LE MATEMATICHE
Vol. LII (1997) – Fasc. I, pp. 169–176

# OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF HIGHER ORDER DIFFERENCE EQUATIONS

BŁAŻEJ SZMANDA

This paper is concerned with the oscillation and asymptotic behaviour of nonoscillatory solutions of nonlinear difference equation of the form

$$\Delta^m(u_n + p_n u_{n-k}) + \delta f(n, u_{\tau_n}) = 0, \quad m \geq 1,\ n \in \mathbb{N},$$

where $\mathbb{N} = \{1, 2, \ldots\}$, $\delta = \pm 1$, $k \in \mathbb{N}$, $\Delta^m$ is the $m$-th order forward difference operator.

## 1. Introduction.

In the past several years there has been a lot of activity concerning the oscillatory and asymptotic behaviour of solutions of difference equations. See for example [1] – [3], [6], [9], [12], [13], [16], and the references cited therein. In particular, there has been an increasing interest in the study of difference equations of the form which can be viewed as a discrete analogues of delay and neutral delay differential equations (see e.g. [5], [8], [10], [11], [14], [15], [17]). For the general theory of difference equations one can refer to [4] and [7].

In this paper we consider the nonlinear difference equation of the form

$$(E) \qquad \Delta^m(u_n + p_n u_{n-k}) + \delta f(n, u_{\tau_n}) = 0, \quad m \geq 1,\ n \in \mathbb{N},$$

Entrato in Redazione il 14 gennaio 1997.

where $\mathbb{N} = \{1, 2, \ldots\}$, $\delta = \pm 1$, $k \in \mathbb{N}$, $\Delta$ is the forward difference operator i.e. $\Delta v_n = v_{n+1} - v_n$ and

$$\Delta^i v_n = \Delta(\Delta^{i-1} v_n),\ i = 1, 2, \ldots, m, \Delta^0 v_n = v_n;$$

$(p_n)$ is a sequence of real numbers, $(\tau_n)$ is a sequence of integers with $\tau_n \le n$ and $\tau_n \to \infty$ as $n \to \infty$, $f : \mathbb{N} \times R \to R = (-\infty, \infty)$, $uf(n, u) > 0$ for $u \neq 0$ and $n \in \mathbb{N}$.

By a solution of $(E)$ we mean a sequence $(u_n)$ which is defined for all $n \ge \min_{i \ge 1}\{i - k, \tau_i\}$ and satisfies $(E)$ for all large $n$. We consider only such solutions which are nontrivial for all large $n$. A nontrivial solution $(u_n)$ is said to be oscillatory if for every $n_0 \in \mathbb{N}$ there exists $n \ge n_0$ such that $u_n u_{n+1} \le 0$. Otherwise it is called nonoscillatory.

Our purpose in this paper is to study the oscillatory and asymptotic properties of the solutions of equation $(E)$. Recently, the above problem for the difference equation of the form $(E)$ in the case of the second order difference operator have been discussed in [15]. The results obtained here extend some of those contained in [15].

## 2. Main results.

We begin with a lemma that will be utilized in the proof of our main results.
We will need the condition that if $(v_n)$ is a real sequence with $v_n > 0$ $(< 0)$ and $\liminf_{n \to \infty} |v_n| > 0$, then

$$\sum_{n=1}^{\infty} f(n, v_n) = \infty(-\infty). \tag{1}$$

Also, we will use the following conditions:

(2) there is a constant $P_1$ such that $-1 < P_1 \le p_n \le 0$;
(3) there is a constant $P_2$ such that $0 \le p_n \le P_2 < 1$.

Let $(u_n)$ be a solution of $(E)$. Set $z_n = u_n + p_n u_{n-k}$.

**Lemma.** *Suppose that (1) holds and $(u_n)$ is an eventually positive (negative) solution of $(E)$.*

(a) *If $\delta = 1$, then*

(i) $(\Delta^{m-1}z_n)$ *is eventually decreasing (increasing) and* $\Delta^{m-1}z_n \to L < \infty(> -\infty)$ *as* $n \to \infty$.

