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THEORY OF MULTIINDEX MULTIVARIABLE BESSEL FUNCTIONS AND HERMITE POLYNOMIALS

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We discuss the theory of multivariable multiindex Bessel functions (B.F.) and Hermite polynomials (H.P.) using the generating function method. We derive addition and multiplication theorems and discuss how generalized H.P. can be exploited as a useful complement to the theory of B.F.. We also discuss the importance of the Poisson-Charlier polynomials in the context of multiindex special functions.

1. Introduction.

The theory of Generalized Bessel functions (G.B.F.) and of generalized Hermite polynomials (G.H.P.) has been summarized in [3]. The importance of this new class of functions has been recognized both in purely mathematical and applied frameworks. The body of problems they rise is however so wide and touches so many branches of research, going from the theory of partial differential equations [13] to the abstract group theory [2] and from crystallographic problems [15] to the theory of squeezed states [6], that it is rather difficult to provide a detailed accounting of their properties.

The expression G.B.F. and G.H.P. are now becoming rather generic. The number of non trivial generalized forms of B.F. or H.P. is so large and continuously proliferating that further specifications are needed. A preliminary and

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very rough distinction between the various classes of generalized special functions is between mono and multiindex functions. For integer indices, the former class is characterized by a one parameter generating function (g.f.) [16] the latter by a multiparameter extension.

In this paper we discuss the problems associated to the definition of B.F. and H.P. with more than one integer index and possibly with more than one variable. We analyze the interplay existing between many index B.F. and H.P. and also point out the possibility of further extensions of the notion of multiindex H.P., for the solution of problems of practical interest. An important point we will touch on in this paper is the role played by Poisson-Charlier polynomials [1] (P.C.P.) and by suitable extensions, within the context of multiindex H.P. We will in particular show that many of the proposed two index H.P. can expressed as infinite series of P.C.P.

Albeit the paper is mainly concerned with the case of integer order many index B.F., we will touch on the more general case of real order indices, also discussing the possible existence of multiindex Anger functions.

2. Two index B.F. and P.C.P.

Before entering into the specific details of the section, let us briefly recall the notion of first kind two variable G.B.F. $J_n(x, y)$, which is defined through the g.f. [3]

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)\right]=\sum_{n=-\infty}^{+\infty} t^n J_n(x, y),\ 0<|t|<\infty \tag{2.1}$$

and $J_n(x, y)$ is expressed as the following series of ordinary B.F.

$$J_n(x, y)=\sum_{\ell=-\infty}^{+\infty} J_{n-2\ell}(x) J_\ell(y). \tag{2.2}$$

A two variable G.H.P. of the Appell-Kampé de Fériet type is provided by the g.f. [4]

$$\exp[xt+yt^2]=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x, y). \tag{2.3}$$

Recalling that the g.f. of ordinary H.P. is [1]

$$\exp\left[xt-\frac{1}{2}t^2\right]=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x) \tag{2.4}$$

we can conclude that

$$H_n(x, y) = i^n (2y)^{n/2} H_n\Big(\frac{x}{i\sqrt{2y}}\Big). \tag{2.5}$$

In addition it can also be proved that

$$\frac{\partial}{\partial y} H_n(x, y) = \frac{\partial^2}{\partial x^2} H_n(x, y). \tag{2.6}$$

The link between $J_n(x, y)$ and $H_n(x, y)$ is easily realized, just inspecting (2.1) and (2.3) we find

$$J_n(x, y) = \sum_{s=0}^{\infty} \frac{H_{s+n}(x/2, y/2) H_s(-x/2, -y/2)}{(s+n)!s!}, \quad n \geq 0. \tag{2.7}$$

Two index B.F. type functions have been defined in [1], [3], [10] and are specified by the g.f.

$$e^{\frac{1}{2}x[(u-1/u)+(v-1/v)+(uv-1/uv)]} = \sum_{m,n=-\infty}^{+\infty} u^m v^n J_{m,n}(x), \qquad 0 < |u|, |v| < \infty. \tag{2.8}$$

Two index polynomials can be realized using the following two variable g.f.

$$e^{x(u+v)+uv} = \sum_{m,n=0}^{+\infty} \frac{u^m v^n}{m!n!} Q_{m,n}(x) \tag{2.9}$$

with $Q_{m,n}(x)$ being explicitly provided by the series

$$Q_{m,n}(x) = \sum_{q=0}^{\min(m,n)} q! \binom{m}{q} \binom{n}{q} x^{m+n-2q}. \tag{2.10}$$

The recurrence properties of the $Q_{m,n}(x)$ polynomials are derived either from (2.9) and (2.10) and writes

$$\begin{cases} \dfrac{d}{dx} Q_{m,n}(x) = m Q_{m-1,n}(x) + n Q_{m,n-1}(x) \\ x Q_{m,n}(x) + n Q_{m,n-1}(x) = Q_{m+1,n}(x) \\ x Q_{m,n}(x) + m Q_{m-1,n}(x) = Q_{m,n+1}(x). \end{cases} \tag{2.11}$$

It is obvious that the g.f. (2.9) can be extended to the multivariable case, so that

$$e^{xu+yv+zuv} = \sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!n!} Q^*_{m,n}(x, y, z) \tag{2.12}$$

where

$$\begin{aligned} Q^*_{m,n}(x, y, z) &= \sum_{q=0}^{\min(m,n)} q! \binom{m}{q} \binom{n}{q} x^{m-q} y^{n-q} z^{+q} \\ Q^*_{m,n}(x, x, 1) &= Q_{m,n}(x). \end{aligned} \tag{2.13}$$

