type of sets, in the construction of the energy form, it is not possible to rely on the by-now well-established "classical" approaches due to Goldestein (see [7]) and Kusuoka (see [15]) which are deeply based on the self-similarity of the underlying set. Here, in the construction of the energy form, we shall take advantage of the underlying geometry of S and the structure of the energy functional (on S) will reflect it. A semigroup approach will be adopted to study the evolution problem both in the fractal and prefractal case and it will turn crucial in the study of the asymptotics of the approximating problems $\overline{P}_h$ when h goes to infinity i.e. in the study of the convergence of the approximating solutions to the solution of the limit problem $\overline{P}$.

In this analysis, we will point out how the jump of the geometric dimension – which is 2 in the prefractal case and a number $D_f \in (2,3)$ in the fractal case – plays a role in the study of the asymptotic behaviour.

The model problem, in the fractal case, can be formally stated as:

$$(\overline{P}) \quad \begin{cases} u_t(t,P) - \Delta u(t,P) = f(t,P) & \text{in } [0,T]\times Q^i,\ i=1,2 & j) \\ -c_0\Delta_S u(t,P) = \left[\frac{\partial u(t,P)}{\partial n}\right] & \text{on } [0,T]\times S, & jj) \\ u(t,P) = 0 & \text{on } [0,T]\times \partial Q, & jjj) \\ u^1(t,P) = u^2(t,P) & \text{on } [0,T]\times\ S, & jv) \\ u(t,P) = 0 & \text{on } [0,T]\times\ \partial S & v) \\ u(0,P) = 0 & \text{on } Q & vj), \end{cases}$$

where Q is the parallelepiped $Q = (-1,1)^2 \times (0,1)$, S denotes a "cylindrical" fractal layer in Q (see Section 1) dividing Q in two subsets Q^1 and Q^2, u^i denotes the restriction of u to Q^i, $[u] = u^1 - u^2$ denotes the jump of u across S, Δ_S denotes the Laplace operator defined on the layer S (see (7) in Section 3), $\left[\frac{\partial u}{\partial n}\right] = \frac{\partial u^1}{\partial n_1} + \frac{\partial u^2}{\partial n_2}$ the jump of the normal derivatives across S, to be intended in a suitable sense, and $f(t,P)$ is a given function in $C^\theta([0,T];L^2(Q))$, $\theta \in (0,1)$. Condition $jj)$ is the so-called *second order transmission condition.* In view of numerical approximations (see [18]), we will study the approximating problems $(\overline{P}_h)$ – with layer the pre-fractal interface S_h approximating S– and their asymptotic behavior as $h \to \infty$, with the idea that the asymptotic limit problem is indeed the transmission problem $(\overline{P})$. Problem $(\overline{P}_h)$ can be formally written as follows

$$(\overline{P}_h) \quad \begin{cases} u_t(t,P) - \Delta u(t,P) = f(t,P) & \text{in } [0,T]\times Q_h^i,\ i=1,2 & j) \\ -\Delta_{S_h} u(t,P) = \left[\frac{\partial u(t,P)}{\partial n}\right] & \text{on } [0,T]\times S_h, & jj) \\ u(t,P) = 0 & \text{on } [0,T]\times \partial Q, & jjj) \\ u^1(t,P) = u^2(t,P) & \text{on } [0,T]\times\ S_h, & jv) \\ u(t,P) = 0 & \text{on } [0,T]\times\ \partial S_h & v) \\ u(0,P) = 0 & \text{on } Q & vj), \end{cases}$$

where Q is the parallelepiped as above, S_h denotes a "polyhedral " layer in Q (see Section 1) dividing Q in two subsets Q_h^1 and Q_h^2, u^i denotes the restriction of u to Q_h^i, $[u] = u^1 - u^2$ denotes the jump of u across S_h, Δ_{S_h} denotes the piecewise-tangential Laplacian defined on the layer S_h (see Section 6.2) $\left[\frac{\partial u}{\partial n}\right] = \frac{\partial u^1}{\partial n_1} + \frac{\partial u^2}{\partial n_2}$ the jump of the normal derivatives across S_h.

2. Geometry

Let Q denote a bounded open set in $\mathbb{R}^3$; in our basic model Q denotes the parallelepiped $Q = (-1,1)^2 \times (0,1)$ and S denotes a "cylindrical" layer in Q of the type $S = F \times I$, where $I = [0,1]$ and F is the so-called Koch snowflake (it can be regarded as the union of three Koch curves K_i) . We assume that S is located in a median position inside Q and divides Q in two sub-domains Q^1 and Q^2 (see Figure 1).