(ii) *If* $L > -\infty(< \infty)$, *then* $\liminf_{n\to\infty} |u_n| = 0$.

(iii) *If* $z_n \to 0$ *as* $n \to \infty$, *then* $(\Delta^i z_n)$ *is monotonic and*

$$\Delta^i z_n \to 0 \text{ as } n \to \infty \text{ and } \Delta^i z_n \Delta^{i+1} z_n < 0 \tag{4}$$

*for* $i = 0, 1, 2, \ldots, m-1$.

(iv) *Let* $z_n \to 0$ *as* $n \to \infty$. *If* $m$ *is even, then* $z_n < 0(z_n > 0)$ *for* $u_n > 0(u_n < 0)$. *If* $m$ *is odd, then* $z_n > 0(z_n < 0)$ *for* $u_n > 0(u_n < 0)$.

(v) *If, in addition,* (2) *holds, then* $z_n \to 0$ *as* $n \to \infty$.

(b) *If* $\delta = -1$, *then*

(i) $(\Delta^{m-1}z_n)$ *is eventually increasing (decreasing) and* $\Delta^{m-1}z_n \to L > -\infty(< \infty)$ *as* $n \to \infty$.

(ii) *If* $L < \infty(> -\infty)$, *then* $\liminf_{n\to\infty} |u_n| = 0$.

(iii) *Let* $z_n \to 0$ *as* $n \to \infty$. *If* $m$ *is even, then* $z_n > 0(z_n < 0)$ *for* $u_n > 0(u_n < 0)$. *If* $m$ *is odd, then* $z_n < 0(z_n > 0)$ *for* $u_n > 0(u_n < 0)$.

(iv) *If, in addition,* (2) *holds, then either* $|u_n| \to \infty$ *as* $n \to \infty$, *or* $(\Delta^i z_n)$ *is monotonic and* (4) *holds.*

*Proof.* Let $(u_n)$ be an eventually positive solution of $(E)$. Then there is $n_0 \in \mathbb{N}$ such that $u_{n-k} > 0$ and $u_{\tau_n} > 0$ for $n \geq n_0$.

(a) From $(E)$ we have $\Delta^m z_n = -f(n, u_{\tau_n}) < 0$, so $(\Delta^{m-1}z_n)$ is decreasing and converges to $L < \infty$ as $n \to \infty$. Thus (i) holds.

If $L > -\infty$, then summing $(E)$ from $n_0$ to $n$ and then letting $n \to \infty$, we have

$$\sum_{i=n_0}^{\infty} f(i, u_{\tau_i}) = \Delta^{m-1}z_{n_0} - L < \infty.$$

The last inequality and condition (1) imply that $\liminf_{n\to\infty} u_n = 0$, so (ii) holds.

To prove (iii), suppose $z_n \to 0$ as $n \to \infty$. Then we see that $\Delta^i z_n \to 0$ as $n \to \infty$ for $i = 1, 2, \ldots, m-1$. By (i), $(\Delta^{m-1}z_n)$ is decreasing, so $\Delta^{m-1}z_n > 0$ for $n \geq n_0$. Hence, if $m \geq 2$, then $(\Delta^{m-2}z_n)$ is increasing and so $\Delta^{m-2}z_n < 0$ for $n \geq n_0$. Continuing in this manner we obtain (iii).

Part (iv) follows immediately from (iii).

In order to prove (v), first note that from (i) and (ii) we have that $(\Delta^{m-1}z_n)$ is decreasing, $\Delta^{m-1}z_n \to L \geq -\infty$ as $n \to \infty$ and $\liminf_{n\to\infty} u_n = 0$ if $L > -\infty$. If $L = -\infty$, then successive summations show that $z_n \to -\infty$ as $n \to \infty$ so

