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PARTIAL HÖLDER CONTINUITY FOR SECOND ORDER NON LINEAR NON VARIATIONAL PARABOLIC SYSTEMS WITH CONTROLLED GROWTH

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Let $u \in L^2(-T, 0, H^2(\Omega, \mathbb{R}^N)) \cap H^1(-T, 0, L^2(\Omega, \mathbb{R}^N))$ be a solution in $Q = \Omega \times (-T, 0)$ of the second order non linear non variational system

$$-a(X, u, Du, H(u)) + \frac{\partial u}{\partial t} = b(X, u, Du).$$

We prove that if $b(X, u, p)$ has strictly controlled growth and $a(X, u, p, \xi)$ satisfies the condition (A), then the vector Du is partial Hölder continuous in Q.

1. Introduction.

Let Ω be an open, bounded set of $\mathbb{R}^n$, $n > 2$. We denote a generic point of Ω by $x = (x_1, x_2, ..., x_n)$, the cylinder $\Omega \times (-T, 0)$, with $T > 0$, by Q, and the point $(x, t) \in \mathbb{R}^n_x \times \mathbb{R}_t$ by X.

If $u : Q \to \mathbb{R}^N$, N an integer ≥ 1, we set

$$D_i u = \frac{\partial u}{\partial x_i} \quad , \quad Du = (D_1 u, D_2 u, ..., D_n u),$$
$$H(u) = \{D_i D_j u\} = \{D_{ij} u\}, \qquad i, j = 1, 2, ..., n;$$

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Du is a vector in $\mathbb{R}^{nN}$ and $H(u)$ is an element of $\mathbb{R}^{n^2N}$.

We denote a generic vector in $\mathbb{R}^{nN}$ by $p = (p_1, p_2, ...p_n)$, $p_i \in \mathbb{R}^N$, and a generic element of $\mathbb{R}^{n^2N}$ by $\xi = \{\xi_{ij}\}$, $i, j = 1, 2, ..., n$, $\xi_{ij} \in \mathbb{R}^N$.

We shall study in Q the second order non linear non variational system

$$-a(X, u, Du, H(u)) + \frac{\partial u}{\partial t} = b(X, u, Du), \tag{1.1}$$

where $a(x, u, p, \xi)$ is a vector in $\mathbb{R}^N$, measurable in X, continuous in (u, p, ξ), satisfying the conditions:

$$a(X, u, p, 0) = 0; \tag{1.2}$$

(A) *there exist three positive constants* α, γ *and* δ, *with* $\gamma + \delta < 1$, *such that* $\forall u \in \mathbb{R}^N$, $\forall p \in \mathbb{R}^{nN}$, $\forall \tau, \eta \in \mathbb{R}^{n^2N}$ *and for a.e.* $X \in Q$, *it results*

$$\left\| \sum_{i=1}^{n} \tau_{ii} - \alpha[a(X, u, p, \tau + \eta) - a(X, u, p, \eta)] \right\|^2 \leq \gamma \|\tau\|^2 + \delta \left\| \sum_{i=1}^{n} \tau_{ii} \right\|^2;$$

and $b(X, u, p)$ is a vector in $\mathbb{R}^N$, measurable in X and continuous in (u, p), satisfying the condition (of strictly controlled growth)

$$\|b(X, u, p)\| \leq f(X) + c(\|u\|^{\beta} + \|p\|^{\beta'}) \tag{1.3}$$

$\forall u \in \mathbb{R}^N$, $\forall p \in \mathbb{R}^{nN}$ and for a.e. $X \in Q$, where $f \in L^2(Q)$, $1 \leq \beta < \frac{n+2}{n-2}$ and $1 \leq \beta' < \frac{n+2}{n}$.

Let $H^{s,q}(\Omega, \mathbb{R}^N)$ and $H_0^{s,q}(\Omega, \mathbb{R}^N)$ be the usual Sobolev spaces, we define the following functional spaces

$$W^q(Q, \mathbb{R}^N) = \left\{ u : u \in L^q(-T, 0, H^{2,q}(\Omega, \mathbb{R}^N)), \frac{\partial u}{\partial t} \in L^q(Q, \mathbb{R}^N) \right\},$$

$$W_0^q(Q, \mathbb{R}^N) = \left\{ u \in W^q(Q, \mathbb{R}^N) : u \in L^q(-T, 0, H_0^{1,q}(\Omega, \mathbb{R}^N)), \right.$$
$$\left. u(x, -T) = 0 \right\}.$$

A function $u \in W^2(Q, \mathbb{R}^N)$ is a solution of the system (1.1) if u satisfies the (1.1) for a.e. $X \in Q$.

In this paper we prove the partial Hölder continuity of the spatial gradient of the solutions of system (1.1) with the strictly controlled growth (1.3); that is, we prove some results that will be a generalization of the results, obtained by M. Marino and A. Maugeri in [8] for second order non linear non variational parabolic systems with linear growth, that is $\beta = \beta' = 1$.

S. Campanato obtained similar results for second order non linear but variational parabolic systems in [3] and [5].

2. Preliminary lemmas.

Let $X^0 = (x^0, t^0)$ be a point of $\mathbb{R}^n_x \times \mathbb{R}_t$, $\sigma > 0$, and $B(x^0, \sigma) = \{x \in \mathbb{R}^n : \|x - x^0\| < \sigma\}$.