The above introduced polynomials satisfy the recurrences

$$\begin{cases} \frac{\partial}{\partial x} Q^*_{m,n}(x, y, z) = m Q^*_{m-1,n}(x, y, z) \\ \frac{\partial}{\partial y} Q^*_{m,n}(x, y, z) = n Q^*_{m,n-1}(x, y, z) \\ \frac{\partial}{\partial z} Q^*_{m,n}(x, y, z) = mn Q^*_{m-1,n-1}(x, y, z) \end{cases} \tag{2.14 a}$$

and

$$\begin{aligned} x Q^*_{m,n}(x, y, z) + nz Q^*_{m,n-1}(x, y, z) &= Q^*_{m+1,n}(x, y, z) \\ y Q^*_{m,n}(x, y, z) + mz Q^*_{m-1,n}(x, y, z) &= Q^*_{m,n+1}(x, y, z). \end{aligned} \tag{2.14 b}$$

The first three recurrences can be combined, thus getting

$$\frac{\partial}{\partial z} Q^*_{m,n}(x, y, z) = \frac{\partial^2}{\partial x \partial y} Q^*_{m,n}(x, y, z) \tag{2.15}$$

and since

$$Q^*_{m,n}(x, y, 0) = x^m y^n \tag{2.16}$$

we end up with the identity

$$Q^*_{m,n}(x, y, z) = e^{z(\partial^2/\partial x \partial y)} x^m y^n. \tag{2.17}$$

In addition any p.d.e. of the type (2.15) possesses the solution

$$g(x, y, z) = \sum_{m,n=0}^{\infty} a_{m,n} Q^*_{m,n}(x, y, z) \tag{2.18}$$

if the initial condition admits the Taylor series expansion

$$g(x, y, 0) = \sum_{m,n=0}^{\infty} a_{m,n} x^m y^n. \tag{2.19}$$

An important aspect to be emphasized is the link between $Q^*_{m,n}$ polynomials and the P.C.P. [1]. In the limit $x = 1$, $y = 1$ we get indeed

$$Q^*_{m,n}(1, 1, -|z|) = \sum_{q=0}^{\min(m,n)} (-1)^q q! \binom{n}{q}\binom{m}{q} |z|^{+q} = C_n(m; |z|). \tag{2.20}$$

It is also straightforwardly understood that for $x \neq y \neq 1$, the following identity holds

$$Q^*_{m,n}(x, y, -|z|) = x^m \cdot y^n \cdot C_n(m; |z|/xy). \tag{2.21}$$

The $Q^*_{m,n}$ polynomials can be usefully exploited to complement the theory of $J_{m,n}(x)$ functions, which, according to (2.8) and (2.12) can be written as

$$J_{m,n}(x) = \sum_{q,p=0}^{\infty} \frac{1}{p!q!(m+p)!(n+q)!} \cdot Q^*_{m+p,n+q}(x/2, x/2, x/2) Q^*_{p,q}(-x/2, -x/2, -x/2) \tag{2.22 a}$$

or, what is the same

$$J_{m,n}(x) = \left(\frac{x}{2}\right)^{m+n} \cdot \sum_{q,p=0}^{\infty} (-1)^{p+n} \frac{C_{n+q}(m+p, -2/x) C_q(p, 2/x) (\frac{x}{2})^{2(p+q)}}{(m+p)!(n+q)!p!q!}, \quad m, n \geq 0. \tag{2.22 b}$$

A further idea of the interplay between $Q^*_{m,n}$ polynomials and two index B.F. is offered by the following multiplication theorem (*)

$$J_{m,n}(\lambda x) = \lambda^{m+n} \sum_{p,q=0}^{\infty} \lambda^{p+q} \frac{J_{m+p,n+q}(x; 1/\lambda) Q^*_{q,p}\left(\left\{\frac{1-\lambda^2}{2\lambda} x\right\}_s\right)}{p!q!} \tag{2.23}$$

(*) We have denoted by $\{x\}_s$ the three identical arguments of the $Q^*_{m,n}$ polynomials.

where $J_{m,n}(x;\zeta)$ is a one parameter two index B.F. defined by the generating function [8]

$$e^{x/2[(u-1/u)+(v-1/v)+(\zeta uv-(1/\zeta uv))]} = \tag{2.24}$$

$$= \sum_{m,n=-\infty}^{\infty} u^m v^n J_{m,n}(x;\zeta), \quad 0 < |u|, |v| < \infty.$$

The properties of this function will be briefly commented on in the concluding section of the paper.