We give a point $P \in S$ the cartesian coordinates $P = (x,y)$, where $x = (x_1, x_2)$ are the coordinates of the orthogonal projection of P on the plane containing F and y is the coordinate of the orthogonal projection of P on the y–line containing the interval I: $P = (x,y) \in S$, $x = (x_1,x_2) \in F$, $y \in I$.
One can define in a natural way, a finite Borel measure m supported on S as the product measure

$$\mathrm{d}m = \mathrm{d}\mu_F \times \mathrm{d}y \tag{1}$$

where dy denotes the one-dimensional Lebesgue measure on I. The measure m has the property that there exists two positive constants c_1, c_2

$$c_1 r^d \le m(B(P,r) \cap S) \le c_2 r^d, \ \forall P \in S \tag{2}$$

where $\mathrm{d} = d_f + 1 = \frac{\log 12}{\log 3}$ and where $B(P,r)$ denotes the Euclidean ball in $\mathbb{R}^3$. As m is supported on S, it is not ambiguous to write in (2) $m(B(P,r))$ in place of $m(B(P,r) \cap S)$. In the terminology of the following subsection we say that S is a d–set with $\mathrm{d} = d_f + 1$.

By S_h we denote the pre-fractal layer of the type $S_h = F_h \times I$, $h = 1,2,\ldots$, F_h is the piecewise linear pre-fractal approximation of F at the step h. S_h is a surface of polyhedral type. S_h divides Q in two sub-domains Q_h^i, $i = 1,2$.

We give a point $P \in S_h$ the Cartesian coordinates $P = (x,y)$, where $x = (x_1,x_2)$ are the coordinates of the orthogonal projection of P on the plane containing F_h and y is the coordinate of the orthogonal projection P on the y–line containing the interval I.

3. Energy forms and semigroups associated

3.1. The Energy form E

We define the energy forms E_S on the fractal layer $S = F \times I$ by setting

$$E_S[u] = \sigma^1 \int_I \int_F \mathcal{L}_x[u](\mathrm{d}x)\mathrm{d}y + \sigma^2 \int_F \int_I |D_y u|^2 \mathrm{d}y \mu_F(\mathrm{d}x) \tag{3}$$

where σ^1 and σ^2 are positive constants. Here $D_y(\cdot)$ denotes the derivative in the y direction, $\mathcal{L}_x(\cdot,\cdot)(\mathrm{d}x)$ denotes the measure–valued Lagrangian (of the energy form $\mathcal{E}_F$ of F with domain $\mathcal{D}(F)$) now acting on $u(x,y)$ and $v(x,y)$ as function of $x \in F$ for a.e. $y \in I$; $\mu_F(\mathrm{d}x)$ is the Hausdorff measure acting on each section F of S for a.e. $y \in I$ with $d_f = \frac{\log 4}{\log 3}$.

The energy, on F, has the following integral representation (see Definition 4.5 of [5])

$$\mathcal{E}_F(u,v) = \int_F \mathrm{d}\mathcal{L}_F(u,v) \tag{4}$$

$u,v \in \mathcal{D}(F)$ where $\mathcal{D}(F)$, which is a Hilbert space with norm

$$(\|u\|^2_{L^2(F,\mu_F)} + \mathcal{E}_F(u,u))^{\frac{1}{2}}$$

has been characterized in terms of the domains of the energy forms on K_i (see [5] Theorem 4.6).

In the following we will omit the subscript F, the Lagrangian measure will be simply denoted by $\mathcal{L}(u,v)$ and we will set $\mathcal{L}[u] = \mathcal{L}(u,u)$, an analogous notation will be adopted for the energies.

The form E_S is defined for $u \in \mathcal{D}(S)$ where $\mathcal{D}(S)$ is the closure in the intrinsic norm

$$\|u\|_{\mathcal{D}(S)} = (E_S[u] + \|u\|^2_{L^2(S,m)})^{\frac{1}{2}} \tag{5}$$

of the set

$$C_0(S) \cap L^2(0,1;\mathcal{D}(F)) \cap H^1_0(0,1;L^2(F)) \tag{6}$$

where $L^2(F) = L^2(F,\mu_F(\mathrm{d}x))$.