We denote the cylinder of $\mathbb{R}^n_x \times \mathbb{R}_t$, $B(x^0, \sigma) \times (t^0 - \sigma^2, t^0)$, by $Q(X^0, \sigma)$.

The following results will be useful later.

Lemma 2.1. *If* $u \in W^2(Q(X^0, \sigma), \mathbb{R}^N)$, *then* $u \in L^{q_0}(Q(X^0, \sigma), \mathbb{R}^N)$ *and* $Du \in L^{q_1}(Q(X^0, \sigma), \mathbb{R}^{nN})$, *with* $q_0 = \frac{2(n+2)}{n-2}$ *and* $q_1 = \frac{2(n+2)}{n}$. *Therefore there exists a constant c, for which the following inequalities hold*

$$\left(\int_{Q(X^0,\sigma)} \|u\|^{q_0}\, dX\right)^{\frac{2}{q_0}} \leq \tag{2.1}$$
$$\leq c \int_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \sigma^{-4}\|u\|^2\right) dX,$$

$$\left(\int_{Q(X^0,\sigma)} \|Du\|^{q_1}\, dX\right)^{\frac{2}{q_1}} \leq \tag{2.2}$$
$$\leq c \int_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \|u\|^2\right) dX,$$

the constant c does not depend on σ.

Proof. See [7], Lemma 3.3, Chap. II (with $l = 1$, $q = 2$). □

Lemma 2.2. *If* $u \in W^q(Q(X^0, \sigma), \mathbb{R}^N)$, *with* $q > 2$ *and* $\sigma \in (0, 1)$, *then* $\forall \tau \in (0, 1)$ *it results*

$$\int_{Q(X^0,\tau\sigma)} \left(\|u\|^{q_0} + \|Du\|^{q_1}\right) dX \leq \tag{2.3}$$
$$\leq c\left[1 + \int_{Q(X^0,\sigma)} \left(\|H(u)\|^q + \left\|\frac{\partial u}{\partial t}\right\|^q\right) dX\right]^{\frac{8}{q(n-2)}}.$$

$$\cdot\, \sigma^{(n+2)\left(1-\frac{2}{q}\right)\frac{2}{n}} \int_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX +$$

$$+\, c \left[1 + \int_{Q(X^0,\sigma)} \|Du\|^{q_1}\, dX \right]^{\frac{2}{n-2}} \tau^{n+2} \int_{Q(X^0,\sigma)} (\|u\|^{q_0} + \|Du\|^{q_1})\, dX,$$

where c does not depend on σ and τ.

Proof. Let $P_{Q(X^0,\sigma)}$ be the polynomial vector in x, of degree ≤ 1, such that

$$\int_{Q(X^0,\sigma)} D^\alpha (u - P_{Q(X^0,\sigma)})\, dX = 0 \qquad \forall \alpha : |\alpha| \leq 1\, (^1).$$

From Lemma 2.1 it follows that u and $u - P_{Q(X^0,\sigma)}$ belong to $L^{q_0}(Q(X^0,\sigma), \mathbb{R}^N)$, with $q_0 = \frac{2(n+2)}{n-2}$, and Du and $D(u - P_{Q(X^0,\sigma)})$ belong to $L^{q_1}(Q(X^0,\sigma), \mathbb{R}^{nN})$, with $q_1 = \frac{2(n+2)}{n}$.

Let us fix $\tau \in (0,1)$, it results

$$(2.4) \qquad \int_{Q(X^0,\tau\sigma)} \|u\|^{q_0}\, dX \leq$$

$$\leq c \int_{Q(X^0,\sigma)} \|u - P_{Q(X^0,\sigma)}\|^{q_0}\, dX + c \int_{Q(X^0,\tau\sigma)} \|P_{Q(X^0,\sigma)}\|^{q_0}\, dX.$$

On the other hand, from (2.1) and Lemma 2.1 in [8] it follows

$$\int_{Q(X^0,\sigma)} \|u - P_{Q(X^0,\sigma)}\|^{q_0}\, dX \leq$$

$$\leq c \left[\int_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 + \sigma^{-4} \|u - P_{Q(X^0,\sigma)}\|^2 \right) dX \right]^{\frac{q_0}{2}} \leq$$

$$\leq c \left[\int_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX \right] \cdot$$

$$\cdot \left[\int_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX \right]^{\frac{4}{n-2}},$$

(^{1}) If $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multiindex, we set $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n}$, $D_i = \frac{\partial}{\partial x_i}$, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$.

from which, applying the Hölder inequality, one has

$$\text{(2.5)}\qquad \int_{Q(X^0,\sigma)} \|u - P_{Q(X^0,\sigma)}\|^{q_0}\, dX \leq$$
$$\leq c\left[\int_{Q(X^0,\sigma)} \left(\|H(u)\|^q + \left\|\frac{\partial u}{\partial t}\right\|^q\right) dX\right]^{\frac{8}{q(n-2)}} \cdot$$
$$\cdot\, \sigma^{\frac{4}{n-2}\left(1-\frac{2}{q}\right)(n+2)} \int_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX.$$

Now, let us estimate the last integral in the right hand side of (2.4).