Before closing this section we will discuss two further properties of the $Q^*_{m,n}$ polynomials which will be exploited in the forecoming sections. The following addition theorem can be proved using the standard procedure based on the g.f. method (see e.g. [1])

$$Q_{m,n}(x+y) = \frac{1}{2^{(m+n)/2}} \sum_{p,q=0}^{(m,n)} \binom{m}{p}\binom{n}{q} Q_{m-p,n-q}(\sqrt{2}x) Q_{p,q}(\sqrt{2}y). \tag{2.25}$$

Furthermore

$$\begin{aligned}
&Q^*_{m,0}(x,y,z) = x^m,\ Q^*_{0,n}(x,y,z) = y^n \\
&Q^*_{m,n}(x,y,0) = x^m y^n \\
&Q^*_{m,n}(x,0,z) = \begin{cases} n!\binom{m}{n} x^{m-n} z^n & \text{if } m \geq n \\ 0 & \text{if } m < n \end{cases} \\
&Q^*_{m,n}(0,y,z) = \begin{cases} m!\binom{n}{m} y^{n-m} z^m & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases}
\end{aligned} \tag{2.26}$$

In this section we have given a first idea of how the theory of two index $Q^*_{m,n}$ polynomials may be used to complement that of the $J_{m,n}$ functions. In the forecoming part of the paper we will complete the scenario, discussing more general cases.

3. Two variable two index B.F. and H.P.

The theory of many variable many index H.P. was initially developed by Hermite himself [10] and more recently the associated orthogonal functions have been discussed in [3] along with a number of applications to classical and quantum mechanics [6]. The importance of these polynomials for physical applications has been recognized by other authors and a partial list of references is reported in [7], [8], [12], [14]. Restricting ourselves to the case of two indices and two variables only (the multiindex and multivariable extension being straightforward) we define the polynomials $H_{m,n}(x, y)$ using the g.f. [13]

$$(3.1)\quad e^{t(ax+by)+h(bx+cy)-1/2(at^2+2bth+ch^2)} = \\ = \sum_{m,n=0}^{\infty} \frac{t^m h^n}{m!n!} H_{m,n}(x, y); \qquad a,\ c > 0,\ \Delta = ac - b^2 > 0.$$

The explicit expression for the $H_{m,n}(x, y)$ can be obtained in many different ways. Using the already given definition of Appell-Kampé de Fériet polynomials we find

$$(3.2)\quad H_{m,n}(x, y) = \\ = \sum_{q=0}^{\min(m,n)} (-1)^q q! \binom{m}{q} \binom{n}{q} b^q H_{m-q}(ax + by, -a/2) H_{n-q}(bx + cy, -c/2).$$

On the other side using the $Q^*_{m,n}$ polynomials as reference basis we get

$$(3.3)\quad H_{m,n}(x, y) = \\ = m!n! \sum_{q,p=0}^{([m/2],[n/2])} (-1)^{p+q} \left(\frac{a}{2}\right)^q \left(\frac{c}{2}\right)^p \frac{Q^*_{m-2q,n-2p}(ax + by, bx + cy, -b)}{p!q!(m-2q)!(n-2p)!}.$$

Using the already discussed properties of the $H_n(x, y)$ or $Q^*_{m,n}$ polynomials we can infer those of the $H_{m,n}(x, y)$. It is indeed easy to realize that

$$(3.4)\quad \begin{aligned} H_{m,0}(x, y) &= H_m(ax + by, -a/2) = a^{m/2} H_m\left(\frac{ax + by}{\sqrt{a}}\right) \\ H_{0,n}(x, y) &= H_n(bx + cy, -c/2) = c^{n/2} H_n\left(\frac{bx + cy}{\sqrt{c}}\right). \end{aligned}$$

The adjoint Hermite polynomials $G_{m,n}(x, y)$ are introduced by means of the g.f.

$$(3.5)\quad e^{ux+vy-(cu^2-2buv+av^2)/2\Delta} = \sum_{m,n=0}^{\infty} \frac{u^m}{m!} \frac{v^n}{n!} G_{m,n}(x, y)$$

and therefore, according to the previous discussion, we can obtain the following expression

$$(3.6\,a)\quad G_{m,n}(x,y) = m!n! \sum_{q=0}^{\min(m,n)} \frac{1}{q!}\left(\frac{b}{\Delta}\right)^q \frac{H_{m-q}(x,-c/2\Delta)H_{n-q}(y,-a/2\Delta)}{(m-q)!(n-q)!}$$

and

$$(3.6\,b)\quad G_{m,n}(x,y) = m!n! \sum_{p,q=0}^{([m/2],[n/2])} \frac{(-1)^{p+q}}{p!q!} \frac{\left(\frac{c}{2\Delta}\right)^p \left(\frac{a}{2\Delta}\right)^q}{(m-2p)!(n-2q)!} \cdot Q^*_{m-2p,n-2q}(x,y,b/\Delta).$$

Using the recurrence properties (2.11), we find

$$(3.7\,a)\quad \begin{aligned} \frac{\partial}{\partial x} G_{m,n}(x,y) &= mG_{m-1,n}(x,y) \\ \frac{\partial}{\partial y} G_{m,n}(x,y) &= nG_{m,n-1}(x,y) \end{aligned}$$

and

$$(3.7\,b)\quad \begin{aligned} \frac{\partial}{\partial x} H_{m,n}(x,y) &= amH_{m-1,n}(x,y) + bnH_{m,n-1}(x,y) \\ \frac{\partial}{\partial y} H_{m,n}(x,y) &= bmH_{m-1,n}(x,y) + cnH_{m,n-1}(x,y). \end{aligned}$$

The addition theorems for these class of polynomials can be obtained either extending the standard procedure or exploiting those already known for the ordinary case, in any case one gets

$$(3.8)\quad H_{m,n}(x+x',y+y') = \frac{1}{2^{\frac{m+n}{2}}} \sum_{p,q=0}^{(m,n)} \binom{m}{p}\binom{n}{q} H_{m-p,n-q}(\sqrt{2}x,\sqrt{2}y)H_{p,q}(\sqrt{2}x',\sqrt{2}y')$$

and an analogous expression for the adjoint forms.

