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A GENERALIZATION OF LAX-MILGRAM'S THEOREM

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The main purpose of this note is to prove the following statement:

Theorem 1. *Let H be a real Hilbert space and A a linear operator on H. If*

$$\inf_{\|x\|=1}(\langle Ax, x\rangle + \|Ax\|) > 0$$

the operator A is continuous and invertible.

This is a generalization of the well-known theorem of Lax-Milgram since if

$$\inf_{\|x\|=1}\langle Ax, x\rangle > 0$$

the previous inequality holds. In fact we shall even prove the following improvements of Theorem 1:

Theorem 2. *Let H be a real Hilbert space and A a linear operator on H. Let $(y_1, y_2, \dots, y_k)$ be an orthonormal family in H and $\gamma > 0$. If*

$$\inf_{\|x\|=1}\left(\langle Ax, x\rangle + \|Ax\| + \gamma\Big(\sum_{j=1}^{k}\langle Ax, y_j\rangle^2\Big)^{1/2}\right) > 0$$

the operator A is continuous and invertible.

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Theorem 3. *Let H be a real Hilbert space and A a linear operator on H. Let K be a compact linear operator on H. If*

$$\inf_{\|x\|=1}(\langle Ax, x\rangle + \|Ax\| + \|KAx\|) > 0$$

the operator A is continuous and invertible.

Both Theorem 1 and Theorem 2 are answers to questions asked by B. Ricceri.

If A satisfies the hypothesis of Theorem 2, we put

$$\delta = \inf_{\|x\|=1}\left(\langle Ax, x\rangle + \|Ax\| + \gamma\Big(\sum_{j=1}^{k}\langle Ax, y_i\rangle^2\Big)^{1/2}\right) > 0.$$

Then we have for every x in the unit sphere

$$(*) \qquad \langle Ax, x\rangle + \|Ax\|\,\|x\| + \gamma\left(\sum_{j=1}^{k}\langle Ax, y_i\rangle^2\right)^{1/2}\|x\| \geq \delta\|x\|^2$$

and by homogeneity this inequality holds for every $x \in H$.

Notations. We will denote by V the linear space spanned by $(y_1, y_2, \ldots, y_k)$ and by $F = V^{\perp}$ its orthogonal. V is a closed subspace of H of dimension k, and we denote by π the orthogonal projector on V. Then for every u in H we have

$$\sum_{j=1}^{k}\langle u, y_j\rangle^2 = \|\pi u\|^2.$$

So $(*)$ becomes

$$(**) \qquad \langle Ax, x\rangle + \|Ax\|\,\|x\| + \gamma\|\pi Ax\|\,\|x\| \geq \delta\|x\|^2.$$

We denote by I the identity operator on H.

Lemma 4. *Let A be a linear operator on H satisfying* $(**)$. *Then* $\ker(A) = 0$.

Proof. We have

$$\begin{aligned}\delta\|x\|^2 &\leq \langle Ax, x\rangle + \|Ax\|\,\|x\| + \gamma\|\pi Ax\|\,\|x\| \\ &\leq \|Ax\|\,\|x\| + \|Ax\|\,\|x\| + \gamma\|Ax\|\,\|x\| = (\gamma+2)\|Ax\|\,\|x\|\end{aligned}$$

hence $\delta\|x\| \leq (\gamma+2)\|Ax\|$, and $x = 0$ if $Ax = 0$. □

Lemma 5. *Let A be a linear operators on H satisfying* $(**)$. *Then for any real $t \geq 0$ and any $x \in F$ the following inequality holds*

$$\|(A + tI)x\| \geq \frac{\delta}{\gamma+2}\|x\|.$$

Proof. Since $x \in F = V^{\perp}$,

$$\pi(A + tI)x = \pi Ax + t\pi x = \pi Ax,$$

$$\langle (A + tI)x, x\rangle + \|(A + tI)x\|\,\|x\| = \langle Ax, x\rangle + t\|x\|^2 + \|Ax + tx\|\,\|x\| \geq$$
$$\geq \langle Ax, x\rangle + t\|x\|^2 + \|Ax\|\,\|x\| - \|tx\|\,\|x\| = \langle Ax, x\rangle + \|Ax\|\,\|x\|.$$

