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BETTI NUMBERS OF SPACE CURVES BOUNDED BY HILBERT FUNCTIONS

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We study relationships between Hilbert functions and graded Betti numbers of two space curves C and C_0 bilinked by a sequence of basic double linkages; precisely we obtain bounds for the graded Betti numbers of C by means of the Hilbert functions of the two curves and the graded Betti numbers of C_0. On the other hand for every set of integers satisfying these bounds we can construct a curve with these integers as its graded Betti numbers. As a consequence we get a Dubreil-type theorem for a curve C which strongly dominates C_0 at height h which is exactly the Amasaki bound for Buchsbaum curves. Moreover we deduce for biliaison classes of Buchsbaum curves that a strong Lazarsfeld-Rao property holds.

Introduction.

The Dubreil's Theorem for perfect homogeneous ideals I of height two in a polynomial ring $k[x_0, \dots, x_n]$ states that the minimal number of generators

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$\nu(I)$ is bounded by $s + 1$, where s is the least degree of a form in I (see [3],[4] and [12] for more on this subject).

Along this line there is a result of Amasaki (see [1], [6]) which enstablishes an analogous bound for the ideal of a Buchsbaum curve of $\mathbb{P}^3$; precisely, if $C \subset \mathbb{P}^3$ is a Buchsbaum curve, I its defining ideal, $N = l(M(C))$ the length of its Rao module, s the minimal degree of a surface containing C, then $\nu(I) \leq s + 1 + N$.

Another point of view is to relate the Betti numbers to the degree d of C; results in this directions can be found, for instance, in [5], [9], [6], [16], [15].

Since both s and d can be deduced from the Hilbert function of C, one can think to obtain more precise bounds for the graded Betti numbers of C taking into account the whole Hilbert function.

Indeed, using the Hilbert function H of a perfect ideal I of height two in $k[x_0, \dots, x_n]$ one gets bounds for all graded Betti numbers (see [2], [13]); precisely, for every $i > s$:

$$\begin{aligned} \max\{0, -\Delta^{n+1}H(i)\} &\leq \alpha_i \leq -\Delta^n H(i) \\ \max\{0, \Delta^{n+1}H(i)\} &\leq \beta_i \leq -\Delta^n H(i-1) \end{aligned}$$

where

$$\begin{aligned} \alpha_i &= \text{number of generators of degree } i \text{ for } I \\ \beta_i &= \text{number of generators of degree } i \text{ for the syzygy module of } I \end{aligned}$$

and of course $\alpha_s = -\Delta^{n+1}H(s)$.

Moreover, for any set of integers $\{\alpha_i\}$, $\{\beta_i\}$ satisfying the previous conditions and $\alpha_i - \beta_i = -\Delta^{n+1}H(i)$, there exists a perfect ideal of height two with such integers as graded Betti numbers (see [13]).

Of course Dubreil's Theorem is a consequence of the above bounds and one can observe that the lower bound depends on H; precisely we have:

$$\nu(I) \geq \sum -\Delta^{n+1}H(i)$$

where the sum is taken on all i such that $\Delta^{n+1}H(i) < 0$.

In this paper we want to generalize the results on the bounds for the graded Betti numbers to any space curve C and to see if the Hilbert function of C affects the bounds for $\nu(I(C))$.

To obtain our results we make a detailed analysis on the relationship between Hilbert functions and graded Betti numbers of two space curves C_0 and C, where C is obtained from C_0 by "basic double linkages" of total height

h. Precisely, if $H_0 = H(C_0)$, $H = H(C)$, $s_0 = s(C_0)$, $s = s(C)$, α_i^0, β_i^0, γ_i^0, and α_i, β_i, γ_i are the graded Betti numbers for C_0 and C respectively, we define certain integers A_i, B_i which depend on H_0, H and α_i^0 and we get for α_i, β_i, γ_i the bounds described in Theorem 2.1. These bounds are sharp in the sense that, for any given curve C_0 and any admissible sequence H which dominates H_0 at height h, for any set of integers α_i, β_i, γ_i satisfying the conditions of the theorem and $\alpha_i - \beta_i + \gamma_i = -\Delta^4 H(i)$ we can construct a curve C for which $H(C) = H$ and α_i, β_i, γ_i are its graded Betti numbers (see Theorem 2.2).