From the definitions of $P_{Q(X^0,\sigma)}$, $u_{Q(X^0,\sigma)}$ and $(Du)_{Q(X^0,\sigma)}$ (2) and thanks to the Hölder inequality we get

$$\text{(2.6)}\qquad \int_{Q(X^0,\tau\sigma)} \|P_{Q(X^0,\sigma)}\|^{q_0}\, dX \leq$$
$$\leq c(\tau\sigma)^{(n+2)}\left[\sigma^{q_0}\|(Du)_{Q(X^0,\sigma)}\|^{q_0} + \|u_{Q(X^0,\sigma)}\|^{q_0}\right] \leq$$
$$\leq c\tau^{n+2}\left[\left(\int_{Q(X^0,\sigma)} \|Du\|^{q_1}\, dX\right)^{\frac{q_0}{q_1}} + \int_{Q(X^0,\sigma)} \|u\|^{q_0} dX\right] \leq$$
$$\leq c\left(1 + \int_{Q(X^0,\sigma)} \|Du\|^{q_1}\, dX\right)^{\frac{2}{n-2}} \tau^{n+2} \int_{Q(X^0,\sigma)} \left(\|u\|^{q_0} + \|Du\|^{q_1}\right) dX.$$

Thanks to (2.5) and (2.6), (2.4) gives us the following inequality

$$\text{(2.7)}\quad \int_{Q(X^0,\tau\sigma)} \|u\|^{q_0}\, dX \leq c\left[\int_{Q(X^0,\sigma)} \left(\|H(u)\|^q + \left\|\frac{\partial u}{\partial t}\right\|^q\right) dX\right]^{\frac{8}{q(n-2)}} \cdot$$
$$\cdot\, \sigma^{\frac{4}{n-2}\left(1-\frac{2}{q}\right)(n+2)} \int_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX +$$
$$+\, c\left(1 + \int_{Q(X^0,\sigma)} \|Du\|^{q_1}\, dX\right)^{\frac{2}{n-2}} \tau^{n+2} \int_{Q(X^0,\sigma)} \left(\|u\|^{q_0} + \|Du\|^{q_1}\right) dX.$$

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(2) If $E \subset \mathbb{R}^k$ is a measurable set with positive measure and $f \in L^1(E, \mathbb{R}^h)$, we set

$$f_E = \fint_E f\, dx = \frac{\int_E f\, dx}{\text{meas}(E)}\,.$$

It results $P_{Q(X^0,\sigma)} = \sum_{i=1}^n (D_i u)_{Q(X^0,\sigma)}(x_i - x_i^0) + u_{Q(X^0,\sigma)}$.

Similarly, making use of (2.2), one gets

$$\int_{Q(X^0,\tau\sigma)} \|Du\|^{q_1}\,dX \leq \tag{2.8}$$

$$\leq c\int_{Q(X^0,\sigma)} \|D(u-P_{Q(X^0,\sigma)})\|^{q_1}\,dX + c\int_{Q(X^0,\tau\sigma)} \|DP_{Q(X^0,\sigma)}\|^{q_1}\,dX \leq$$

$$\leq c\left[\int_{Q(X^0,\sigma)}\left(\|H(u)\|^q + \left\|\frac{\partial u}{\partial t}\right\|^q\right)dX\right]^{\frac{4}{qn}}\cdot$$

$$\cdot\,\sigma^{(n+2)\left(1-\frac{2}{q}\right)\frac{2}{n}}\int_{Q(X^0,\sigma)}\left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right)dX +$$

$$+\,c\tau^{n+2}\int_{Q(X^0,\sigma)} \|Du\|^{q_1}\,dX.$$

From (2.7) and (2.8), in virtue of the hypothesis $\sigma < 1$, the inequality (2.3) follows. □

Lemma 2.3. *If $u \in W^2(Q,\mathbb{R}^N)$ is a solution of the system* (1.1)*, if the hypotheses* (1.2)*,* (1.3) *and* (*A*) *hold, and if $f \in L^p(Q)$, $p > 2$, then there exists $\overline{q} \in (2,\overline{p}]$, $\overline{p} = \min\left\{p, \frac{2(n+2)}{(n-2)\beta}, \frac{2(n+2)}{n\beta'}\right\}$*(3)*, such that $\forall q \in [2,\overline{q})$*

$$u \in W^q_{loc}(Q,\mathbb{R}^N),$$

and $\forall\, Q(X^0,2\sigma) \subset\subset Q$, with $\sigma < 1$, it results:

$$\left[\fint_{Q(X^0,\sigma)}\left(\|H(u)\|^q + \left\|\frac{\partial u}{\partial t}\right\|^q\right)dX\right]^{\frac{1}{q}} \leq \tag{2.9}$$

$$\leq c\left[\fint_{Q(X^0,2\sigma)}\left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right)dX\right]^{\frac{1}{2}} +$$

$$+\,c\left[\fint_{Q(X^0,2\sigma)} \|b(X,u,Du)\|^q\,dX\right]^{\frac{1}{q}},$$

where c does not depend on σ.

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(3) It results $\overline{p} > 2$.

Proof. Fixed the cylinder $Q(X^0, 2\sigma) \subset\subset Q$, with $\sigma < 1$, the Lemma 2.1 ensures that $u \in L^{\frac{2(n+2)}{n-2}}(Q(X^0, 2\sigma), \mathbb{R}^N)$, $Du \in L^{\frac{2(n+2)}{n}}(Q(X^0, 2\sigma), \mathbb{R}^{nN})$; then, from the hypothesis (1.3) and since $f \in L^p(Q)$, it follows

$$\|b(X, u, Du)\| \in L^{\overline{p}}(Q(X^0, 2\sigma))$$

with $\overline{p} = \min\left\{p, \frac{2(n+2)}{(n-2)\beta}, \frac{2(n+2)}{n\beta'}\right\} > 2$.