Hence

$$\langle (A + tI)x, x\rangle + \|(A + tI)x\|\,\|x\| + \gamma\|\pi(A + tI)x\|\,\|x\| \geq$$
$$\geq \langle Ax, x\rangle + \|Ax\|\,\|x\| + \gamma\|\pi Ax\|\,\|x\| \geq \delta\|x\|^2$$

and since

$$\langle (A + tI)x, x\rangle + \|(A + tI)x\|\,\|x\| + \gamma\|\pi(A + tI)x\|\,\|x\| \leq$$
$$\leq \|(A + tI)x\|\,\|x\| + \|(A + tI)x\|\,\|x\| + \gamma\|(A + tI)x\|\,\|x\| =$$
$$= (\gamma + 2)\|(A + tI)x\|\,\|x\|$$

we get

$$(\gamma + 2)\|(A + tI)x\|\,\|x\| \geq \delta\|x\|^2$$

and this completes the proof of the Lemma. □

Corollary 6. *Let A be a continuous linear operator on H satisfying $(**)$. Then for every real $t \geq 0$ the space $(A + tI)(F)$ is closed in H. Moreover $A + tI$ is one-to-one on F.*

Proof. From Lemma 5 we get clearly $F \cap \ker(A + tI) = \{0\}$. Hence $A + tI$ is one-to-one on F. Let (u_n) be a sequence in $(A + tI)(F)$ converging to u in H. If $u_n = (A + tI)x_n$ with $x_n \in F$, we have by Lemma 5

$$\|x_n - x_m\| \leq \frac{\gamma + 2}{\delta}\|u_n - u_m\|$$

and this shows that the sequence (x_n) is a Cauchy sequence. Since F is complete, the sequence (x_n) converges to some $x \in F$, and since A is continuous,

$$(A + tI)x = \lim_{n\to\infty}(A + tI)x_n = \lim_{n\to\infty} u_n = u.$$

Thus $u \in (A + tI)(F)$. □

Lemma 7. *Let A be a continuous linear operator on H satisfying $(**)$. Then for every real $t \geq 0$ the space $(A + tI)(H)$ has codimension k in H.*

Proof. Denote by T the set

$$T = \{t \geq 0 : (A + tI)(F) \text{ is not of codimension } k \text{ in } H\}.$$

We want to prove $T = \theta$.

First of all, notice that, for $t > \|A\|$, the operator $A+tI$ is invertible on H. Hence the subspaces $(A + tI)(F)$ and $(A + tI)(V)$ are direct summands. And since $\dim((A + tI)(V)) = \dim(V) = k$, t does not belong to T. This shows that $T \subset [0, \|A\|]$.

Thus if T is not empty, it has an upper bound θ. Denote by E_θ the orthogonal subspace of $(A + \theta I)(F)$ and by S_t the operator from $F \times E_\theta$ to H defined by

$$S_t(x, u) = (A + tI)x + u.$$

Then S_t is continuous for each $t \geq 0$. Moreover S_θ is one-to-one since $H = (A+\theta I)(F) \oplus E_\theta$ and since $A+\theta I$ is one-to-one from F to $(A+\theta I)(F)$ by Corollary 6.

Since S_θ is invertible, there exists some $\rho > 0$ such that S_t is invertible for $|t - \theta| < \rho$. In particular for $t = \theta + \frac{\rho}{2} > \theta$, $(A + tI)(F)$ has codimension k in H. And since S_t is invertible, $(A + tI)(F)$ and E_θ are direct summands. Hence $\dim(E_\theta) = k$. Similarly for $\theta - \rho < t \leq \theta (A + tI)(F)$ and E_θ are direct summands. Hence $(A + tI)(F)$ has codimension $\dim(E_\theta) = k$, and $T \cap]\theta - \rho, \theta] = \theta$. This shows that $\theta = \sup T \leq \theta - \rho$, a contradiction. This completes the proof. □

Theorem 8. *Let A be a continuous linear operator on H satisfying* $(**)$. *Then A is invertible.*

Proof. By Lemma 7, $A(F)$ has codimension k. By Lemma 4, A is one-to-one. Hence $A(V)$ has dimension k and $A(F) \cap A(V) = \{0\}$. Thus $A(H) = A(F) + A(V) = H$. Since A is continuous and one-to-one from H to H, it is invertible. □

Theorem 9. *Let A and K be continuous linear operators on H, K being compact. If*

$$\inf_{\|x\|=1} (\langle Ax, x\rangle + \|Ax\| + \|KAx\|) > 0$$

then A is invertible.