We can now wonder whether the $H_{m,n}$ or $G_{m,n}$ can be generalized in some useful way, as e.g. extended forms of the Appell-Kampé de Fériet polynomials. We consider therefore the following g.f.

$$e^{t(ax+by)+h(bx+cy)+(at^2+2bth+ch^2)z} = \sum_{m,n=0}^{\infty} \frac{t^m h^n}{m!n!}\phi_{m,n}(x,y,z) \tag{3.9 a}$$

and

$$e^{ux+vy+(cu^2-2buv+av^2)z/\Delta} = \sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!n!}\psi_{m,n}(x,y,z). \tag{3.9 b}$$

The polynomials $\phi_{m,n}$ and $\psi_{m,n}$ are linked to $H_{m,n}$ and $G_{m,n}$ by the relations

$$\begin{aligned} \phi_{m,n}(x,y,z) &= i^{m+n}(2z)^{(m+n)/2} H_{m,n}\Big(\frac{x}{i\sqrt{2z}}, \frac{y}{i\sqrt{2z}}\Big) \\ \psi_{m,n}(x,y,z) &= i^{m+n}(2z)^{(m+n)/2} G_{m,n}\Big(\frac{x}{i\sqrt{2z}}, \frac{y}{i\sqrt{2z}}\Big) \end{aligned} \tag{3.10}$$

which are clearly recognized as a generalization of (2.5). A noticeable property of the above polynomials can be derived noting that

$$\begin{aligned} \frac{\partial}{\partial z}\psi_{m,n}(x,y,z) &= \frac{c}{\Delta}m(m-1)\psi_{m-2,n}(x,y,z) - \\ &- 2\frac{b}{\Delta}mn\psi_{m-1,n-1}(x,y,z) + \frac{a}{\Delta}n(n-1)\psi_{m,n-2}(x,y,z) \end{aligned} \tag{3.11}$$

in addition since $\psi_{m,n}$ satisfies the same recurrences (3.7 a) we end up with (see also [4])

$$\Delta\frac{\partial}{\partial z}\psi_{m,n}(x,y,z) = \Big[c\frac{\partial^2}{\partial x^2} - 2b\frac{\partial^2}{\partial x \partial y} + a\frac{\partial^2}{\partial y^2}\Big]\psi_{m,n}(x,y,z) \tag{3.12}$$

which is a kind of extended heat equation. It is now easy to prove that $\phi_{m,n}(x,y,z)$ satisfies the same equation. Defining indeed the variables

$$\xi = ax + by\ , \quad \eta = bx + cy \tag{3.13 a}$$

and recalling that

$$\frac{\partial}{\partial \xi} = \frac{1}{\Delta}\Big(c\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\Big)\ , \quad \frac{\partial}{\partial \eta} = \frac{1}{\Delta}\Big(a\frac{\partial}{\partial y} - b\frac{\partial}{\partial x}\Big) \tag{3.13 b}$$

we find

$$(3.14\,a)\qquad \begin{aligned} &\frac{\partial}{\partial \xi}\phi_{m,n}(x,y,z) = m\phi_{m-1,n}(x,y,z)\\ &\frac{d}{d\eta}\phi_{m,n}(x,y,z) = n\phi_{m,n-1}(x,y,z)\end{aligned}$$

and

$$(3.14\,b)\qquad \frac{\partial}{\partial z}\phi_{m,n}(x,y,z) = a\,m(m-1)\phi_{m-2,n}(x,y,z) + \\ + 2b\,mn\phi_{m-1,n-1}(x,y,z) + cn(n-1)\phi_{m,n-2}$$

which once combined yields

$$(3.15)\qquad \frac{\partial}{\partial z}\phi_{m,n}(x,y,z) = \Big(a\frac{\partial^2}{\partial \xi^2} + 2b\frac{\partial^2}{\partial \xi \partial \eta} + c\frac{\partial^2}{\partial \eta^2}\Big)\phi_{m,n}(x,y,z).$$

Using the identities $(3.13\,b)$ it is easily checked that also $\phi_{m,n}(x,y,z)$ satisfies (3.12). A further important consequence one may draw from the previous results is the following: any p.d.e. of the type (3.12) admits the formal solution

$$(3.16\,a)\qquad \begin{aligned} &g(x,y,z) = \exp\left\{z\underline{\partial}_x^T\widehat{M}^{-1}\underline{\partial}_x\right\}g(x,y,0)\\ &\underline{\partial}_x = \begin{pmatrix}\partial/\partial x\\ \partial/\partial y\end{pmatrix}, \quad \widehat{M} = \begin{pmatrix} a & b\\ b & c\end{pmatrix}\end{aligned}$$

if $g(x,y,0)$ can be written in the form (2.19), then

$$(3.16\,b)\qquad g(x,y,z) = \sum_{m,n=0}^{\infty} a_{m,n}\psi_{m,n}(x,y,z).$$

An obvious extension of the previous results is suggested by the g.f.s.