$z_n < 0$ for $n \geq n_1 \geq n_0$. Since $p_n > -1$, we have $u_n < -p_n u_{n-k} < u_{n-k}$. This implies that $(u_n)$ is bounded contradicting the fact that $z_n \to -\infty$ as $n \to \infty$. If $-\infty < L < 0$, then by summations as above, we see that $z_n \leq L_1$ for some constant $L_1 < 0$ and sufficiently large $n$. By (2) we have

$$P_1 u_{n-k} \leq p_n u_{n-k} < z_n \leq L_1 < 0,$$

which contradicts $\liminf_{n\to\infty} u_n = 0$. Hence $L \geq 0$. If $L > 0$, then we have $\Delta^{m-1} z_n \geq L$ and a summation shows that $z_n \to \infty$ as $n \to \infty$. Since $u_n \geq z_n$ hence $u_n \to \infty$ as $n \to \infty$, a contradiction. Thus, we have $L = 0$, i.e. $\Delta^{m-1} z_n \to 0$ as $n \to \infty$. Since $(\Delta^{m-1} z_n)$ is decreasing, hence $\Delta^{m-1} z_n > 0$, so $(\Delta^{m-2} z_n)$ is increasing. Moreover, $\Delta^{m-2} z_n < 0$ since otherwise $(\Delta^{m-2} z_n)$ would be eventually positive and increasing, which in turn implies that $(z_n)$ has a positive lower bound and contradicts $\liminf_{n\to\infty} u_n = 0$. If $\Delta^{m-2} z_n \to L_2 < 0$ as $n \to \infty$, then $\Delta^{m-2} z_n \leq L_2$ and a summation shows that eventually $z_n \leq L_3$ for some negative constant $L_3$. This again contradicts $\liminf_{n\to\infty} u_n = 0$. Therefore, $(\Delta^{m-2} z_n)$ is increasing and tends to zero as $n \to \infty$, continuing this form of argument we see that $z_n \to 0$ as $n \to \infty$.

(b) The proofs of (i) – (iii) are similar to the proofs of the corresponding parts in (a) and will be omitted.

To prove (iv), let $(u_n)$ be an eventually positive solution of $(E)$ and let $u_{n-k} > 0$ and $u_{\tau_n} > 0$ for $n \geq n_0 \in \mathbb{N}$. By parts (i) and (ii) of (b), we have that $(\Delta^{m-1} z_n)$ is increasing, $\Delta^{m-1} z_n \to L \leq \infty$ as $n \to \infty$ and $\liminf_{n\to\infty} u_n = 0$ if $L < \infty$. If $L = \infty$, then $z_n \leq u_n \to \infty$ as $n \to \infty$. If $L < 0$, then eventually $z_n \leq L_1$ for some $L_1 < 0$. But then $P_1 u_{n-k} \leq p_n u_{n-k} < z_n \leq L_1 < 0$ contradicting $\liminf_{n\to\infty} u_n = 0$. Hence $L \geq 0$. If $L > 0$, then eventually $u_n \geq z_n \geq L_2$ for some constant $L_2 > 0$, which again contradicts $\liminf_{n\to\infty} u_n = 0$. Thus, $\Delta^{m-1} z_n \to 0$ as $n \to \infty$. Moreover, $\Delta^{m-1} z_n < 0$ since $(\Delta^{m-1} z_n)$ is increasing. Therefore, $(\Delta^{m-2} z_n)$ is decreasing. Also, $\Delta^{m-2} z_n > 0$ since otherwise $(\Delta^{m-2} z_n)$ is eventually negative and decreasing, which implies $(z_n)$ has a negative upper bound, contradicting $\liminf_{n\to\infty} u_n = 0$. Furthermore, if $\Delta^{m-2} z_n \to L_3 > 0$ as $n \to \infty$, then eventually $z_n \geq L_4$ for some $L_4 > 0$. But this again contradicts $\liminf_{n\to\infty} u_n = 0$. Therefore, $(\Delta^{m-2} z_n)$ is decreasing and tends to zero as $n \to \infty$. Continuing in this manner we see that (4) holds.

The proof when $(u_n)$ is eventually negative is similar and will be omitted. □

**Theorem 1.** *Suppose that conditions* (1) *and* (2) *hold.*

(i) *If* $\delta = 1$, *then any solution* $(u_n)$ *of* $(E)$ *is either oscillatory or satisfies* $u_n \to 0$ *as* $n \to \infty$.

(ii) *If* $\delta = -1$, *then either* $(u_n)$ *is oscillatory,* $u_n \to 0$ *as* $n \to \infty$ *or* $|u_n| \to \infty$ *as* $n \to \infty$.