In the following we shall also use the form $E_S(u,v)$ which is obtained from $E_S[u]$ by the polarization identity.

It can be proved as in Proposition 3.1 of [27], that:

Proposition 1. In the previous notations and assumptions the form E_S with domain $\mathcal{D}(S)$ is a regular Dirichlet form in $L^2(S,m)$ and the space $\mathcal{D}(S)$ is a Hilbert space under the intrinsic norm (5).

For the definition and properties of regular Dirichlet forms we refer to [6]. We now define the Laplace operator on S. As $(E_S, \mathcal{D}(S))$ is a closed, bilinear form on $L^2(S,m)$, there exists (see Chap. 6, Theorem 2.1 in [14]) a unique self–adjoint, non positive operator Δ_S on $L^2(S,m)$—with domain $\mathcal{D}(\Delta_S) \subseteq \mathcal{D}(S)$ dense in $L^2(S,m)$—such that

$$E_S(u,v) = -\int_S (\Delta_S u) v \mathrm{d}m, \; u \in \mathcal{D}(\Delta_S), v \in \mathcal{D}(S). \tag{7}$$

Let $(\mathcal{D}(S))'$ denote the dual of the space $\mathcal{D}(S)$. We now introduce the Laplace operator on the fractal S as a variational operator from $\mathcal{D}(S) \to (\mathcal{D}(S))'$ by

$$E_S(z,w) = -\langle \Delta_S z, w \rangle_{(\mathcal{D}(S))',\mathcal{D}(S)} \tag{8}$$

for $z \in \mathcal{D}(S)$ and for all $w \in \mathcal{D}(S)$, where $\langle \cdot,\cdot \rangle_{(\mathcal{D}(S))',\mathcal{D}(S)}$ is the duality pairing between $(\mathcal{D}(S))'$ and $\mathcal{D}(S)$. We use the same symbol Δ_S to define the Laplace operator both as a self-adjoint operator in (7) and as a variational operator in (8). It will be clear from the context to which case we refer.

Consider now the space of functions $u : Q \to \mathbb{R}$

$$V(Q,S) = \left\{ u \in H^1_0(Q) \ : \ u|_S \in \mathcal{D}(S) \right\}. \tag{9}$$

The space $V(Q,S)$ is non trivial, see Proposition 3.3 of [17]. We now introduce the energy form

$$E[u] = \int_Q |Du|^2 dQ + c_0 E_S[u|_S] \tag{10}$$

defined on the domain $V(Q,S)$.
In the following, by $E(u,v)$, we will denote the corresponding bilinear form

$$E(u,v) = \int_Q DuDv dQ + c_o E_S(u|_S, v|_S) \tag{11}$$

defined on $V(Q,S) \times V(Q,S)$.

As in Theorem 3.2 of [27], it can be proved

Proposition 2. The form E defined in (10) is a regular Dirichlet form in $L^2(Q)$ and the space $V(Q,S)$ is a Hilbert space equipped with the scalar product

$$(u,v)_{V(Q,S)} = E(u,v). \tag{12}$$

We denote by $\|u\|_{V(Q,S)}$ the norm in $V(Q,S)$, associated with (12)), that is

$$\|u\|_{V(Q,S)} = \left(c_0 E_S[u|_S] + \int_Q |Du|^2 dQ \right)^{\frac{1}{2}}. \tag{13}$$

As in Proposition (3.6) and (3.1) in [17], it can be proved

Proposition 3. The space $\mathcal{D}(S)$ is embedded in $B^{2,2}_{\beta,0}$, $\beta = \frac{d_f}{2}$.

Proposition 4. The space $\mathcal{D}(S)$ is embedded in $B^{2,2}_{\alpha}$, $\alpha < 1$.

3.1.1. Resolvent and semigroup associated to the energy form E

As $(E, V(Q,S))$ is a closed bilinear form on $L^2(Q)$ with domain $V(Q,S)$ dense in $L^2(Q)$ there exists (see chap. 6 Theorem 2.1 in [14]) a unique self-adjoint non positive operator A on $L^2(Q)$ with domain $\mathcal{D}(A) \subseteq V(Q,S)$ dense in $L^2(Q)$ such that

$$E(u,v) = -\int_Q Auv\,\mathrm{d}Q, \; u \in \mathcal{D}(A), v \in V(Q,S) \tag{14}$$

Moreover in Theorem 13.1 of [6] it is proved that to each closed symmetric form E a family of linear operators $\{G_\alpha, \alpha > 0\}$ can be associated with the property

$$E(G_\alpha u, v) + \alpha(u,v) = (u,v), \; u \in L^2(Q) \text{ and } v \in V(Q,S)$$

and this family is a strongly continuous resolvent with generator A, which also generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$.