$$(3.17)\qquad \begin{aligned} &e^{t(ax+by)+h(bx+cy)+(azt^2+2\sqrt{zw}bht+cwh^2)} = \sum_{m,n=0}^{\infty}\frac{t^m h^n}{m!n!}\phi_{m,n}(x,y,z,w)\\ &e^{ux+vy+1/\Delta(czu^2-2b\sqrt{zw}uv+awv^2)} = \sum_{m,n=0}^{\infty}\frac{u^m v^n}{m!n!}\psi_{m,n}(x,y,z,w)\end{aligned}$$

and it is also easily proved that

$$
\begin{aligned}
\phi_{m,n}(x, y, z, w) &= i^{m+n} 2^{(m+n)/2} z^{m/2} w^{n/2} \cdot H_{m,n}\Big(\frac{x}{i\sqrt{2z}}, \frac{y}{i\sqrt{2w}}\Big) \\
\psi_{m,n}(x, y, z, w) &= i^{m+n} 2^{(m+n)/2} z^{m/2} w^{n/2} \cdot G_{m,n}\Big(\frac{x}{i\sqrt{2z}}, \frac{y}{i\sqrt{2w}}\Big).
\end{aligned} \tag{3.18}
$$

The recurrence relations can be obtained from e.g. (3.17), differentiating indeed both sides with respect to z and w we find

$$
\begin{aligned}
\Delta \frac{\partial}{\partial z} \psi_{m,n}(x, y, z, w) &= \Big[c \frac{\partial^2}{\partial x^2} - b\sqrt{w/z} \frac{\partial^2}{\partial x \partial y}\Big] \psi_{m,n}(x, y, z, w) \\
\Delta \frac{\partial}{\partial w} \psi_{m,n}(x, y, z, w) &= \Big[a \frac{\partial^2}{\partial y^2} - b\sqrt{z/w} \frac{\partial^2}{\partial x \partial y}\Big] \psi_{m,n}(x, y, z, w)
\end{aligned} \tag{3.19}
$$

and an analogous expression for the $\phi_{m,n}(x, y, z, w)$ counterpart.

A final example of multivariable two index H.P. is offered by the following g.f.

$$
e^{ux+vy+zu^2+huv+wv^2} = \sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!n!} R_{m,n}(x, y, z, h, w) \tag{3.20}
$$

and

$$
R_{m,n}(x, y, h, w) = m!n! \sum_{q=0}^{\min(m,n)} \frac{h^q}{q!} \frac{H_{m-q}(x, z) H_{n-q}(y, w)}{(m-q)!(n-q)!} \, . \tag{3.21}
$$

The polynomial $R_{m,n}$ satisfies the differential identities

$$
\begin{aligned}
\frac{\partial}{\partial z} R_{m,n} &= \frac{\partial^2}{\partial x^2} R_{m,n} \\
\frac{\partial}{\partial h} R_{m,n} &= \frac{\partial^2}{\partial x \partial y} R_{m,n} \\
\frac{\partial}{\partial w} R_{m,n} &= \frac{\partial^2}{\partial y^2} R_{m,n} \, .
\end{aligned} \tag{3.22}
$$

We can now use this new class of polynomials to define *a* further type of two index B.F. which is usually introduced using the g.f. [11]

$$\exp\Big\{\frac{x}{2}\Big(u-\frac{1}{u}\Big)+\frac{y}{2}\Big(v-\frac{1}{v}\Big)+\frac{z}{2}\Big(u^2-\frac{1}{u^2}\Big)+\frac{w}{2}\Big(v^2-\frac{1}{v^2}\Big)+ +\frac{h}{2}\Big(uv-\frac{1}{uv}\Big)\Big\}=\sum_{m,n=-\infty}^{+\infty}u^m v^n j_{m,n}(x,y,z,h,w). \tag{3.23}$$

Confronting this last relation with (3.20) we also get

$$j_{m,n}(x,y,z,h,w)=\sum_{p,q=0}^{\infty}\frac{R_{m+p,n+q}(x/2,y/1,z/2,h/2,w/2)}{(m+p)!(n+q)!}\cdot \cdot\frac{R_{p,q}(-x/2,-y/2,-z/2,-h/2,-w/2)}{p!q!}. \tag{3.24}$$

The importance of this new class of functions of the Bessel type has been discussed in [11], in connection with problems associated to the radiation emitted by relativistic electrons moving in complex magnetic structures.

4. Concluding remarks.

In this paper we have discussed some aspects of the theory of many index B.F. and H.P., which in the authors' opinion is not well developed and widespread known as it should be. In this concluding section we will touch on some further points which may provide seminal elements for future investigations.

We have introduced two variable B.F. which should be recognized as belonging to the class of first kind cylinder type. It is clear that we can introduce the modified form, which is provided by the g.f.

$$e^{x/2[(u+1/u)+(v+1/v)+(uv+1/uv)]}=\sum_{m,n=-\infty}^{\infty}u^m v^n I_{m,n}(x),\ 0<|u|,|v|<\infty. \tag{4.1}$$

Both $I_{m,n}(x)$ and $J_{m,n}(x)$ can be expressed in the form of the following series

$$\begin{aligned} J_{m,n}(x)&=\sum_{s=-\infty}^{+\infty}J_{m-s}(x)J_{n-s}(x)J_s(x)\\ I_{m,n}(x)&=\sum_{s=-\infty}^{+\infty}I_{m-s}(x)I_{n-s}(x)J_s(x). \end{aligned} \tag{4.2}$$