Then we apply these results to obtain a bound for the number $\nu(I(C))$ of minimal generators of a curve C obtained from C_0 by a sequence of basic double linkages of total height h:

$$(*) \qquad \nu(I(C)) \leq s(C) - s(C_0) + \nu(I(C_0)).$$

This number is the Amasaki's bound when C is a Buchsbaum curve and C_0 is minimal in its biliaison class.

Since in the biliaison classes of Buchsbaum curves the invariants of the minimal curves are determined by the integers n_i, the dimension of the i-th graded component of the Rao module, one can apply the previous results to obtain sharp bounds for the graded Betti numbers of Buchsbaum curves. These bounds can be found in terms of the Hilbert function of the curve and the numbers n_i (see Theorem 2.6).

Recall that any curve in a biliaison class $\mathscr{L}_M$ can be obtained from a minimal one by a sequence of basic double linkages and a deformation which preserves cohomologies and the Rao module (the Lazarsfeld-Rao property).

If in $\mathscr{L}_M$ any curve can be obtained from a minimal curve by a sequence of basic double linkages and a deformation which preserves Rao module and graded Betti numbers (hence cohomology), i.e. if a strong L-R property holds, then the upper bounds of Theorem 2.1 apply to all curves in $\mathscr{L}_M$ with respect to a minimal curve C_0. Vice versa, if $\mathscr{L}_M$ is a biliaison class in which every curve verifies the upper bounds of Theorem 2.1 with respect to a minimal curve then, by Theorem 2.2 we can deduce that in $\mathscr{L}_M$ the strong L-R property holds. This happens, for instance, for the biliaison classes of Buchsbaum curves; in that case the number $(*)$ is exactly the Amasaki's bound.

For all general facts not explicitly mentioned we refer to Hartshorne's book [7].

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1. Notation and preliminaries.

Let $\mathbb{P}^3 = \mathbb{P}^3_k$ be the projective 3-space over an algebraically closed field k of any characteristic. The homogeneous coordinate ring of $\mathbb{P}^3$ will be denoted $R = k[x_0, x_1, x_2, x_3]$. For M a homogeneous R-module, M_t will denote its graded component of degree t. For $\mathscr{F}$ a sheaf of $\mathscr{O}_{\mathbb{P}}$-modules, $H^i_*(\mathscr{F})$ will denote $\oplus_n H^i(\mathscr{F}(n))$ with the natural R-module structure. A curve C in $\mathbb{P}^3$ will mean a 1-dimensional closed subscheme of $\mathbb{P}^3$ without isolated or embedded points. Hence if $\mathscr{I}_C$ is the ideal sheaf of a curve C in $\mathbb{P}^3$ we denote:

$$I(C) = \oplus_n H^0(\mathscr{O}_{\mathbb{P}}, \mathscr{I}_C(n)), \quad M(C) = \oplus_n H^1(\mathscr{O}_{\mathbb{P}}, \mathscr{I}_C(n))$$

the saturated defining ideal of C in R and its Hartshorne-Rao module respectively. Further, we denote

$$H(C, n) = \dim_k(R/I(C))_n = h^0(\mathscr{O}_{\mathbb{P}}(n)) - h^0(\mathscr{I}_C(n))$$

the Hilbert function of the curve C and
$\Delta H(C, n) = H(C, n) - H(C, n-1)$,
$\Delta^i H(C, n) = \Delta^{i-1} H(C, n) - \Delta^{i-1} H(C, n-1)$
its successive differences.

The minimal degree of a surface containing C will be denoted by $s(C) = \min\{i \mid I(C)_i \neq 0\} = \min\{i \mid \Delta^3 H(C, i) \leq 0\}$. A sequence of integers H will be said an *admissible* sequence if there exists a curve which has H as its Hilbert function; in this case we put $s(H) = \min\{i \mid \Delta^3 H(i) \leq 0\}$.

A curve C is said *arithmetically Cohen-Macaulay* if $M(C) = 0$ and *arithmetically Buchsbaum* if $R_1 \cdot M(C) = 0$.

For any R-module M of finite length $\mathscr{L}_M$ will denote the biliaison class of curves C with $M(C) = M$ up to shift.