Proof. Put $\gamma = \|K\|$ and $\delta = \inf_{\|x\|=1}(\langle Ax, x\rangle + \|Ax\| + \|KAx\|)$. We have, as at the beginning of this paper,

$$(***) \qquad \langle Ax, x\rangle + \|Ax\| \, \|x\| + \|KAx\| \, \|x\| \geq \delta \|x\|^2.$$

Then K^*K is compact, symmetric and positive, and so is $K_1 = (K^*K)^{1/2}$. Moreover $\gamma = \|K\| = \|K_1\|$. For every $y \in H$ we have

$$\|K_1 y\|^2 = \langle K_1 y, K_1 y\rangle = \langle K_1^2 y, y\rangle = \langle K^*K y, y\rangle = \langle K y, K y\rangle = \|K y\|^2.$$

So $(***)$ can be rewritten as

$$\langle Ax, x\rangle + \|Ax\|\,\|x\| + \|K_1 Ax\|\,\|x\| \geq \delta\|x\|^2.$$

Hence we can and do assume that K is symmetric. There exists an orthonormal sequence (y_i) and a sequence of non-negative eigenvalues (λ_j) such that, for every $y \in H$

$$\|K y\|^2 = \sum_{j=1}^{\infty} \lambda_j^2 \langle y, y_j\rangle^2.$$

Moreover the sequence (λ_j) converges to 0. Put $\varepsilon = \dfrac{\delta}{2\|A\|}$. Thus there exists some k such that $\lambda_j < \varepsilon$ for $j > k$. And $\lambda_j \leq \gamma$ for all j. Then for every x in H we have

$$\begin{aligned}
\|K Ax\|^2 &= \sum_{j=1}^{\infty} \lambda_j^2 \langle Ax, y_j\rangle^2 \\
&\leq \sum_{j=1}^{k} \gamma^2 \langle Ax, y_j\rangle^2 + \sum_{j=k+1}^{\infty} \varepsilon^2 \langle Ax, y_j\rangle^2 \\
&\leq \gamma^2 \sum_{j=1}^{k} \langle Ax, y_j\rangle^2 + \varepsilon^2 \sum_{j=1}^{\infty} \langle Ax, y_j\rangle^2 \\
&\leq \gamma^2 \sum_{j=1}^{k} \langle Ax, y_j\rangle^2 + \varepsilon^2 \|Ax\|^2 \\
&\leq \gamma^2 \sum_{j=1}^{k} \langle Ax, y_j\rangle^2 + \varepsilon^2 \|A\|^2 \|x\|^2 \\
&\leq \gamma^2 \sum_{j=1}^{k} \langle Ax, y_j\rangle^2 + \frac{\delta^2}{4} \|x\|^2
\end{aligned}$$

hence

$$\|KAx\| \leq \frac{\delta}{2}\|x\| + \gamma \sqrt{\sum_{j=1}^{k} \langle Ax, y_j \rangle^2}$$

and from (∗∗∗) we get

$$\langle Ax, x \rangle + \|Ax\| \, \|x\| + \gamma \sqrt{\sum_{j=1}^{k} \langle Ax, y_j \rangle^2} \, \|x\| \geq$$

$$\geq \langle Ax, x \rangle + \|Ax\| \, \|x\| + \Big(\|KAx\| - \frac{\delta}{2}\|x\| \Big) \|x\| \geq \frac{\delta}{2}\|x\|^2$$

This shows that A satisfies (∗∗) (with $\delta/2$ instead of δ). Then the conclusion follows from Theorem 8. □

From now on, the operator A is not assumed to be continuous. We denote by W the closed linear subspace of H defined by

$$W = \Big\{ w \in H : \text{ there exists a sequence } (x_n) \text{ such that } \lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} Ax_n = w \Big\}.$$

Lemma 10. *Let X and Y be two Banach spaces and Φ a continuous linear mapping from X onto Y. Then for every closed linear subspace G of X containing the kernel of Φ, $\Phi(G)$ is closed in Y.*

Proof. By Banach's Theorem, Φ is an open mapping. Thus $\Phi(X \setminus G)$ is an open subset of Y. Now it is enough to notice that

$$Y \setminus \Phi(G) = \Phi(X \setminus G)$$

for completing the proof. □

For A a linear operator on H, we denote by q the orthogonal projector on $W^\perp$, and we put $A' = qA$.