Using the definition (2.24) and its extension to the modified case, we can state the following analytic continuation formula (**)

$$I_{m,n}(ix; i\xi) = i^{m+n} J_{m,n}(x; \xi). \tag{4.3}$$

Equations (4.2) suggest however the existence of further modified forms, provided e.g. by

$$II_{m,n}(x) = \sum_{s=-\infty}^{+\infty} I_{m-s}(x) I_{n-s}(x) J_s(x) \tag{4.4}$$

which can be rewritten as

$$II_{m,n}(x) = \sum_{s=-\infty}^{+\infty} (-i)^s I_{m-s}(x) I_{n-s}(x) I_s(ix) = I_{m,n}(x, x, ix; -i). \tag{4.5}$$

An idea of the importance of the two index B.F. in applications is offered by the following Jacobi-Anger expansions, obtained setting $u = e^{i\phi}$, $v = e^{i\psi}$, $\xi = e^{i\beta}$ in (2.24) and (4.1),

$$\begin{aligned} e^{ix[\sin\phi + \sin\psi + \sin(\beta+\phi+\psi)]} &= \sum_{m,n=-\infty}^{+\infty} e^{im\phi} e^{in\psi} J_{m,n}(x; e^{i\beta}) \\ e^{x[\cos\phi + \cos\psi + \cos(\beta+\phi+\psi)]} &= \sum_{m,n=-\infty}^{+\infty} e^{im\phi} e^{in\psi} I_{m,n}(x; e^{i\beta}) = \\ &= I_{0,0}(x) + 2 \sum_{m,n=1}^{+\infty} \cos(m\phi + n\psi) I_{m,n}(x; e^{i\beta}) \end{aligned} \tag{4.6}$$

which suggest that two index B.F. can be exploited in problems relevant to the emission by charges oscillating at two distinct frequencies and at their mutual sum.

This type of situation occurs in the theory of the emission by relativistic electrons moving in two frequency undulator magnet [4].

(**) Recall that in the ordinary case we have $I_n(ix) = i^n J_n(x)$.

The extension of the above functions to a larger number of indices is almost straightforward and in fact

$$e^{x/2\left[\sum_{j=1}^{N}\left(u_j-\frac{1}{u_j}\right)+\left(\prod_{j=1}^{N}u_j-\prod_{j=1}^{N}\frac{1}{u_j}\right)\right]}= \tag{4.7}$$

$$=\sum_{m_j=-\infty}^{+\infty}\prod_{j=1}^{N}u_j^{m_j}J_{m_1,\ldots,m_N}(x)\,,\quad 0<|u_j|,|v_j|<\infty,$$

with $J_{m_1,\ldots,m_N}(x)$ being provided by

$$J_{m_1,\ldots,m_N}(x)=\sum_{s=-\infty}^{+\infty}\left(\prod_{j=1}^{N}J_{m_j-s}(x)\right)J_s(x). \tag{4.8}$$

The properties of this class of functions will be discussed elsewhere.

Before closing this first part of the concluding comments, we want to stress the possibility of extending the definition of many index B.F. to the real order case, an example might be provided by the two index Anger function

$$A_{\mu,\nu}(x)=\sum_{s=-\infty}^{+\infty}A_{\mu-s}(x)A_{\nu-s}(x)J_s(x) \tag{4.9}$$

where $A_\mu(x)$ is the ordinary Anger function which is discussed in [12].

The recurrence relations of $A_{\mu,\nu}(x)$ are provided by

$$\begin{aligned}
\frac{d}{dx}A_{\mu,\nu}(x)&=\frac{1}{2}\Big[\Big(A_{\mu-1,\nu}(x)-A_{\mu+1,\nu}(x)\Big)+\\
&\quad+\Big(A_{\mu,\nu-1}(x)-A_{\mu,\nu+1}(x)\Big)+\Big(A_{\mu-1,\nu-1}(x)-A_{\mu+1,\nu+1}(x)\Big)\Big]\\
2\mu A_{\mu,\nu}(x)&=\frac{2\sin(\pi\mu)}{\pi}A_\nu(x;x)+x\Big[A_{\mu-1,\nu}(x)+A_{\mu+1,\nu}(x)\Big]+\\
&\quad+x\Big[A_{\mu+1,\nu+1}(x)+A_{\mu-1,\nu-1}(x)\Big]\\
2\nu A_{\mu,\nu}(x)&=\frac{2\sin(\pi\nu)}{\pi}A_\mu(x;x)+x\Big[A_{\mu,\nu-1}(x)+A_{\mu,\nu+1}(x)\Big]+\\
&\quad+x\Big[A_{\mu-1,\nu-1}(x)+A_{\mu+1,\nu+1}(x)\Big],\\
A_\mu(x;x)&=\sum_{s=-\infty}^{+\infty}(-1)^sJ_s(x)A_{\mu-s}(x)=\frac{\sin(\pi\mu)}{\mu\pi}\,.
\end{aligned} \tag{4.10}$$

In the previous section we have discussed forms of H.P. which generalize the two variable two index cases defined on quadratic forms.

Further generalizations can be proposed and in the following we will suggest a number of extended forms, following the point of view of [7], [17].