For the reader's convenience we recall here the main properties of the semigroup $\{T(t)\}_{t\geq 0}$.

Proposition 5. Let $\{T(t)\}_{t\geq 0}$ be the semigroup generated by the operator A associated to the energy form in (14). Then $\{T(t)\}_{t\geq 0}$ is an analytic contraction semigroup in $L^2(Q)$.

The contraction property follows from Lumer Phillips Theorem on dissipative operators (Chap.1 Theorem 4.3 in [30]). As the form E is coercive in V(Q,S) the analyticity follows from Theorem 6.A Chap.4 in [32].

3.2. The energy forms E_{S_h}

By Q we denote the parallelepiped as defined in Section 2 and by S_h we denote the pre-fractal layer of the type $S_h = F_h \times I$, $h = 1,2,\ldots$, F_h is the pre-fractal approximation of F at the step h. S_h divides Q in two sub-domains Q_h^i, $i = 1,2$. We give a point $P \in S_h$ the Cartesian coordinates $P = (x,y)$, where $x = (x_1,x_2)$ are the coordinates of the orthogonal projection of P on the plane containing F_h and y is the coordinate of the orthogonal projection of P on the y–line containing the interval I.

We first construct the energy forms E_{S_h} on the *pre–fractal* layers $S_h = F_h \times I$, $h \in \mathbb{N}$. By ℓ we denote the natural arc–length coordinate on each edge of F_h and we introduce the coordinates $x_1 = x_1(\ell)$, $x_2 = x_2(\ell)$, $y = y$ on every affine "face" $S_h^{(j)}$ of S_h. By $\mathrm{d}\ell$ we denote the one–dimensional measure given by the arc–length ℓ and by $\mathrm{d}\sigma$ the surface measure on each face $S_h^{(j)}$ of S_h, that is $\mathrm{d}\sigma = \mathrm{d}\ell\mathrm{d}y$. We define $E_{S_h}[u]$ by setting

$$E_{S_h}[u] = \sum_j \left(\int_{S_h^{(j)}} \left(\sigma_h^1 |D_\ell u|^2 + \sigma_h^2 |D_y u|^2 \right) \mathrm{d}\sigma \right) \tag{15}$$

where σ_h^1, σ_h^2 are positive constants and $u \in H^1(S_h)$, the Sobolev space of functions on the piece–wise affine set S_h (see Section 2.1). By Fubini theorem, we can write this functional in the form

$$E_{S_h}[u] = \sigma_h^1 \int_I \left(\int_{F_h} |D_\ell u|^2 \mathrm{d}\ell \right) \mathrm{d}y + \sigma_h^2 \int_{F_h} \left(\int_I |D_y u|^2 \mathrm{d}y \right) \mathrm{d}\ell. \tag{16}$$

We denote the corresponding bilinear form by $E_{S_h}(u,v)$.
Consider now the space of functions $u : Q \to \mathbb{R}$

$$V(Q,S_h) = \left\{ u \in H_0^1(Q) \ : \ u|_{S_h} \in H_0^1(S_h) \right\}, \tag{17}$$

it is not trivial as it contains $\mathcal{D}(Q)$ (the smooth functions with compact support on Q).
Consider now the energy form

$$E^{(h)}[u] = \int_Q |\mathrm{D}u|^2 \mathrm{d}Q + E_{S_h}[u|_{S_h}] \tag{18}$$

defined on the domain $V(Q,S_h)$.
By $E^{(h)}(u,v)$ we will denote the corresponding bilinear form

$$E^{(h)}(u,v) = \int_Q \mathrm{D}u \mathrm{D}v \, \mathrm{d}Q + E_{S_h}(u|_{S_h}, v|_{S_h}) \tag{19}$$

defined on $V(Q,S_h) \times V(Q,S_h)$.