*Proof.* Let $(u_n)$ be a nonoscillatory solution of $(E)$ such that $u_{n-k} > 0$ and $u_{\tau_n} > 0$ for $n \geq n_0 \in \mathbb{N}$.

(i) Part (iii) and (v) of Lemma (a) imply that (4) holds. For $m$ even, condition (2) and Lemma (a) (iv) imply that $u_n < -P_1 u_{n-k}$ for $n \geq n_0$. Hence $u_{n+k} < -P_1 u_n$, and by induction we have $u_{n+ik} < (-P_1)^i u_n$ for each positive integer $i$. This implies $u_n \to 0$ as $n \to \infty$ since $0 < -P_1 < 1$. If $m$ is odd, then (2) and parts (iv) and (v) of Lemma (a) imply that $0 < z_n < M$ for some constant $M > 0$, from which it follows that $0 < u_n < -P_1 u_{n-k} + M$. If $(u_n)$ is unbounded, then there exists an increasing sequence $(n_i)$ such that $n_1 > n_0$, $n_i \to \infty$ and $u_{n_i} \to \infty$ as $i \to \infty$ and $u_{n_i} = \max_{n_1 \leq n \leq n_i} u_n$. For each $i$ we have

$$u_{n_i} < -P_1 u_{n_i - k} + M \leq -P_1 u_{n_i} + M$$

so $(1 + P_1)u_{n_i} \leq M$ which is impossible since (2) holds. Therefore, $(u_n)$ is bounded and there exists $a > 0$ such that $\limsup_{n\to\infty} u_n = a$. Thus, there is a subsequence of $(u_n)$, say $(u_{s_i})$ with $s_1 > n_0$ and $u_{s_i} \to a$ as $i \to \infty$. From (2) it follows that $-P_1 u_{s_i - k} \geq u_{s_i} - z_{s_i}$. Since $a > 0$, there exists $\varepsilon > 0$ such that $(1 - P_1)\varepsilon < (1 + P_1)a$ which implies $0 < -P_1(a + \varepsilon) < a - \varepsilon$. But for $i$ sufficiently large, $u_{s_i - k} < a + \varepsilon$, so

$$a - \varepsilon > -P_1 u_{s_i - k} \geq u_{s_i} - z_{s_i}.$$

As $i \to \infty$, $z_{s_i} \to 0$ so we obtain a contradiction to $u_{s_i} \to a$ as $i \to \infty$.

(ii) First notice that by Lemma (b) (iv) we have that either $u_n \to \infty$ as $n \to \infty$ or (4) holds. To complete the proof, we show that if (4) holds, then $u_n \to 0$ as $n \to \infty$. When (4) holds and $m$ is odd, part (iii) of Lemma (b) implies that $z_n < 0$, and hence from (2) we see that $u_n < -P_1 u_{n-k}$. It then follows, as before, that $u_{n+ik} < (-P_1)^i u_n$ for each integer $i \geq 1$. Since $0 < -P_1 < 1$, we have $u_n \to 0$ as $n \to \infty$. When (4) holds and $m$ is even, parts (iii) and (iv) of Lemma (b) imply that $z_n > 0$ and $\Delta z_n < 0$. Therefore, $0 < z_n < M$ for some constant $M > 0$, and from (2) we have $0 < u_n < -P_1 u_{n-k} + M$. The remainder of the proof that $u_n \to 0$ as $n \to \infty$ is the same as in part (i). □

**Theorem 2.** *Suppose that conditions* (1) *and* (3) *hold.*

(i) *If* $\delta = 1$ *and* $m$ *is even, then every solution of* $(E)$ *is oscillatory, while if* $m$ *is odd, then any solution* $(u_n)$ *of* $(E)$ *is either oscillatory or satisfies* $u_n \to 0$ *as* $n \to \infty$.

(ii) *If* $\delta = -1$ *and* $m$ *is even, then either* $(u_n)$ *is oscillatory,* $|u_n| \to \infty$ *or* $u_n \to 0$ *as* $n \to \infty$*, while if* $m$ *is odd, then either* $(u_n)$ *is oscillatory or* $|u_n| \to \infty$ *as* $n \to \infty$.