In virtue of Lemma 3.2 in [8], it results

$$\fint_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX \leq \tag{2.10}$$

$$\leq c\left[\fint_{Q(X^0,2\sigma)} \left(\|H(u)\|^{\frac{2(n+2)}{n+4}} + \left\|\frac{\partial u}{\partial t}\right\|^{\frac{2(n+2)}{n+4}}\right) dX\right]^{\frac{n+4}{n+2}} + c\fint_{Q(X^0,2\sigma)} \|b\|^2\, dX,$$

where c does not depend on σ.

Now the thesis is consequence of (2.10) and of the lemma of Gehring - Giaquinta - G. Modica (see [8], Lemma 3.3), written setting

$$U = \left(\|H(u)\| + \left\|\frac{\partial u}{\partial t}\right\|\right)^{\frac{2(n+2)}{n+4}}, \qquad G = \|b\|^{\frac{2(n+2)}{n+4}},$$

$$r = \frac{n+4}{n+2}, \qquad s = \frac{n+4}{2(n+2)}\overline{p}. \qquad \square$$

3. Partial Hölder continuity of vector Du.

Let $u \in W^2(Q, \mathbb{R}^N)$ be a solution in Q of the second order non linear non variational system

$$-a(X, u, Du, H(u)) + \frac{\partial u}{\partial t} = b(X, u, Du), \tag{3.1}$$

where $a(X, u, p, \xi)$ and $b(X, u, p)$ are vectors in $\mathbb{R}^N$ with the following properties

(3.2) *$b(X, u, p)$ is measurable in X, continuous in (u, p) with strictly controlled growth*

$$\|b(X, u, p)\| \leq f(X) + c(\|u\|^{\beta} + \|p\|^{\beta'})$$

$\forall u \in \mathbb{R}^N$, $\forall p \in \mathbb{R}^{nN}$ and for a.e. X in Q, with $f \in L^2(Q)$, $1 \leq \beta < \frac{n+2}{n-2}$ and $1 \leq \beta' < \frac{n+2}{n}$;

(3.3) *$a(X, u, p, \xi)$ is continuous in (X, u, p), of class C^1 in ξ, with derivatives $\dfrac{\partial a}{\partial \xi_{ij}}$ uniformly continuous and bounded in $Q \times \mathbb{R}^N \times \mathbb{R}^{nN} \times \mathbb{R}^{n^2N}$ satisfying the conditions*

$$a(X, u, p, 0) = 0\,;$$

(A) *there exist three positive constants α, γ and δ, with $\gamma + \delta < 1$, such that $\forall u \in \mathbb{R}^N$, $\forall p \in \mathbb{R}^{nN}$, $\forall \tau, \eta \in \mathbb{R}^{n^2N}$ and for a.e. X in Q*

$$\left\| \sum_{i=1}^{n} \tau_{ii} - \alpha[a(X, u, p, \eta + \tau) - a(X, u, p, \eta)] \right\|^2 \leq \gamma \|\tau\|^2 + \delta \left\| \sum_{i=1}^{n} \tau_{ii} \right\|^2;$$

(B) *there exists a positive, continuous, bounded, concave function $\omega(t)$, defined for $t \geq 0$, with $\omega(0) = 0$, such that $\forall X, Y \in Q$, $\forall u, v \in \mathbb{R}^N$, $\forall p, q \in \mathbb{R}^{nN}$ and $\forall \xi, \tau \in \mathbb{R}^{n^2N}$*

$$\|a(X, u, p, \xi) - a(Y, v, q, \xi)\| \leq \omega(d^2(X, Y) + \|u - v\|^2 + \|p - q\|^2)\|\xi\| \,(^4)$$

and

$$\left\| \frac{\partial a(X, u, p, \xi)}{\partial \xi} - \frac{\partial a(X, u, p, \tau)}{\partial \xi} \right\| \leq \omega(\|\xi - \tau\|^2) \,(^5).$$

Let us prove the following lemma

Lemma 3.1. *Let $u \in W^2(Q, \mathbb{R}^N)$ be a solution of system* (3.1).
If hypotheses (3.2) *and* (3.3) *hold and if $f \in L^p(Q)$, where $p > n + 2$, then $\forall Q(X^0, \sigma) \subset\subset Q$, with $0 < \sigma \leq min\{2, \frac{d_0}{2}\}$(6), $\forall \tau \in (0, 1)$ and $\forall \epsilon \in \left(0, (n+2)\left(1 - \frac{2}{p} - \frac{2}{n} + \frac{4}{nq}\right)\right]$, it results*

(3.4)
$$\begin{aligned}&\Phi(u, X^0, \tau\sigma) \leq \\ &\leq A\Phi(u, X^0, \sigma)\left\{\tau^{(n+2)(1-\frac{2}{p})-\epsilon} + \sigma^{(n+2)(1-\frac{2}{q})\frac{2}{n}} + [\omega(c\sigma^{-n}\Phi(u, X^0, \sigma))]^{1-\frac{2}{q}} + \right.\\ &\left. + \left[\omega\left(\fint_{Q(X^0,\sigma)} \|H(u) - (H(u))_{Q(X^0,\sigma)}\|^2\, dX\right)\right]^{1-\frac{2}{q}}\right\},\end{aligned}$$

(4) $d(X, Y) = \max\left\{\|x - y\|, |t - \tau|^{\frac{1}{2}}\right\}$, $X = (x, t)$, $Y = (y, \tau)$.