Theorem 6. The form $E^{(h)}$, defined in (18), with domain $V(Q,S_h)$ is a regular Dirichlet form in $L^2(Q)$ and the space $V(Q,S_h)$ is a Hilbert space equipped with the scalar product

$$(u,v)_{V(Q,S_h)} = E^{(h)}(u,v).$$

For the proof see Theorem 4.1 in [17].

We denote by $\|u\|_{V(Q,S_h)}$ the corresponding energy norm in $V(Q,S_h)$, that is

$$\|u\|_{V(Q,S_h)} = \left(\int_Q |\mathrm{D}u|^2 \mathrm{d}Q + E_{S_h}[u|_{S_h}] \right)^{\frac{1}{2}}. \tag{20}$$

3.2.1. Resolvents and associated semigroups

Proceeding as in Section(3.1.1) we denote by $\{G_\alpha^h, \alpha > 0\}$, A_h and $\{T_h(t)\}_{t\geq 0}$ the resolvents, the generators and the semigroups associated to $E^{(h)}$, for every $h \in \mathbb{N}$ respectively.
As in proposition (5) it can be proved that:

Proposition 7. Let $\{T_h(t)\}_{t\geq 0}$ be the semigroup generated by the operator A_h associated to the energy form in (19). Then $\{T_h(t)\}_{t\geq 0}$ is an analytic contraction semigroup in $L^2(Q)$.

4. The convergence of forms and semigroups

In this Section we study the convergence of the approximating energy forms $E^{(h)}$ to the fractal energy E. In this asymptotic behaviour the factors σ_h^1 and σ_h^2 have a key role and can be regarded as a sort of renormalization factors of the approximating energies. These factors take into account the non rectifiability of the curve F and hence the irregularity of the surface S and in particular the effect of the $d-$dimensional length intrinsic to the curve F, for details see [19]. The convergence of functionals is here intended in the sense of the M-convergence which we define below.

4.1. The M-convergence of forms

We recall, for the sake of completeness, the definition of M-convergence of forms introduced by Mosco in [25].

We extend the form E defined in (10) and $E^{(h)}$ defined in (18) on the whole space $L^2(Q)$ by defining

$$E[u] = +\infty \text{ for every } u \in L^2(Q) \setminus V(Q,S)$$

and

$$E^{(h)}[u] = +\infty \text{ for every } u \in L^2(Q) \setminus V(Q,S_h).$$

Definition 1. A sequence of form $\{E^{(h)}\}$ M-converges to a form E in $L^2(Q)$ if
(a) for every $\{v_h\}$ converging weakly to u in $L^2(Q)$

$$\underline{lim}\, E^{(h)}[v_h] \geq E[u], \text{ as } \qquad h \to \infty$$

(b) for every $u \in L^2(Q)$ there exists $\{w_h\}$ converging strongly to u in $L^2(Q)$ such that

$$\overline{lim}\, E^{(h)}[w_h] \leq E[u], \text{ as } \qquad h \to \infty. \tag{21}$$

According to Definition 2.3.1 in [26], we say that

Definition 2. The sequence of forms $\{E^{(h)}\}$ is asymptotically compact in $L^2(Q)$ if every sequence $\{u_h\}$ with

$$\underline{lim}\, E^{(h)}[u_h] + \int_Q |u_h|^2 \mathrm{d}Q < \infty \tag{22}$$

has a subsequence strongly convergent in $L^2(Q)$.

Proposition 8. The sequence of forms (18) is asymptotically compact in $L^2(Q)$.

Remark 1. We point out that, as the sequence of forms (18) is asymptotically compact in $L^2(Q)$, M-convergence is equivalent to the Γ-convergence (see Lemma 2.3.2 in [26]), thus we can take in (a) v_h strongly converging to u in $L^2(Q)$.

We can now state the main theorem of this section:

Theorem 9. Let $\sigma_h^1 = \sigma_1 c_0 (3^{d_f - 1})^h$ and $\sigma_h^2 = \sigma_2 c_0 (3^{1-d_f})^h$, then the sequence of forms $\{E^{(h)}\}$ defined in (18) M-converges in the space $L^2(Q)$ to the form E defined in (10).

The proof is long and delicate (for details see Theorem 3.5 in [19]). It takes into account the fact that the set S is the cartesian product of F and the unit interval I; it is deduced by using an analogous 2-dimensional result (see Theorem 4.1 in [20]) and by using some technical tools such as trace and extension theorems for Besov spaces on d-sets and on arbitrary closed sets (see [11], [12] and [33]) and classical arguments as Fatou's lemma and convergence results.