We recall the definition of *basic double linkage* (ascending trivial elementary biliaison in the terminology of [MDP]) for any curve C of $\mathbb{P}^3$: let S and T be two surfaces containing C, meeting properly, of degrees s and t respectively; let Γ be the curve linked to C by S and T; now let H be a surface of degree h which meets S properly. The curve C' linked to Γ by S and $T' = T \cdot H$ is a 'basic double linkage' of C of degree s and height h. We call a basic double linkage *of type* (s, h) *and surface* S when we need to specify the surface used.

In each biliaison class $\mathscr{L}_M$ there is a curve C_0 with the leftmost shift; the Lazarsfeld-Rao property says that any curve $C \in \mathscr{L}_M$ can be obtained by C_0 using a finite sequence of basic double linkages and a deformation with constant cohomologies and Rao-module (cf. [10] or Th. 5.1 in Chap. IV of [11]). We shall construct sharp bounds for the Betti numbers of all curves in $\mathscr{L}_M$ which can be obtained from C_0 by a finite sequence of basic double linkages. If all the curves in $\mathscr{L}_M$ can be obtained from a minimal curve by basic double linkages and a deformation which preserves the graded Betti numbers, we say that in $\mathscr{L}_M$ the strong L-R property holds. Of course the bounds that we find apply to any curve in these classes. At this moment we do not know if every curve in $\mathscr{L}_M$ can be obtained using just deformations of this type; indeed, we will use our results to deduce that this is the case for biliaison classes of Buchsbaum curves.

Since a curve C' which is a basic double linkage of C of type (s, h) can also be obtained by h basic double linkages of type $(s_i, 1)$, $i = 1, 2, \dots, h$ ([11], Chap. III, Rem. 2.10), we will use very often this kind of linkages; in this case we will say that C' is obtained from C by the sequence $\underline{s} = \{s_i\}$.

In the next proposition we recall how the Hilbert function changes under a basic double linkage of type $(s, 1)$.

Lemma 1.1. *If a curve C' is obtained from a curve C by a double basic linkage of type $(s, 1)$ then*

$$\textit{1)}\quad \Delta^3 H(C', n) = \begin{cases} \Delta^3 H(C, n-1) & \textit{for } n \neq s \\ \Delta^3 H(C, n-1) - 1 & \textit{for } n = s \end{cases}$$

$$\textit{2)}\quad \Delta^4 H(C', n) = \begin{cases} \Delta^4 H(C, n-1) & \textit{for } n \neq s, s+1 \\ \Delta^4 H(C, n-1) - 1 & \textit{for } n = s \\ \Delta^4 H(C, n-1) + 1 & \textit{for } n = s+1 \end{cases}$$

Proof. See [11], Chap. III, Prop. 3.4. $\square$

If C is a curve of $\mathbb{P}^3$ we will denote by

$$0 \to \oplus R(-i)^{\gamma_i} \to \oplus R(-i)^{\beta_i} \to \oplus R(-i)^{\alpha_i} \to I(C) \to 0$$

a minimal free resolution of its defining ideal $I(C)$, where $\alpha_i, \beta_i, \gamma_i$ are the graded Betti numbers of C.

The following properties for the graded Betti numbers are well known and will be used without comments.

1) $\sum \alpha_i + \sum \gamma_i = \sum \beta_i + 1$
2) $\alpha_i - \beta_i + \gamma_i = -\Delta^4 H(C, i)$ for any $i > 0$.

We will write also $\alpha_i(C), \beta_i(C), \gamma_i(C)$ to specify the graded Betti numbers of a curve C. The relationship between graded Betti numbers of bilinked curves is stated in the next lemma.

Lemma 1.2. *Let C' be a curve obtained from a curve C by a basic double linkage of type $(s, 1)$ and surface S, $I = I(C)$ the ideal of C. Then the graded*