An interesting set of polynomials is provided by the following example

$$\begin{aligned}(4.11)\quad & \exp\Big[h\Big(a_1x+b_1y\Big)+t\Big(b_1x+c_1y\Big)-\frac{1}{2}\Big(a_1h^2+2b_1th+c_1t^2\Big)\Big]+\\ & +h^2\Big(a_2z+b_2w\Big)+t^2\Big(b_2z+c_2w\Big)-\frac{1}{2}\Big(a_2h^4+2b_2t^2h^2+c_2t^4\Big)=\\ & =\sum_{n,m=0}^{\infty}\frac{h^mt^n}{m!n!}{}^{(2)}H_{m,n}(x,y;z,w),\end{aligned}$$

and

$$(4.12)\qquad {}^{(2)}H_{m,n}(x,y;z,w)=m!n!\sum_{r,s=0}^{([m/2],[n/2])}\frac{H_{m-2r,n-2s}(x,y)H_{r,s}(z,w)}{r!s!(n-2s)!(m-2r)!}\,.$$

The above polynomial is a clear generalization of the two variable one index case introduced in [17], and provided by

$$(4.13)\qquad \begin{aligned}&\exp\Big[xt-\frac{1}{2}t^2+yt^2-\frac{1}{2}t^4\Big]=\sum_{n=0}^{\infty}\frac{t^n}{n!}{}^{(2)}H_n(x;y)\\ &{}^{(2)}H_n(x;y)=n!\sum_{r=0}^{[n/2]}\frac{H_{n-2r}(x)H_r(y)}{(n-2r)!r!}\,.\end{aligned}$$

A further element of interest for the H.P. (4.12), is the possibility of exploiting them to construct four variable biorthogonal basis. It has already been shown that the functions [17]

$$(4.14)\qquad \phi_n(x,y)=A_n^{1/2(2)}H_n(x;y)e^{-1/4(x^2+y^2)}$$

provide an orthonormal two variable basis. In the case of the extension (4.12), the situation is slightly more complicated and, perhaps, more interesting.

We should first note that generalized forms of two variable two index harmonic oscillator functions exist, namely [3], [4]

$$(4.15)\qquad \mathcal{H}_{m,n}(x,y)=\sqrt{\Delta^{1/2}/2\pi}\,\frac{1}{\sqrt{m!n!}}H_{m,n}(x,y)e^{-1/4(ax^2+2bxy+cy^2)}$$

which is biorthogonal to

$$\mathcal{G}_{m,n}(x,y) = \sqrt{\Delta^{1/2}/2\pi}\, G_{m,n}(x,y)^{-1/4(ax^2+2bxy+cy^2)} \tag{4.16}$$

i.e.

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, \mathcal{H}_{m,n}(x,y)\mathcal{G}_{r,s}(x,y) = \delta_{m,r}\delta_{n,s}\,. \tag{4.17}$$

We can therefore introduce the adjoint of (4.12) denoted by ${}^{(2)}G_{m,n}(x, y; z, w)$ (***), and define the biorthogonal functions

$$\begin{aligned} {}^{(2)}\mathcal{H}_{m,n}(x,y;z,w) &= A_{m,n}^{1/2}\, {}^{(2)}H_{m,n}(x,y;z,w)e^{-1/4\left(\underline{f}_1^T \hat{M}_1 \underline{f}_1 + \underline{f}_2^T \hat{M}_2 \underline{f}_2\right)} \\ {}^{(2)}\mathcal{G}_{m,n}(x,y;z,w) &= A_{m,n}^{1/2}\, {}^{(2)}G_{m,n}(x,y;z,w)e^{-1/4\left(\underline{f}_1^T \hat{M}_1 \underline{f}_1 + \underline{f}_2^T \hat{M}_2 \underline{f}_2\right)} \\ \underline{f}_1 = \begin{pmatrix} x \\ y \end{pmatrix}, \quad & \underline{f}_2 = \begin{pmatrix} z \\ w \end{pmatrix}, \quad M_\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ b_\alpha & c_\alpha \end{pmatrix}, \quad \alpha = 1, 2\,, \end{aligned} \tag{4.18}$$

and $A_{m,n}$ is a normalization constant we will specify below.

Using the condition (4.17), we find

$$\begin{aligned} &\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} dw\, {}^{(2)}\mathcal{H}_{m,n}(x,y;z,w)\cdot \\ &\qquad \cdot {}^{(2)}\mathcal{G}_{m',n'}(x,y;z,w) = \\ &= (m!n!)^2 \frac{(2\pi)^2}{\sqrt{\Delta_1\Delta_2}} \sum_{r,s=0}^{([m/2],[n/2])} \frac{1}{r!s!(m-2r)!(n-2s)!} A_{m,n}\delta_{m,m'}\cdot\delta_{n,n'} \end{aligned} \tag{4.19 a}$$

Δ_α being the determinant of the matrix M_α, $\alpha = 1, 2$.
The normalization constant $A_{m,n}$ is therefore provided by

$$A_{m,n} = \frac{\sqrt{\Delta_1\Delta_2}}{(2\pi)^2 m!n!} \quad \frac{(i/\sqrt{2})^{m+n}}{H_m(i/\sqrt{2})H_n(i/\sqrt{2})}\,. \tag{4.19 b}$$

(***) The definition of ${}^{(2)}G_{m,n}(x, y; z, w)$ follows directly from (3.5), using the prescription (4.11).