4.2. Convergence of semigroups and resolvents

From Theorem 9 we deduce the following.

Proposition 10. Let $E^{(h)}$ and E be the energy forms defined in (18) and (10) and $G_\alpha^{(h)}$ and G_α the resolvents associated to $E^{(h)}$ and E respectively. If $\{E^{(h)}\}$ is M-convergent to E, then for every $\alpha > 0$ the sequence $\{G_\alpha^h\}$ converges to the operator G_α in the strong operator topology of $L^2(Q)$.

The proof follows from Theorem 2.4.1 part (i) in [26]. As a consequence of the well known Trotter-Kato Theorem, which characterizes the convergence of semigroups in terms of convergence of the related resolvents, the following theorem holds:

Theorem 11. Let $E^{(h)}$ and E be as in Corollary 10 then the sequence of semigroups $\{T_h(t)\}$ associated with the form $E^{(h)}$ converges in $L^2(Q)$ to the semigroup $T(t)$ associated with the form E in the strong operator topology of $L^2(Q)$ uniformly on every interval $[0, t_1]$.

The proof easily follows from Proposition 10 and Theorem 4.2 Chapter 3 in [30].

5. Evolution problems and convergence of the solutions

We study the solvability of the Cauchy problems:

$$(P) \qquad \begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t), & 0 \le t \le T \\ u(0) = 0 \end{cases} \tag{23}$$

and for every $h \in \mathbb{N}$

$$(P_h) \qquad \begin{cases} \frac{\mathrm{d}u_h(t)}{\mathrm{d}t} = A_h\, u_h(t) + f(t), & 0 \le t \le T \\ u_h(0) = 0 \end{cases} \tag{24}$$

where $A : \mathcal{D}(A) \subset L^2(Q) \to L^2(Q)$ and $A_h : \mathcal{D}(A_h) \subset L^2(Q) \to L^2(Q)$ are the generators associated respectively to the energy form E and the energy forms $E^{(h)}$ introduced in (10) and (18), of the previous sections, T is a fixed positive real number, and f is a given function in a suitable Banach space.

In the following we shall make use of functions with values in Hilbert spaces (see [24]).
A "strict" solution of problem (P) is a function

$$u \in C^1([0,T];\, L^2(Q)) \cap C([0,T];\, \mathcal{D}(A)) \tag{25}$$

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t), \text{ for every } t \in [0,T] \qquad \textit{and } u(0) = 0.$$

Theorem 12. Let $0 < \theta < 1,\ f \in C^\theta([0,T], L^2(Q))$ and let

$$u(t) = \int_0^t T(t-s)\, f(s)\, \mathrm{d}s, \tag{26}$$

where $T(t)$ is the analytic semigroup generated by A. Then u is the unique "strict" solution of (P).
Furthermore, there exists c such that

$$\|u\|_{C^1([0,T],L^2(Q))} + \|u\|_{C^0([0,T],\mathcal{D}(A))} \le c\|f\|_{C^\theta([0,T],L^2(Q))} \tag{27}$$

Theorem 13. Let $0 < \theta < 1,\ f \in C^\theta([0,T], L^2(Q))$ and let

$$u_h(t) = \int_0^t T_h(t-s)\, f(s)\, \mathrm{d}s \qquad \text{for every } h \in \mathbb{N} \tag{28}$$

where $T_h(t)$ is the analytic semigroup generated by A_h. Then u_h is the unique "strict" solution of (P_h).
Furthermore there exists c, independent from h, such that

$$\|u_h\|_{C^1([0,T],L^2(Q))} + \|u_h\|_{C^0([0,T],\mathcal{D}(\mathrm{A_h}))} \le c\|f\|_{C^\theta([0,T],L^2(Q))}. \tag{29}$$

For the proof see Theorem 4.3.1 page 134 in [24].
Now we are interested in the behavior of the sequence $\{u_h\}$ when h goes to ∞ in view of the numerical approximation of these problems (see [18]).