Betti numbers of the two curves are related as follows:

$$\alpha_i(C') = \begin{cases} \alpha_{i-1}(C) & \text{for } i \neq s, s+1 \\ \alpha_{i-1}(C)+1 & \text{for } i = s \\ \alpha_{i-1}(C)-1 & \text{for } i = s+1,\ S \text{ a minimal generator of } I \\ \alpha_{i-1}(C) & \text{for } i = s+1,\ S \text{ not a minimal generator of } I \end{cases}$$

$$\beta_i(C') = \begin{cases} \beta_{i-1}(C) & \text{for } i \neq s+1 \\ \beta_{i-1}(C) & \text{for } i = s+1,\ S \text{ a minimal generator of } I \\ \beta_{i-1}(C)+1 & \text{for } i = s+1,\ S \text{ not a minimal generator of } I \end{cases}$$

$$\gamma_i(C') = \gamma_{i-1}(C) \quad \text{for any } i.$$

Proof. See [11], Chap. III, n. 4. □

If H, H' are two admissible sequences and $h \geq 0$ is an integer we say that *H' dominates H at height h* ([11], Chap. V, Def. 2.1 and [14], Def. 2.5.2) and we will write $H' \geq_h H$ when

$$\begin{cases} s(H) \leq s(H') \\ \Delta^3 H'(i) \leq 0 & \text{for } s(H') \leq i < s(H)+h \\ \Delta^3 H'(i) \leq \Delta^3 H(i-h) & \text{for } s(H)+h \leq i \end{cases}$$

We will say that C' dominates C at height h ($C' \geq_h C$) if C' can be obtained from C by a sequence of basic double links with heights summing to h, followed by a deformation which preserves cohomology and Rao module ([14], Def. 2.5.3). If moreover we require that the deformation preserves also the graded Betti numbers then we say that C' *strongly dominates* C ($C' \succeq_h C$). Note that if $C' \geq_h C$ or $C' \succeq_h C$ then $H(C') \geq_h H(C)$ (see [11], Chap. V, Prop. 2.4).

Let $M = \oplus_{i=1}^d k^{n_i}$ be a graded R-module of finite length such that $R_1 \cdot M = 0$. In this case the biliaison class $\mathscr{L}_M$ of Buchsbaum curves will be denoted by $\mathscr{L}_{n_1 \dots n_d}$; the leftmost shift and the graded Betti numbers of a minimal curve C_0 in this class are completely determined by the integers n_i, and in particular by $N = \sum n_i$. Precisely we have (see [11], Chap. IV, n. 6 c)):

Lemma 1.3. *Let $\mathscr{L}_M$ be a biliaison class of Buchsbaum curves; with the previous notation*

1) If $C \in \mathscr{L}_M$ then $M(C)_i = 0 \quad \forall i < 2N-2$

2) $\exists\, C_0 \in \mathscr{L}_M$ such that $M(C_0)_{2N-2} \neq 0$ (i.e. there is a minimal curve)

3) $$\Delta^3 H(C_0, i) = \begin{cases} 1 & \text{for } 0 \leq i < 2N \\ -3n_{i-2N+1} + n_{i-2N} & \text{for } 2N \leq i \leq 2N+d \\ 0 & \text{otherwise} \end{cases}$$

(convention: $n_j = 0 \quad \forall j < 1, j > d$*)*

4) $$\Delta^4 H(C_0, i) = \begin{cases} 1 & \textit{for } i = 0 \\ 0 & \textit{for } 0 < i < 2N \\ -3n_1 - 1 & \textit{for } i = 2N \\ 4n_{i-2N} - n_{i-2N-1} - 3n_{i-2N+1} & \textit{for} \begin{cases} N < i \\ i \le 2N + d + 1 \end{cases} \\ 0 & \textit{otherwise} \end{cases}$$

5) *the graded Betti numbers for* C_0 *are*

$$\alpha_i(C_0) = \begin{cases} 3n_1 + 1 & \textit{for } i = 2N \\ 3n_{i-2N+1} & \textit{for } 2N + 1 \le i \le 2N + d - 1 \end{cases}$$

$$\beta_i(C_0) = 4n_{i-2N} \quad \textit{for} 2N + 1 \le i \le 2N + d$$

$$\gamma_i(C_0) = n_{i-2N-1} \quad \textit{for} 2N + 2 \le i \le 2N + d + 1.$$

If C is a Buchsbaum curve with Rao module M as above then $C \in \mathscr{L}^h_{n_1 \dots n_d}$ will mean that C belongs to $\mathscr{L}_M$ and that the first nonzero shift of M occurs in degree $2N - 2 + h$.

2. Betti numbers of bilinked curves.

In this section we study the relationships between the graded Betti numbers of two curves C_0, C belonging to the same biliaison class and such that C strongly dominates C_0 at height h.

Throughout this section we will use the following notation.