(5) $\frac{\partial a(X,u,p,\eta)}{\partial \xi} = \left\{\frac{\partial a(X,u,p,\eta)}{\partial \xi_{ij}}\right\}$, $i, j = 1, 2, ..., n$.

(6) d_0 is the parabolic distance from X^0 to the boundary of Q.

where $2 < q < \overline{q}$ (7) *and*

$$\Phi(u, X^0, \sigma) = \sigma^{(n+2)(1-\frac{2}{p})} + \\ + \int_{Q(X^0,\sigma)} \left(\|u\|^{q_0} + \|Du\|^{q_1} + \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX.$$

Proof. Fixed the cylinder $Q(X^0, 2\sigma) \subset\subset Q$, with $\sigma \leq \min\{1, \frac{d_0}{4}\}$, let w be the solution in $Q(X^0, \sigma)$ of the Cauchy-Dirichlet problem

$$\begin{cases} w \in W_0^2(Q(X^0,\sigma), \mathbb{R}^N) \\ -\sum_{i,j=1}^{n} \dfrac{\partial \tilde{a}(X^0, u_\sigma, (Du)_\sigma, (H(u))_\sigma)}{\partial \xi_{ij}} D_{ij} w + \dfrac{\partial w}{\partial t} = B_2 + B_1, \end{cases}$$

where

$$\frac{\partial \tilde{a}(X, u, p, \eta)}{\partial \xi_{ij}} = \left\{ \int_0^1 \frac{\partial a^h(X, u, p, t\eta)}{\partial \xi_{ij}^k} dt \right\}, \qquad h, k = 1, 2, ..., N,$$

$u_\sigma = u_{Q(X^0,\sigma)}$, $(Du)_\sigma = (Du)_{Q(X^0,\sigma)}$, $(H(u))_\sigma = (H(u))_{Q(X^0,\sigma)}$,

$$B_1 = a(X, u, Du, H(u)) - a(X^0, u_\sigma, (Du)_\sigma, H(u)),$$

$$B_2 = \sum_{i,j=1}^{n} \left(\frac{\partial \tilde{a}(X^0, u_\sigma, (Du)_\sigma, H(u))}{\partial \xi_{ij}} - \frac{\partial \tilde{a}(X^0, u_\sigma, (Du)_\sigma, (H(u))_\sigma)}{\partial \xi_{ij}} \right) D_{ij} u,$$

it results in $Q(X^0, \sigma)$, $u = w + v$, with $v \in W^2(Q(X^0, \sigma), \mathbb{R}^N)$ solution of the linear system

$$-\sum_{i,j=1}^{n} \frac{\partial \tilde{a}(X^0, u_\sigma, (Du)_\sigma, (H(u))_\sigma)}{\partial \xi_{ij}} D_{ij} v + \frac{\partial v}{\partial t} = b(X, u, Du).$$

For v the following inequality holds (see [1])

$$\int_{Q(X^0,\tau\sigma)} \left(\|H(v)\|^2 + \left\| \frac{\partial v}{\partial t} \right\|^2 \right) dX \leq \\ \leq c\tau^{n+2} \int_{Q(X^0,\sigma)} \left(\|H(v)\|^2 + \left\| \frac{\partial v}{\partial t} \right\|^2 \right) dX + \\ + c \int_{Q(X^0,\sigma)} \|b(X, u, Du)\|^2 dX, \qquad \forall \tau \in (0, 1). \tag{3.5}$$

(7) $\overline{q}$ is the constant (> 2) that appears in Lemma 2.3.

On the other hand, from (3.2) it follows

$$\|b(X,u,Du)\|^2 \le c\{|f|^2 + \|u\|^{2\beta} + \|Du\|^{2\beta'}\} \le \\ \le c\{1 + |f|^2 + \|u\|^{q_0} + \|Du\|^{q_1}\},$$

from which and by means of the Hölder inequality we get

$$\text{(3.6)} \qquad \int_{Q(X^0,\sigma)} \|b(X,u,Du)\|^2\, dX \le \\ \le c\Big\{\sigma^{n+2} + \sigma^{(n+2)\left(1-\frac{2}{p}\right)} \left(\int_Q |f|^p\, dX\right)^{\frac{2}{p}} + \\ + \int_{Q(X^0,\sigma)} \left(\|u\|^{q_0} + \|Du\|^{q_1}\right)\, dX\Big\} \le \\ \le c\left\{\sigma^{(n+2)\left(1-\frac{2}{p}\right)} + \int_{Q(X^0,\sigma)} \left(\|u\|^{q_0} + \|Du\|^{q_1}\right)\, dX\right\} = cF(u,X^0,\sigma).$$

From (3.5) and (3.6) we deduce, taking account that $\tau < 1$

$$\text{(3.7)} \qquad \int_{Q(X^0,\tau\sigma)} \left(\|H(v)\|^2 + \left\|\frac{\partial v}{\partial t}\right\|^2\right) dX \le \\ \le c\tau^{(n+2)\left(1-\frac{2}{p}\right)} \int_{Q(X^0,\sigma)} \left(\|H(v)\|^2 + \left\|\frac{\partial v}{\partial t}\right\|^2\right) dX + cF(u,X^0,\sigma)$$

$\forall\, \tau \in (0,1)$, where c does not depend on X^0, σ and τ.