Lemma 11. *A' is continuous.*

Proof. Denote by G_A (resp. $G_{A'}$) the graph of A (resp. A') in $H \times H$. If $\Phi(x, y) = (x, qy)$, we have $G_{A'} = \Phi(G_A)$.

By the definition of W, we have $W_0 = \{0\} \times W \subset \overline{G_A}$, hence $G_A + W_0 \subset \overline{G_A}$. Conversely, if $(x, y) \in \overline{G_A}$, there exists a sequence $((x_n, y_n))$ in G_A

converging to (x, y). Then $(x_n - x, Ax_n - Ax) \in G_A$, $x_n - x \to 0$ and $Ax_n - Ax \to y - Ax$. We conclude that $y - Ax \in W$ and that $(x, y) = (x, Ax) + (0, y - Ax) \in G_A + W_0$. Hence $\overline{G_A} = G_A + W_0$.

Since $\overline{G_A} \supset W_0 = \ker(\Phi)$, Lemma 10 implies that $\Phi(\overline{G_A})$ is closed, but

$$\Phi(\overline{G_A}) = \Phi(G_A + (\{0\} \times W)) = \Phi(G_A) + \Phi(W_0) = \Phi(G_A) = G_{A'}$$

Hence $G_{A'}$ is closed, and A' is continuous. □

Lemma 12. *Let H_1 and H_2 be Hilbert spaces, A a linear operator from H_1 to H_2, K a compact linear operator on H_2. If H_1 is infinite-dimensional, then for every $\varepsilon > 0$ there exists some u in the unit sphere of H_1 such that $\|KAu\| < \varepsilon$.*

Proof. If not we have $\|KAx\| \geq \varepsilon\|x\|$ for any $x \in H_1$. Then for $x \in H_1$, $w = A'x - Ax \in W$, and there exists some sequence (x_n) converging to 0 such that $Ax_n \to w$. Then the sequence $(KA(x + x_n))$ converges to $K(Ax + w) = KA'x$, and since

$$\|KA(x + x_n)\| \geq \varepsilon\|x + x_n\|$$

we conclude that

$$\|KA'x\| \geq \varepsilon\|x\| \tag{1}$$

but, since A' is continuous by Lemma 11, KA' is compact, and this contradicts (1). □

Lemma 13. *If A satisfies $(**)$ or $(***)$, so does A'.*

Proof. Since $\gamma\pi$ is a compact operator, it is enough to prove the Lemma for $(***)$. Let $x \in H$. Then $w = A'x - Ax \in W$. There exists a sequence (x_n) converging to 0 such that $w = \lim_{n\to\infty} Ax_n$. Applying $(***)$ to $x + x_n$ we get

$$\langle A(x + x_n), x + x_n\rangle + \|A(x + x_n)\|\,\|x + x_n\| + \|KA(x + x_n)\|\,\|x + x_n\| \geq$$
$$\geq \delta\|x + x_n\|^2$$

and by letting n go to the infinity

$$\langle Ax + w, x\rangle + \|Ax + w\|\,\|x\| + \|K(Ax + w)\|\,\|x\| \geq \delta\|x\|^2,$$

$$\langle A'x, x\rangle + \|A'x\|\,\|x\| + \|KA'x\|\,\|x\| \geq \delta\|x\|^2.$$

This last inequality shows that A' satisfies $(***)$. □

Proof of Theorem 3. If A satisfies the hypothesis of Theorem 3, it satisfies $(***)$. Then A' is continuous by Lemma 11 and satisfies $(***)$ by Lemma 13. Thus A' invertible by Theorem 9. In particular

$$H = A'(H) = q(A(H)) \subset W^{\perp}.$$

Hence $W^{\perp} = H$, $q = I$ and $A' = qA = A$. This proves that A is continuous and invertible. □

Proof of Theorem 2. If A satisfies the hypothesis of Theorem 2, it satisfies $(**)$. And since $\gamma\pi$ is a compact operator, the conclusion follows from Theorem 3.