Let $C_0 \subset \mathbb{P}^3$ be a curve with Hilbert function H_0, call $s_0 = s(C_0)$ the minimal degree of a surface containing C_0 and $\alpha_i^0, \beta_i^0, \gamma_i^0$ the graded Betti numbers of degree i of $I(C_0)$. If h is a non-negative integer and C is a curve in the biliaison class of C_0 which strongly dominates C_0 at height h, we will denote by H its Hilbert function, by $s = s(C)$ the minimal degree of a surface containing C, $m = s - s_0$ and by $\alpha_i, \beta_i, \gamma_i$ its graded Betti numbers of degree i. Set now

(1) $$A_i = A_i(H) = \begin{cases} -\Delta^4 H(i) & \text{for } s \le i < s_0 + h \\ -\Delta^4 H(i) + \Delta^4 H_0(i - h) & \text{for } i \ge s_0 + h \end{cases}$$
$$B_i = B_i(H) = \begin{cases} -\Delta^3 H(i) & \text{for } s \le i < s_0 + h \\ -\Delta^3 H(i) + \Delta^3 H_0(i - h) & \text{for } i \ge s_0 + h \end{cases}$$

Observe that $B_i \geq 0$ and $A_i \leq B_i$ for any $i > s$ since C dominates C_0 at height h ([11], Cap. V, Prop. 2.4). Moreover, when $s = s_0 + h$ then $A_s = B_s$ and when $s < s_0 + h$ then $A_s = B_s + 1$ and $A_{s_0+h} < B_{s_0+h}$.

When $\sum_{i \geq s} B_i > 0$ we define $t = \max\{i \mid B_{s+i} \neq 0\}$. Note that the numbers B_i in Nollet's thesis are called $\theta(i)$ (see [14]).

Theorem 2.1. *Let $C_0 \subset \mathbb{P}^3$ be a curve, h a non-negative integer, C any curve in the biliaison class of C_0 which strongly dominates C_0 at height h. Then, with the above notation, for any $i > s$ ($i \neq s_0 + h$ if $s < s_0 + h$) we have:*

$$\alpha_s = \alpha^0_{s-h} + A_s$$

$$\max\{0, \alpha^0_{i-h} + A_i\} \leq \alpha_i \leq \alpha^0_{i-h} + B_i$$

$$\beta^0_{i-h} + \max\{0, -\alpha^0_{i-h} - A_i\} \leq \beta_i \leq \beta^0_{i-h} + B_{i-1}$$

$$\gamma_i = \gamma^0_{i-h}\ .$$

(If $i = s_0 + h$ and $s < s_0 + h$ the upper bounds for α_i and β_i decrease by 1).

Proof. Since $\beta_s = \gamma_s = 0$ we have $\alpha_s = -\Delta^4 H(s)$. So, if $s = s_0 + h$ then $\alpha_s = -\Delta^4 H_0(s_0) + A_s = \alpha^0_{s_0} + B_s$; if $s < s_0 + h$ then $\alpha_s = A_s = B_s + 1$.

Now we prove the rest of the theorem by induction on h; we need to prove both the cases $h = 0$ and $h = 1$ to have the base of the induction since the case $s < s_0 + h$ can occur only for $h > 0$. If $h = 0$ then C is by definition a deformation of C_0 which does not change graded Betti numbers and Rao module. Hence $\alpha_i = \alpha^0_i$, $A_i = A_i(H_0)$ and $B_i = B_i(H_0)$; the theorem follows since $B_i(H_0) = A_i(H_0) = 0$ for any $i \geq s_0$.

If $h = 1$, C is a deformation which does not change graded Betti numbers and Rao module of a curve $\overline{C}$ obtained by a bilinkage of type $(r, 1)$ and surface F from C_0. Now, if $r = s_0$, then $s = s_0$ hence $\alpha_s = 1$, $\alpha_{s_0+1} = \alpha^0_{s_0} - 1$ since F must be a minimal generator for $I(C_0)$, $\alpha_i = \alpha^0_{i-1}$ according to Lemma 1.2 for $i > s_0 + 1$. The conclusion follows since $A_i = B_i = 0$ for $i > s_0$ except for $A_{s_0+1} = -1$. If $r > s_0$, then $s = s_0 + 1$ and for $i > s$ we have $A_r = 1, A_{r+1} = -1, A_i = 0$ for $i \neq r, r + 1$; $B_r = 1, B_i = 0$ for $i \neq r$; $\alpha_r = \alpha^0_{r-1} + 1, \alpha_{r+1} = \alpha^0_r$ or $\alpha^0_r - 1, \alpha_i = \alpha^0_{i-1}$ for $i \neq r, r + 1$. Therefore the conclusion follows.