Lemma 2.3 ensures that

$$u \in W^q\left(Q\left(X^0, \frac{d_0}{2}\right), \mathbb{R}^N\right), \qquad \forall\, q \in (2, \overline{q});$$

then it's possible to apply Lemma 2.2 that gives us the inequality $\forall\, \tau \in (0,1)$

$$F(u,X^0,\tau\sigma) \le c\left[1 + \int_{Q(X^0,\sigma)} \|Du\|^{q_1} dX\right]^{\frac{2}{n-2}} \tau^{(n+2)\left(1-\frac{2}{p}\right)} F(u,X^0,\sigma) + \\ + c\left[1 + \int_{Q(X^0,\sigma)} \left(\|H(u)\|^q + \left\|\frac{\partial u}{\partial t}\right\|^q\right) dX\right]^{\frac{8}{q(n-2)}} \cdot \\ \cdot \sigma^{(n+2)\left(1-\frac{2}{q}\right)\frac{2}{n}} \int_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX,$$

from which, recalling that $\sigma \le \frac{d_0}{4}$, it follows

$$F(u, X^0, \tau\sigma) \le c_1(u)\tau^{(n+2)\left(1-\frac{2}{p}\right)} F(u, X^0, \sigma) + \\ + c_2(u)\sigma^{(n+2)\left(1-\frac{2}{q}\right)\frac{2}{n}} \Phi(u, X^0, \sigma), \tag{3.8}$$

$\forall \tau \in (0, 1)$, where

$$c_1(u) = c\left[1 + \int_{Q\left(X^0, \frac{d_0}{4}\right)} \|Du\|^{q_1}\, dX\right]^{\frac{2}{n-2}},$$

$$c_2(u) = c\left[1 + \int_{Q\left(X^0, \frac{d_0}{4}\right)} \left(\|H(u)\|^q + \left\|\frac{\partial u}{\partial t}\right\|^q\right) dX\right]^{\frac{8}{q(n-2)}}.$$

From the hypothesis $p > n + 2$ it follows easily that $(n + 2)\left(1 - \frac{2}{p}\right) > (n + 2)\left(1 - \frac{2}{q}\right)\frac{2}{n}$; inequalities (3.7) and (3.8) allow us to apply Lemma 1.II of the Chap. I in [2] and then to obtain $\forall \tau \in (0, 1)$ and $\forall \epsilon \in \left(0, (n + 2)\left(1 - \frac{2}{p} - \frac{2}{n} + \frac{4}{nq}\right)\right]$ the inequality

$$\int_{Q(X^0,\tau\sigma)} \left(\|H(v)\|^2 + \left\|\frac{\partial v}{\partial t}\right\|^2\right) dX \le \\ \le c\tau^{(n+2)\left(1-\frac{2}{p}\right)-\epsilon} \int_{Q(X^0,\sigma)} \left(\|H(v)\|^2 + \left\|\frac{\partial v}{\partial t}\right\|^2\right) dX + \\ + c\Phi(u, X^0, \sigma)\left[\tau^{(n+2)\left(1-\frac{2}{p}\right)-\epsilon} + \sigma^{(n+2)\left(1-\frac{2}{q}\right)\frac{2}{n}}\right]. \tag{3.9}$$

For w the following inequality holds

$$\int_{Q(X^0,\sigma)} \left(\|H(w)\|^2 + \left\|\frac{\partial w}{\partial t}\right\|^2\right) dX \le \\ \le c\int_{Q(X^0,\sigma)} \|B_1\|^2 dX + c\int_{Q(X^0,\sigma)} \|B_2\|^2\, dX. \tag{3.10}$$

Let us estimate the integrals in the right hand side of (3.10). From hyphoteses (3.3)-(B), with the same technique used in [8], it follows

$$\int_{Q(X^0,\sigma)} \|B_1\|^2\, dX \le \tag{3.11}$$

$$\leq c\Big\{ \int_{Q(X^0,2\sigma)} \left(\|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX +$$

$$+ \sigma^{(n+2)\left(1-\frac{2}{q}\right)} \left(\int_{Q(X^0,2\sigma)} \|b\|^q \, dX \right)^{\frac{2}{q}} \Big\} \cdot$$

$$\cdot \left[\omega \left(\fint_{Q(X^0,\sigma)} (\sigma^2 + \|u - u_\sigma\|^2 + \|Du - (Du)_\sigma\|^2) \, dX \right) \right]^{1-\frac{2}{q}} .$$

Now, in virtue of (3.2), one has

$$\text{(3.12)} \qquad \left(\int_{Q(X^0,2\sigma)} \|b\|^q \, dX \right)^{\frac{2}{q}} \leq c\Big\{ \left(\int_{Q(X^0,2\sigma)} |f|^q \, dX \right)^{\frac{2}{q}} +$$

$$+ \left(\int_{Q(X^0,2\sigma)} \|u\|^{\beta q} \, dX \right)^{\frac{2}{q}} + \left(\int_{Q(X^0,2\sigma)} \|Du\|^{\beta' q} \, dX \right)^{\frac{2}{q}} \Big\}.$$