The possibility of realizing this type of orthogonal function opens further classes of problems, like those involving the polynomials

$$^{(2)}HG_{m,n} = m!n! \sum_{r,s=0}^{([m/2],[n/2])} \frac{H_{m-2r,n-2s}(x, y)G_{r,s}(z, w)}{r!s!(n-2s)!(m-2r)!} \ . \tag{4.20}$$

This aspect of the matter will be however discussed elsewhere.

A final example we will touch on in this paper is a two variable two index polynomial providing a generalization of the Gould-Hopper polynomials [7]. We have indeed the g.f.

$$e^{(ax+by)h+(bx+cy)t-1/2(ah^{2r}+2bh^r t^s+ct^{2s})} = \sum_{m,n=0}^{\infty} \frac{h^m t^n}{m!n!} H_{m,n}^{(r,s)}(x, y) \, . \tag{4.21}$$

The recurrence relations of $H_{m,n}^{(r,s)}(x, y)$ can be derived from (4.21) itself and write

$$\begin{aligned}
&\frac{\partial}{\partial x} H_{m,n}^{(r,s)}(x, y) = amH_{m-1,n}^{(r,s)}(x, y) + bnH_{m,n-1}^{(r,s)}(x, y)\\
&\frac{\partial}{\partial y} H_{m,n}^{(r,s)}(x, y) = bmH_{m-1,n}^{(r,s)}(x, y) + cnH_{m,n-1}^{(r,s)}(x, y)\\
&H_{m+1,n}^{(r,s)}(x, y) = (ax+by)H_{m,n}^{(r,s)}(x, y) - rm! \cdot\\
&\qquad \cdot \left[\frac{aH_{m-2r+1,n}^{(r,s)}(x, y)}{(m-2r+1)!} + \frac{bn!H_{m-r+1,n-s}^{(r,s)}(x, y)}{(m-r+1)!(n-s)!} \right]\\
&H_{m,n+1}^{(r,s)}(x, y) = (bx+cy)H_{m,n}^{(r,s)}(x, y) - sn! \cdot\\
&\qquad \left[\frac{cH_{m,n-2s+1}^{(r,s)}(x, y)}{(n-2s+1)!} + \frac{bm!H_{m-r,n-s+1}^{(r,s)}(x, y)}{(n-s+1)!(m-r)!} \right]
\end{aligned} \tag{4.22}$$

and further implications will be discussed in forecoming investigations.

REFERENCES

[1] P. Appell - J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques. Polynomes d'Hermite,* Gauthier-Villars, Paris, 1926.

[2] C. Chiccoli - S. Lorenzutta - G. Maino - G. Dattoli - A. Torre, *Generalized Bessel functions: a group theoretic view,* Reports on Mathematical Physics, 33 (1993), pp. 241–252.

[3] G. Dattoli - S. Lorenzutta - G. Maino - A. Torre, *Generalized forms of Bessel functions and Hermite polynomials,* Annals of Numerical Math., 2 (1995), pp. 211–232.

[4] G. Dattoli - S. Lorenzutta - G. Maino - A. Torre - G. Voykov - C. Chiccoli, *Theory of two-index Bessel functions and applications to physical problems,* J. Math. Phys., 35 (1994), pp. 3636–3649.

[5] G. Dattoli - C. Chiccoli - S. Lorenzutta - G. Maino - A. Torre, *Theory of Generalized Hermite polynomials,* Computers Math. Applic., 28 (1994), pp. 71–83.

[6] G. Dattoli - A. Torre - S. Lorenzutta - G. Maino, *Coupled harmonic oscillators, generalized harmonic-oscillator eigenstates and coherent states,* Nuovo Cimento, 111B (1996), pp. 811-823.

[7] V.V. Dodonov - V.I. Man'ko, *The Density Matrix of the Canonically Transformed Multidimensional Hamiltonian in the Fock Basis,* Nuovo Cimento, 83B (1984), p. 145.

[8] V.V. Dodonov - V.I. Man'ko, *New relations for two-dimensional Hermite polynomials,* J. Math. Phys., 35 (1994), pp. 4277–4294.

[9] H.W. Gould - A.T. Hopper, *Operational formulas connected with two generalizations of Hermite polynomials,* Duke Math. J., 29 (1962), pp. 51–63.

[10] Ch. Hermite, *Sur un nouveau développement en séries de fonctions,* C.R.A.S., 58 (1864), p. 93.

[11] P. Humbert, *Sur les fonctions du troisieme order,* C.R.A.S., 190 (1930), pp. 159–160.

[12] M. Kauderer, *Modes in n-dimensional first-order systems,* J. Math. Phys., 34 (1993), pp. 4221–4250.

[13] S. Lorenzutta - G. Maino - G. Dattoli - M. Richetta - A. Torre - C. Chiccoli, *Generalized Bessel functions and exact solutions of partial differential equations,* Rendiconti di Matematica, (7) 12 (1992), pp. 1053–1069.

[14] I.A. Malkin - V.I. Man'ko - D.A. Trifonov, *Linear adiabatic invariants and coherent states,* J. Math. Phys., 14 (1973), pp. 576–582.

[15] W.A. Paciorek - G. Chapuis, *Generalized Bessel functions in incommensurate structure analysis,* Acta Cryst., A 50 (1994), pp. 194–203.

[16] H.M. Srivastava - H.L. Manocha, *A treatise on Generating Functions,* Ellis Horwood Limited, New York, 1984.

[17] G.N. Watson, *A treatise on the Theory of Bessel Functions,* Cambridge Univ. Press, London, 1958.

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