*Proof.* Assume that $(u_n)$ is an eventually positive solution of $(E)$, say $u_{n-k} > 0$ and $u_{\tau_n} > 0$ for $n \geq n_0$. The proof when $(u_n)$ is eventually negative is similar and will be omitted.

In order to prove (i), observe that by Lemma (a) (i) we have that $(\Delta^{m-1} z_n)$ is decreasing and converges to $L \geq -\infty$ as $n \to \infty$. Clearly, if $L < 0$, then $(z_n)$ is eventually negative which contradicts $u_n > 0$ for $n \geq n_0$. Hence, $L \geq 0$, and from Lemma (a) (ii), we have $\liminf_{n\to\infty} u_n = 0$. It is also clear, since $\Delta^m z_n < 0$, that $\Delta^i z_n$ is monotonic for $i = 0, 1, \ldots, m-1$.

Now $(z_n)$ is monotonic, so $z_n \to l$ as $n \to \infty$. Observe that $l \geq 0$ since $l < 0$ implies $u_n < 0$. First suppose $l > 0$. If $(z_n)$ is increasing, we have

$$z_n = u_n + p_n u_{n-k} \leq u_n + p_n z_{n-k} \leq u_n + P_2 z_n,$$

so $z_n(1 - P_2) \leq u_n$. Since (3) holds, we have a contradiction to $\liminf_{n\to\infty} u_n = 0$. On the other hand, if $(z_n)$ is decreasing, let $1 - P_2 = \varepsilon > 0$. Then $z_n \leq u_n + P_2 z_{n-k}$, and since $l$ is finite, $z_n/z_{n-k} \leq u_n/l + P_2$. Since $P_2 + \varepsilon/2 < 1$, there exists $n_1 > n_0$ such that $z_n/z_{n-k} \geq P_2 + \varepsilon/2$ for $n \geq n_1$. Therefore, $u_n \geq \frac{l\varepsilon}{2}$ for $n \geq n_1$ which again contradicts $\liminf_{n\to\infty} u_n = 0$. Hence, we have $z_n \to 0$ as $n \to \infty$ and, by Lemma (a) (iii), that (4) holds. In order to complete the proof, just observe that Lemma (a) (iv) implies that $z_n < 0$ for $m$ even and $z_n > 0$ for $m$ odd. But $z_n < 0$ contradicts $u_n > 0$ and $p_n \geq 0$, while if $z_n > 0$, then $u_n \leq z_n \to 0$ as $n \to \infty$.

To prove (ii), we first see that $(\Delta^{m-1} z_n)$ is increasing and $\Delta^{m-1} z_n \to L > -\infty$ as $n \to \infty$. Now, if $L < 0$, then eventually $z_n < 0$ contradicting $u_n > 0$. Thus $L \geq 0$. If $L = \infty$, then clearly $z_n \to \infty$ as $n \to \infty$. Moreover, from (3) we get

$$u_n \leq z_n \leq u_n + P_2 u_{n-k} \leq u_n + P_2 z_{n-k} \leq u_n + P_2 z_n \tag{5}$$

and therefore, $(1 - P_2) z_n \leq u_n \to \infty$ as $n \to \infty$. If $0 \leq L < \infty$, then Lemma (b) (ii) implies that $\liminf_{n\to\infty} u_n = 0$. Again, since $(z_n)$ is monotonic and positive $z_n \to l_1 \geq 0$ as $n \to \infty$. Now if $(z_n)$ is increasing, then $l_1 > 0$ and (5) holds contradicting $\liminf_{n\to\infty} u_n = 0$. If $(z_n)$ is decreasing and $l_1 > 0$, then $l_1$ is finite, and we have $z_n/z_{n-k} \to 1$ as $n \to \infty$. Let $\varepsilon = 1 - P_2 > 0$. Then there exists $n_1 \geq n_0$ such that $z_n/z_{n-k} > 1 - \varepsilon/2$ for $n \geq n_1$. We then have

$$\begin{aligned} l_1 < z_n &\leq u_n + P_2 u_{n-k} \leq u_n + P_2 z_{n-k} \\ &< u_n + \frac{P_2 z_n}{1 - \frac{\varepsilon}{2}} = u_n + \frac{2P_2 z_n}{1 + P_2}. \end{aligned}$$