On the other hand, since $2 < q < \min\left\{p, \frac{q_0}{\beta}, \frac{q_1}{\beta'}\right\}$, it results

$$\text{(3.13)} \qquad \left(\int_{Q(X^0,2\sigma)} |f|^q \, dX \right)^{\frac{2}{q}} \leq$$

$$\leq c\sigma^{2(n+2)\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\int_{Q(X^0,2\sigma)} |f|^p \, dX \right)^{\frac{2}{p}} \leq c(f)\sigma^{2(n+2)\left(\frac{1}{q}-\frac{1}{p}\right)}$$

and

$$\text{(3.14)} \quad \left(\int_{Q(X^0,2\sigma)} \|u\|^{\beta q} \, dX \right)^{\frac{2}{q}} \leq c \left(\int_{Q(X^0,2\sigma)} \|u\|^{q_0} \, dX \right)^{\frac{2\beta}{q_0}} \sigma^{2(n+2)\left(\frac{1}{q}-\frac{\beta}{q_0}\right)} \leq$$

$$\leq c\sigma^{(n+2)\frac{2}{q}} \left(1 + \fint_{Q(X^0,2\sigma)} \|u\|^{q_0} \, dX \right)^{\frac{2\beta}{q_0}} \leq$$

$$\leq c\sigma^{(n+2)\frac{2}{q}} \left(1 + \fint_{Q(X^0,2\sigma)} \|u\|^{q_0} \, dX \right) \leq$$

$$\leq c\sigma^{(n+2)\left(\frac{2}{q}-1\right)} \left(\sigma^{n+2} + \int_{Q(X^0,2\sigma)} \|u\|^{q_0} \, dX \right).$$

Similarly we obtain

$$\left(\int_{Q(X^0,2\sigma)} \|Du\|^{\beta' q}\, dX\right)^{\frac{2}{q}} \leq \tag{3.15}$$
$$\leq c\sigma^{(n+2)\left(\frac{2}{q}-1\right)}\left(\sigma^{n+2} + \int_{Q(X^0,2\sigma)} \|Du\|^{q_1}\, dX\right).$$

Thanks to (3.13), (3.14) and (3.15) , from (3.12) it follows

$$\left(\int_{Q(X^0,2\sigma)} \|b\|^{q}\, dX\right)^{\frac{2}{q}} \leq$$
$$\leq c\sigma^{(n+2)\left(\frac{2}{q}-1\right)}\left[\sigma^{(n+2)\left(1-\frac{2}{p}\right)} + \sigma^{n+2} + \int_{Q(X^0,2\sigma)} \left(\|u\|^{q_0} + \|Du\|^{q_1}\right)\, dX\right]$$

and hence

$$\sigma^{(n+2)\left(1-\frac{2}{q}\right)}\left(\int_{Q(X^0,2\sigma)} \|b\|^{q}\, dX\right)^{\frac{2}{q}} \leq \tag{3.16}$$
$$\leq c\left[\sigma^{(n+2)\left(1-\frac{2}{p}\right)} + \int_{Q(X^0,2\sigma)} \left(\|u\|^{q_0} + \|Du\|^{q_1}\right)\, dX\right],$$

where c does not depend on X^0 and σ.

Furthermore it results, taking into account (2.6) and (2.7) in [6]

$$⨍_{Q(X^0,\sigma)} \left(\sigma^2 + \|u-u_\sigma\|^2 + \|Du-(Du)_\sigma\|^2\right)\, dX \leq$$
$$\leq c\sigma^{-n}\left\{\sigma^{n+2} + \int_{Q(X^0,\sigma)} \|Du\|^2 dX + \int_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX\right\} \leq$$
$$\leq c\sigma^{-n}\left\{\sigma^{n+2} + \sigma^{n+2}\left(⨍_{Q(X^0,\sigma)} \|Du\|^{q_1}\, dX\right)^{\frac{2}{q_1}} +\right.$$
$$\left. + \int_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right)\, dX\right\} \leq$$
$$\leq c\sigma^{-n}\left\{\sigma^{n+2} + \int_{Q(X^0,\sigma)} \|Du\|^{q_1}\, dX + \int_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right)\, dX\right\} \leq$$

$$\leq c\sigma^{-n}\Phi(u, X^0, \sigma),$$

from which, recalling that ω is not decreasing, it follows

$$\left[\omega\left(\fint_{Q(X^0,\sigma)} \left(\sigma^2 + \|u - u_\sigma\|^2 + \|Du - (Du)_\sigma\|^2\right) dX\right)\right]^{1-\frac{2}{q}} \leq \tag{3.17}$$

$$\leq \left[\omega\left(c\sigma^{-n}\Phi(u, X^0, \sigma)\right)\right]^{1-\frac{2}{q}}.$$

Then (3.11), (3.16) and (3.17) ensure that

$$\int_{Q(X^0,\sigma)} \|B_1\|^2\, dX \leq c\Phi(u, X^0, 2\sigma)\left[\omega\left(c\sigma^{-n}\Phi(u, X^0, 2\sigma)\right)\right]^{1-\frac{2}{q}}. \tag{3.18}$$

Similarly we obtain

$$\int_{Q(X^0,\sigma)} \|B_2\|^2\, dX \leq \tag{3.19}$$

$$\leq c\Phi(u, X^0, 2\sigma)\left[\omega\left(\fint_{Q(X^0,2\sigma)} \|H(u) - (H(u))_{2\sigma}\|^2\, dX\right)\right]^{1-\frac{2}{q}}.$$