Suppose now the theorem true for any curve which strongly dominates C_0 at height h and consider a curve C' which strongly dominates C_0 at height $h' = h + 1$: so C' has the same graded Betti numbers of a curve $\overline{C}'$ which is a bilinkage of type $(r, 1)$ and surface F of a curve $C \succeq_h C_0$. Denoting

H', α'_i, A'_i, B'_i the invariants of the curve C' one has, for any $i > s' = s(C')$:

$$B'_i = \begin{cases} B_{i-1} & \text{for } i \neq r \\ B_{i-1} + 1 & \text{for } i = r \end{cases}$$

$$A'_i = \begin{cases} A_{i-1} & \text{for } i \neq r, i \neq r+1 \\ A_{i-1} + 1 & \text{for } i = r \\ A_{i-1} - 1 & \text{for } i = r+1 \end{cases}$$

The above equalities follow immediately applying Lemma 1.1. For the generators of C' we have the four cases of Lemma 1.2.

If $i \neq r, i \neq r+1$ then $A'_i = A_{i-1}$, $B'_i = B_{i-1}$, $\alpha'_i = \alpha_{i-1}$ and the theorem follows by the inductive hypothesis since $\alpha^0_{i-h-1} = \alpha^0_{i-h'}$.

If $i = r$ then $B'_r = B_{r-1} + 1$, $A'_r = A_{r-1} + 1$, $\alpha'_r = \alpha_{r-1} + 1$.

For the right bound we have: $\alpha'_r = \alpha_{r-1} + 1 \leq \alpha^0_{r-1-h} + B_{r-1} + 1 = \alpha^0_{r-h'} + B'_r$.

For the left side, if $\alpha^0_{r-h'} + A'_r \leq 0$ there is nothing to prove; if $\alpha^0_{r-h'} + A'_r = \alpha^0_{r-1-h} + A_{r-1} + 1 > 0$, by the inductive hypothesis $\alpha_{r-1} \geq \alpha^0_{r-1-h} + A_{r-1}$, therefore $\alpha'_r = \alpha_{r-1} + 1 \geq \alpha^0_{r-1-h} + A_{r-1} + 1 = \alpha^0_{r-h'} + A'_r$.

If $i = r+1$ then $B'_{r+1} = B_r$, $A'_{r+1} = A_r - 1$ and

$$\alpha'_{r+1} = \begin{cases} \alpha_r - 1 & \text{if } F \text{ is a minimal generator of } I(C) \\ \alpha_r & \text{if } F \text{ is not a minimal generator of } I(C) \end{cases}$$

Therefore for the right side we have: $\alpha'_{r+1} \leq \alpha_r \leq \alpha^0_{r-h} + B_r = \alpha^0_{r+1-h'} + B'_{r+1}$.

On the other side, if $\alpha^0_{r+1-h'} + A'_{r+1} \leq 0$ again there is nothing to prove; if $\alpha^0_{r+1-h'} + A'_{r+1} = \alpha^0_{r-h} + A_r - 1 > 0$, by the inductive hypothesis $\alpha'_{r+1} \geq \alpha_r - 1 \geq \alpha^0_{r-h} + A_r - 1 = \alpha^0_{r+1-h'} + A'_{r+1}$. Note that for $i = s_0 + h'$ the proof runs in the same way just substituting $\alpha^0_{i-h'}$ with $\alpha^0_{s_0} - 1$.

This completes the proof for the number of generators.

Regarding the other graded Betti numbers we observe first that applying h times Lemma 1.2 one gets $\gamma_i = \gamma^0_{i-h}$.

From $\max\{0, \alpha^0_{i-h} + A_i\} \leq \alpha_i \leq \alpha^0_{i-h} + B_i$ it follows $\max\{0, \alpha^0_{i-h} + A_i\} + \gamma_i + \Delta^4 H(C, i) \leq \beta_i \leq \alpha^0_{i-h} + B_i + \gamma_i + \Delta^4 H(C, i)$.