It then follows that

$$u_n > \left(1 - \frac{2P_2}{1 + P_2}\right) z_n = \frac{1 - P_2}{1 + P_2} z_n > \frac{\varepsilon l_1}{2}$$

contradicting $\liminf_{n\to\infty} u(n) = 0$. Thus, for $0 \leq L < \infty$ $z_n \to 0$ as $n \to \infty$, and hence $u_n \to 0$ as $n \to \infty$ since $u_n \leq z_n$. Therefore, we have that either $u_n \to \infty$ or $u_n \to 0$ for $m$ even or $m$ odd.

To complete the proof, we need only observe that if $u_n \to 0$ as $n \to \infty$, then $z_n \to 0$ as $n \to \infty$, and for $m$ odd, Lemma (b) (iii) implies that $z_n < 0$, which is impossible in view of (3). □

## REFERENCES

[1] R.P. Agarwal, *Properties of solutions of higher order nonlinear difference equations,* An. Sti. Univ. Iasi, 31 (1985), pp. 165–172.

[2] R.P. Agarwal, *Properties of solutions of higher order nonlinear difference equations II,* An. Sti. Univ. Iasi, 29 (1983), pp. 85–96.

[3] R.P. Agarwal, *Difference calculus with applications to difference equations,* Int. Ser. Num. Math., 71 (1984), pp. 95–110.

[4] R.P. Agarwal, *Difference Equations and Inequalities,* Marcel Dekker, New York, 1992.

[5] D.A. Georgiou - E.A. Grove - G. Ladas, *Oscillations of neutral difference equations,* Appl. Anal., 33 (1989), pp. 243–253.

[6] J.W. Hooker - W.T. Patula, *A second order nonlinear difference equation: Oscillation and asymptotic behaviour,* J. Math. Anal. Appl., 91 (1983), pp. 9–29.

[7] V. Lakshmikantham - D. Trigiante, *Theory of Difference Equations, Numerical Methods and Applications,* Acad. Press, New York, 1988.

[8] B.S. Lalli - B.G. Zhang, *On existence of positive solutions and bounded oscillations for neutral difference equations,* J. Math. Anal. Appl., 166 (1992), pp. 272–287.

[9] H.J. Li - S.S. Cheng, *Asymptotically monotone solutions of a nonlinear difference equation,* Tamkang J. Math., 24 (1993), pp. 269–282.

[10] J. Popenda - B. Szmanda, *On the oscillation of some difference equations,* Demonstr. Math., 17 (1984), pp. 153–164.

[11] A. Sternal - B. Szmanda, *Asymptotic and oscillatory behaviour of certain difference equations,* Le Matematiche, 51 (1996), pp. 77-86.

[12] Z. Szafrański - B. Szmanda, *Oscillatory behaviour of difference equations of second order,* J. Math. Anal. Appl., 150 (1990), pp. 414-424.

[13] B. Szmanda, *Oscillation of solutions of higher order nonlinear difference equations,* Bull. Inst. Math. Acad. Sinica (to appear).

[14] B. Szmanda, *Note on the oscillation of certain difference equations,* Glasnik Mat., 31 (1996), pp. 115–121.

[15] E. Thandapani, *Asymptotic and oscillatory behaviour of solutions of a second order nonlinear neutral delay difference equation,* Riv. Mat. Univ. Parma, (5) 1 (1992), pp. 105–113.

[16] E. Thandapani, *Oscillation theorems for higher order nonlinear difference equations,* Indian J. Pure Appl. Math., 25 (1994), pp. 519–524.

[17] B.G. Zhang - S.S. Cheng, *Oscillation criteria and comparison theorems for delay difference equations,* Fasc. Math., 25 (1995), pp. 13–32.

*Institute of Mathematics,*
*Poznań University of Technology,*
*60–965 Poznań (POLAND)*