From (3.10), (3.18) and (3.19) it follows

$$\int_{Q(X^0,\sigma)} \left(\|H(w)\|^2 + \left\|\frac{\partial w}{\partial t}\right\|^2\right) dX \leq \tag{3.20}$$

$$\leq c\Phi(u, X^0, 2\sigma)\left\{\left[\omega\left(c\sigma^{-n}\Phi(u, X^0, 2\sigma)\right)\right]^{1-\frac{2}{q}} + \right.$$

$$\left. + \left[\omega\left(\fint_{Q(X^0,2\sigma)} \|H(u) - (H(u))_{2\sigma}\|^2\, dX\right)\right]^{1-\frac{2}{q}}\right\}.$$

Hence, for v we have inequality (3.9) and for w (3.20); since $u = v + w$ we obtain $\forall\, \tau \in (0, 1)$ and $\forall \epsilon \in \left(0, (n+2)\left(1 - \frac{2}{p} - \frac{2}{n} + \frac{4}{nq}\right)\right]$

$$\int_{Q(X^0,\tau\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX \leq c\Phi(u, X^0, 2\sigma)\cdot \tag{3.21}$$

$$\cdot\left\{\tau^{(n+2)\left(1-\frac{2}{p}\right)-\epsilon} + \sigma^{\frac{2(n+2)}{n}\left(1-\frac{2}{q}\right)} + \left[\omega\left(c\sigma^{-n}\Phi(u, X^0, 2\sigma)\right)\right]^{1-\frac{2}{q}} + \right.$$

$$\left. + \left[\omega\left(\fint_{Q(X^0,2\sigma)} \|H(u) - (H(u))_{2\sigma}\|^2\, dX\right)\right]^{1-\frac{2}{q}}\right\}.$$

From inequalities (3.21) and (3.8), it deduces $\forall \tau \in (0,1)$ and $\forall \epsilon \in \left(0, (n+2)\left(1-\frac{2}{p}-\frac{2}{n}+\frac{4}{nq}\right)\right]$

$$\Phi(u, X^0, \tau\sigma) \leq c\Phi(u, X^0, 2\sigma)\Bigg\{\tau^{(n+2)\left(1-\frac{2}{p}\right)-\epsilon} + \sigma^{\frac{2(n+2)}{n}\left(1-\frac{2}{q}\right)} + \\ + \left[\omega\left(c\sigma^{-n}\Phi(u, X^0, 2\sigma)\right)\right]^{1-\frac{2}{q}} + \\ + \left[\omega\left(⨍_{Q(X^0,2\sigma)} \|H(u) - (H(u))_{2\sigma}\|^2\, dX\right)\right]^{1-\frac{2}{q}}\Bigg\}.$$

This last inequality is trivially true for $\tau \in [1, 2)$ too.

Then the lemma is proved. $\square$

Set

$$Q_1 = \left\{X \in Q : \lim_{\sigma\to 0}{}'' ⨍_{Q(X,\sigma)} \|H(u) - (H(u))_{Q(X,\sigma)}\|^2\, dY > 0\right\}$$

and

$$Q_2 = \left\{X \in Q : \lim_{\sigma\to 0}{}' \sigma^{-n}\phi(u, X, \sigma) > 0\right\},$$

it results

$$\text{meas } Q_1 = 0$$

and

$$\mathcal{H}_n(Q_2) = 0,$$

where $\mathcal{H}_n$ is the n-dimensional Hausdorff measure with respect to the parabolic metric $d(X, Y)$.

Then

$$\text{meas}(Q_1 \cup Q_2) = 0.$$

Reasoning exactly as in Theorem 5.I of [4] (see Lemma 4.2 in [8] too), one has that, fixed $\epsilon \in \left(0, 1-\frac{n+2}{p}\right)$, it is possible to associate to every $X^0 \in Q \setminus (Q_1 \cup Q_2)$ a cylinder $Q(X^0, R_{X^0}) \subset\subset Q \setminus Q_2$ and a positive number σ_ϵ such that

$$\Phi(u, Y, \tau\sigma_\epsilon) \leq c\tau^{(n+2)\left(1-\frac{2}{p}\right)-2\epsilon}\Phi(u, Y, \sigma_\epsilon),$$

$\forall \tau \in (0, 1)$ and $\forall Y \in Q(X^0, R_{X^0})$, and then

$$H(u) \in L^{2,(n+2)\left(1-\frac{2}{p}\right)-2\epsilon}(Q(X^0, R_{X^0}), \mathbb{R}^{n^2N}),$$
$$\frac{\partial u}{\partial t} \in L^{2,(n+2)\left(1-\frac{2}{p}\right)-2\epsilon}(Q(X^0, R_{X^0}), \mathbb{R}^{N}),$$
$$Du \in \mathcal{L}^{2,(n+2)\left(1-\frac{2}{p}\right)+2-2\epsilon}(Q(X^0, R_{X^0}), \mathbb{R}^{nN}).$$

The partial Hölder continuity of the vector Du is proved.

Theorem 3.1. *If $u \in W^2(Q, \mathbb{R}^N)$ is a solution of system* (3.1)*, if the hypotheses* (3.2) *and* (3.3) *hold and if $f \in L^p(Q)$, $p > n + 2$, then there exists a set Q_0, closed in Q, with*

$$Q_2 \subset Q_0 \subset Q_1 \cup Q_2$$

such that

$$Du \in C^{0,\alpha}(Q \setminus Q_0, \mathbb{R}^{nN}), \qquad \forall \alpha < 1 - \frac{n+2}{p}.$$

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