Now $\gamma_i + \Delta^4 H(C, i) = \gamma^0_{i-h} + \Delta^4 H(C, i) = \beta^0_{i-h} - \alpha^0_{i-h} - \Delta^4 H(C_0, i-h) + \Delta^4 H(C, i) = \beta^0_{i-h} - \alpha^0_{i-h} - A_i$ and substituting in the above inequalities one gets the required result: $\beta^0_{i-h} + \max\{0, -\alpha^0_{i-h} - A_i\} \leq \beta_i \leq \beta^0_{i-h} + B_i - A_i = \beta^0_{i-h} + B_{i-1}$.

A similar computation gives the bounds of β_i for $i = s_0 + h$. $\square$

The above theorem enstablishes precise bounds for the graded Betti numbers of a curve C in terms of the corresponding graded Betti numbers of a curve C_0 such that $C \succeq_h C_0$. Now we want to show that these bounds are sharp in the sense that we can construct curves which have any number of generators allowed by these bounds.

Theorem 2.2. *Let C_0 be a curve with Hilbert function H_0 and minimal number of generators α_i^0 for every $i \geq s_0 = s(C_0)$; let H be an admissible sequence of integers which dominates H_0 at height h (h a non-negative integer) and let α_i be integers satisfying the bounds of the previous theorem for $i \geq s$.*

Then there exists a curve C which (strongly) dominates C_0 at height h such that $H(C) = H$ and $\alpha_i(C) = \alpha_i$ for any i.

Moreover, if β_i^0, γ_i^0 are the remaining Betti numbers of C_0 and β_i, γ_i are integers satisfying the bounds of the above theorem and such that $-\alpha_i + \beta_i - \gamma_i = \Delta^4 H$ then $\beta_i(C) = \beta_i$, $\gamma_i(C) = \gamma_i$.

Proof. We will construct a sequence of h bilinkages which produces the required curve C from C_0. Denote, as usual, $s = s(H)$; if $B_{s+i} = 0$ for any $i \geq 0$ then by definition (1) and Lemma 1.1 this can occur just when we perform h bilinkages of type $(s_0, 1)$. So we have, applying Lemma 1.2 h times:

$$\alpha_i = \begin{cases} 1 & \text{for } i = s_0 \\ 0 & \text{for } s_0 < i < s_0 + h \\ \alpha_{s_0}^0 - 1 & \text{for } i = s_0 + h \\ \alpha_{i-h}^0 & \text{for } i > s_0 + h \end{cases}$$

and the theorem is true. Otherwise set $t = \max\{i \mid B_{s+i} \neq 0\}$, $m = \sum_{j=0}^{t} B_{s+j} = s - s_0$ (see [14], Prop.3.1.7), and, for any $s \leq i \leq s + t$: $z_i = \alpha_{i-h}^0 + B_i - \alpha_i$.

Consider the following h positive integers:

$$s_i = \begin{cases} s_0 & \text{for } 1 \leq i \leq h - m \\ s + i + r_i - h & \text{for } h - m < i \leq h \end{cases} \tag{2}$$

where r_i is the integer defined as follows:

$$r_i = \max\Big\{q \mid i \leq h - m + \sum_{j=q}^{t} B_{s+j}\Big\}.$$

We want to prove that the sequence (2) gives a curve C with $H(C) = H$ and $\alpha_i(C) = \alpha_i$ by successive suitable basic double linkages of type

$(s_i, 1), i = 1, 2, \ldots, h$, starting from C_0. Observe that such a curve C will have $\gamma_i(C) = \gamma_i = \gamma^0_{i-h}$ hence $\beta_i(C) = \beta_i$.

Call now C_i the intermediate bilinked curves, i. e. $C_0 : (s_1, 1) \to C_1 : (s_2, 1) \to \ldots \to C_i : (s_{i+1}, 1) \to \ldots (s_h, 1) \to C_h = C$.

We first remark that $z_i \le \min\{\alpha^0_{i-h} + B_i, B_{i-1}\}$: namely, the only thing to verify is $z_i \le B_{i-1}$; this follows by

$$z_i \le B_i + \alpha^0_{i-h} - \max\{0, A_i + \alpha^0_{i-h}\} \le B_i - A_i = B_{i-1}.$$

The integer r_i can be better understood in the following way. Divide the interval $[1, h]$ in $t + 2$ consecutive disjoint sub-intervals, respectively, of width $h - m, B_{s+t}, B_{s+t-1}, \ldots, B_s$; so, if $i \in [h - m + 1, h]$, i belongs to a unique sub-interval of width B_{s+r_i}.