Theorem 14. Let u and u_h be the solutions of problems (P) and (P_h) according to Theorems (12) and (13). Let σ_h^1 and σ_h^2 be as in Theorem (9). We have:

i) $\{u_h\}$ converges to u in $C([0,T];\, L^2(Q))$

ii) $\{\frac{\mathrm{d}u_h}{\mathrm{d}t}\}$ weakly converges to $\frac{\mathrm{d}u}{\mathrm{d}t}$ in $L^2([0,T]\times Q)$

iii) $\{A_h u_h\}$ weakly converges to Au in $L^2([0,T]\times Q)$

iv) $\{u_h\}$ converges to u in $L^2([0,T];H_0^1(Q))$

For details on the proof see Theorem 5.3 in [19]. The proof of $i)$ is deduced from (28), (26) and Theorem 11; the proof of ii) is deduced from Theorem 9.

6. The strong formulation of the transmission problems

6.1. The case of the fractal layer

Theorem 15. Let u be the solution of problem (P). Then we have for every fixed $t\in[0,T]$

$$\begin{cases} u_t(t,P)-\triangle u(t,P)=f(t,P) & \text{for } a.e.\ P\in Q_i\ i=1,2 \\ \frac{\partial u^i}{\partial n_i}\in\left((B^{2,2}_{\beta,0})(S)\right)', & \beta=\frac{d_f}{2},\ i=1,2 \\ -c_0\langle\triangle_S u|_S, z\rangle_{(\mathcal{D}(S))'\mathcal{D}(S)}=\left\langle\left[\frac{\partial u}{\partial n}\right],\, z\right\rangle_{(\mathcal{D}(S))'\mathcal{D}(S)}, & \text{for every } z\in\mathcal{D}(S) \\ u(t,P)=\ 0 & \text{for } P\in\partial Q \end{cases} \tag{30}$$

where u^i is the restriction of u to Q_i, $\frac{\partial u^i}{\partial n_i}$, $i=1,2$ is the inward "normal derivative", to be defined in a suitable sense, $\left[\frac{\partial u}{\partial n}\right]=\frac{\partial u^1}{\partial n_1}+\frac{\partial u^2}{\partial n_2}$ is the jump of the normal derivative and $\triangle_S$ is the fractal Laplacian defined in (8) of Section 3.1. Moreover $\frac{\partial u^i}{\partial n_i}\in C([0,T];(B^{2,2}_{\beta,0})(S)')$.

The proof is complicated, it makes use of Gauss-Green formula for domains with fractal boundaries (see [16]) and it makes use of sophisticated subspaces of Besov spaces on S and on F; in particular, the embedding of $\mathcal{D}(S)$ in $B^{2,2}_{\beta,0}(S)$ (see Proposition 3.6 in [17]). The space $B^{2,2}_{\beta}(S)$ is defined as the space of functions $f\in L^2(S,m)$ for which is finite the norm

$$\|f\|_{B^{2,2}_{\beta}(S)}:=\left(\|f\|^2_{L^2(S,m)}+\int\!\!\int_{|P-P'|<1}\frac{|f(P)-f(P')|^2}{|P-P'|^{2\beta+d}}dm(P)dm(P')\right)^{1/2}$$

here $d = 1 + d_f$ and $\beta = d_f/2$.

We point out that the definition of Besov spaces on d-sets with smoothness index greater or equal to one is more complicated than the one given above (see [12])). The space $B^{2,2}_{\beta,0}(S)$ is a subspace of $B^{2,2}_{\beta}(S)$ and it is the fractal analogue of the Lions-Magenes space $H^{\frac{1}{2}}_{0,0}(S)$, more precisely,

$$B^{2,2}_{\beta,0}(S) = \{u \in L^2(S,m) | \text{there exists } v \in H^1_0(Q) \text{ such that } v|_S = u \text{ on} S\}, \quad (31)$$

equipped with the norm

$$\|u\|_{B^{2,2}_{\beta,0}(S)} = \inf\{\|v\|_{H^1(Q)} : v \in H^1_0(Q),\ v|_S = u,\ \text{on} S\}.$$

We also use the duals of Besov spaces on S. These spaces, as shown in [13], coincide with a subspace of Schwartz distributions $D'(\mathbb{R}^3)$, which are supported in S. They are built by means of atomic decomposition. Actually, Jonsson and Wallin [13] proved this result in the general framework of d–sets. Here we do not give a detailed description of the duals of Besov spaces on d–sets and we refer to [13] for a complete discussion.