□

We finish this note by reproving an earlier (unpublished) result of the author:

Theorem 14. *Let A and K be linear operators on the Hilbert space H, K being compact. If*

$$\inf_{\|x\|=1} \Big(|\langle Ax, x\rangle| + \|KAx\| \Big) > 0$$

then A is continuous and invertible.

Proof. If A satisfies the previous hypotheses, it is clearly one-to-one. So we can assume H is infinite-dimensional. It is enough to prove that either A or $-A$ satisfies the hypotheses of Theorem 3.

Put $\delta = \inf_{\|x\|=1} \Big(|\langle Ax, x\rangle| + \|KAx\| \Big)$, $\gamma = \|K\|$ and $\varepsilon = \dfrac{\delta}{\gamma + 2}$. Then

$$\begin{aligned} \delta\|x\|^2 &\le |\langle Ax, x\rangle| + \|KAx\|\,\|x\| \le \\ &\le \|Ax\|\,\|x\| + \|K\|\,\|Ax\|\,\|x\| \le (\gamma+1)\|Ax\|\,\|x\| \end{aligned}$$

and

$$\|Ax\| \ge \frac{\delta}{\gamma+1}\|x\|.$$

If A does not satisfy the hypotheses of Theorem 3, there is some x_1 in the unit sphere of H such that

$$\langle Ax_1, x_1\rangle + \|Ax_1\| + \|KAx_1\| < \varepsilon$$

and thus

$$\text{(2)} \quad \|KAx_1\| < \varepsilon \quad \text{and} \quad \langle Ax_1, x_1\rangle < \varepsilon - \|Ax_1\| \le \frac{\delta}{\gamma+2} - \frac{\delta}{\gamma+1} < 0.$$

Similarly if $-A$ does not satisfy the hypotheses of Theorem 3, there is some x_2 in the unit sphere such that

$$\langle -Ax_2, x_2\rangle + \|-Ax_2\| + \|-KAx_2\| < \varepsilon$$

and thus

$$\text{(3)}\quad \|KAx_2\| < \varepsilon \quad \text{and} \quad \langle Ax_2, x_2\rangle > -\varepsilon + \|Ax_2\| \geq -\frac{\delta}{\gamma+2} + \frac{\delta}{\gamma+1} > 0.$$

Let V be the linear space spanned by x_1 and x_2. By Lemma 12 applied to $H_1 = V^{\perp}$ and $H_2 = H$, there exists some $u \in V^{\perp}$ such that $\|u\| = 1$ and $\|KAu\| \leq \frac{\delta}{6}$. Then we put for $t \in [0, 1]$

$$\psi(t) = tx_2 + (1-t)x_1 + \Big(1 - \|tx_2 + (1-t)x_1\|^2\Big)^{1/2} u.$$

We have $\|\psi(t)\| = 1$ for all t, and

$$\langle A\psi(0), \psi(0)\rangle = \langle Ax_1, x_1\rangle < 0 < \langle A\psi(1), \psi(1)\rangle = \langle Ax_2, x_2\rangle.$$

Thus since the restriction of A to the space $V \oplus \mathbb{R}u$ is continuous, there is some $t^* \in [0, 1]$ such that $\langle A\psi(t^*), \psi(t^*)\rangle = 0$ and

$$\begin{aligned}\|KA\psi(t^*)\| &\leq t^*\|KAx_2\| + (1-t^*)\|KAx_1\| + \\ &\quad + \Big(1 - \|t^*x_2 + (1-t^*)x_1\|^2\Big)^{1/2}\|KAu\| \leq \\ &\quad \leq t^*\varepsilon + (1-t^*)\varepsilon + \|KAu\| \leq \varepsilon + \frac{\delta}{6} \leq \frac{\delta}{2} + \frac{\delta}{6} < \delta\end{aligned}$$

hence

$$|\langle A\psi(t^*), \psi(t^*)\rangle| + \|KA\psi(t^*)\| < \delta$$

a contradiction. Thus either A or $-A$ satisfies the hypothesis of Theorem 3 and the proof is complete. $\square$

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