6.2. The case of the prefractal layer

Theorem 16. Let u_h be the solution of problem (P_h). Then we have, for every fixed $t \in [0,T]$,

$$\begin{cases} (u_h)_t(t,P) - \triangle u_h(t,P) = f(t,P) & \text{for } P \in Q^i_h,\ a.e.\ i = 1,2 \\ \frac{\partial u^i_h}{\partial n_i} \in L^2(S_h), & i = 1,2 \\ -\triangle_{S_h} u_h|_{S_h} = \left[\frac{\partial u_h}{\partial n}\right], & \text{in} L^2(S_h) \\ u(t,P) = 0 & \text{for } P \in \partial Q \end{cases} \quad (32)$$

where u^i_h is the restriction of u_h to Q^i_h, $\left[\frac{\partial u_h}{\partial n}\right] = \frac{\partial u^1_h}{\partial n_1} + \frac{\partial u^2_h}{\partial n_2}$ is the jump of the normal derivatives across S_h, n_i, $i = 1,2$, being the inward normal vector and $\triangle_{S_h} = \sigma^1_h D^2_\ell + \sigma^2_h D^2_y$ is the piece-wise tangential Laplacian associated to the Dirichlet form E_{S_h}. Moreover $\frac{\partial u^i_h}{\partial n_i} \in C([0,T]; L^2(S_h))$.

Here D^2_ℓ is the piece-wise second order derivative along the sides of F_h and D^2_y the usual second order partial derivative in y.

The proof relies on some regularity results for u^i_h in the polyhedral domains Q^i_h and on the use of Green's formula for domains with Lipschitz boundary and some trace theorems. In particular, we use weighted estimates in Sobolev spaces (see [28]), which take into account the singularities due to the presence of wedges and corners (for details on the proof see Theorem 6.3 in [19] and Section 4.2 of [17]).

We remark that there are many definitions of trace Sobolev spaces on polyhedral domains. If $0 \leq s \leq 1$ the Sobolev space $H^s(S_h)$ defined in [1] coincides, with equivalent norms, with the Sobolev space defined in [29] by local Lipschitz charts. For large s the definitions of trace spaces are more complicated (see [8]); in this paper we confine ourselves to the case which best fits our situation following [1].

From Theorem 16, it follows that the solution of problem (P_h) is the solution of the following transmission problem. For every $t \in [0,T]$,

$$\begin{cases} u_t^i = \Delta u^i + f & \text{in } L^2(Q_h^i),\ i = 1,2 & j) \\ -\Delta_{S_h} u = [\frac{\partial u}{\partial n}] & \text{in } L^2(S_h) & jj) \\ u = 0 & \text{in } H^{\frac{1}{2}}(\partial Q) & jjj) \\ u^1 = u^2 & \text{in} H^1(S_h) & jv) \\ u = 0 & \text{in } H^{\frac{1}{2}}(\partial S_h) & v). \end{cases}$$

Remark 2. By proceeding as in the proof of Theorem 4.4 in [17] we can claim that for every fixed $t \in [0,T]$, $u_h^1 \in H^{s_1}(Q_h^1), s_1 < \frac{8}{5}, u_h^2 \in H^{s_2}(Q_h^2), s_2 < \frac{7}{4}$ and $u \in C(\overline{Q})$.

By proceeding as in Theorem 8.2 of [21] it is possible to prove the asymptotic convergence of the transmission conditions.

Theorem 17. In the same assumptions of Theorem 14, we have that, for every fixed $t \in [0,T]$, each normal derivative is weakly convergent to the corresponding normal derivative on S, *i.e.*

(j) $\int_{S_h} \frac{\partial u_h^i}{\partial n_i} v\, d\sigma \rightarrow < \frac{\partial u^i}{\partial n_i}, v >_{(B^{2,2}_{\beta,0}(S))', B^{2,2}_{\beta,0}(S)}$, for every $v \in H_0^1(Q)$; $\beta = \frac{d_f}{2}$.

Moreover the both sides of the transmission condition in (32) converge to the corresponding terms in (30) in the dual space of $V(Q,S)$, more precisely:

(jj) $\int_{S_h} [\frac{\partial u_h}{\partial n}] v\, d\sigma \rightarrow < [\frac{\partial u}{\partial n}], v >_{(\mathcal{D}(S))', \mathcal{D}(S)}$, for every $v \in V(Q,S)$.

(jjj) $\int_{S_h} \Delta_{S_h} u_h v\, d\sigma \rightarrow < \Delta_S u, v >_{(\mathcal{D}(S))', \mathcal{D}(S)}$, for every $v \in V(Q,S)$.

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