Now we see that for all $i > h - m$ one has $s_i > s(C_{i-1})$: in fact, the least degree of a surface containing C_j can increase at most by 1 at each bilinkage and does not increase in the first $h - m$ bilinkages; hence $s(C_{i-1}) \le s_0 + (i - 1) - (h - m)$ and $s_i = s + i + r_i - h > s_0 + m + i - h - 1$.

This in particular says that any sequence of bilinkages of type $(s_i, 1)$ will produce a curve C such that $s(C) = s_0 + m = s(H) = s$.

Now let $i \ge s$ be any integer and compute $\Delta^3 H(C, i)$ for such a curve; according to Lemma 1.1 we have

$$\Delta^3 H(C, i) = \Delta^3 H(C_0, i - h) - \tau$$

where $\tau = \#\{x \mid i - h + x = s_x = s - h + x + r_x\} = \#\{x \mid r_x = i - s\}$, i.e. $\tau = B_i$. Therefore,

$$\Delta^3 H(C, i) = \Delta^3 H(C_0, i - h) - B_i = \Delta^3 H(i).$$

Finally, we observe that according to Lemma 1.2, for all $i \ge s$, when we perform h bilinkages of type $(s_i, 1)$ with surfaces S_i, we obtain a curve C for which

$$\alpha_i(C) = \alpha^0_{i-h} + \tau - \psi$$

where τ, as before, is the number of the integers x such that $i - h + x = s_x$, i.e. $\tau = B_i$ and $\psi = \#\{x \mid (a)\ i - h + x - 1 = s_x$ and $(b)\ S_x$ is a minimal generator for $I(C_{x-1})\}$. Now, the number of x satisfying (a) is B_{i-1} (use the same computation as for τ); moreover, the curve C_{y_i} obtained from C_0 by the first $y_i = h - m + \sum_{j=i}^{s+t} B_j$ bilinkages has $\alpha_{i-h+y_i}(C_{y_i}) = \alpha^0_{i-h} + B_i$. In fact, using again Lemma 1.2 we have $\alpha_{i-h+y_i}(C_{y_i}) = \alpha^0_{i-h+y_i-y_i} + \tau_{y_i} - \psi_{y_i}$ where τ_{y_i}, ψ_{y_i} are defined as before.

Now, with a similar computation we get $\tau_{y_i} = \#\{x \le y_i \mid i - h + x = s_x\} = B_i$; $\psi_{y_i} = \#\{x \le y_i \mid (a) \text{ and } (b)\} = 0$ since $x \le y_i$ implies $r_x \ge r_{y_i} = i - s$ and $i - h + x = s + x + r_x - h + 1$ implies $r_x = i - s - 1$. Thus, performing the B_{i-1} bilinkages of type $(s_j, 1)$ with $j \in [y_i + 1, +B_{i-1}]$, since $z_i \le \min\{B_{i-1}, \alpha^0_{i-h} + B_i\}$, we can use exactly z_i times a minimal generator of the ideal of the curve C_{j-1}. Therefore, with this choice, $\psi = z_i$. Consequently, $\alpha_i(C) = \alpha^0_{i-h} + B_i - z_i = \alpha_i$. $\square$

Remark 2.3. Given an admissible sequence H which dominates $H_0 = H(C_0)$ at height h, we defined in Theorem 2.2 a sequence $\underline{s}$ of integers such that any curve C built up with $\underline{s}$ has $H(C) = H$; indeed, one can construct $\frac{h!}{(h-m)! \cdot B_s! \cdot \ldots \cdot B_{s+t}!}$ sequences of h integers with the same property. But not all of these sequences permit to find suitable bilinkages of type $(s_i, 1)$ in order to fill up the whole range of degrees of generators described in Theorem 2.1 as that we used in the proof of Theorem 2.2. Precisely, with each of these sequences we can construct curves with maximal graded Betti numbers but not necessarily with the minimal ones. For instance, the sequence

$$s_i = \begin{cases} s_0 & \text{for } 1 \le i \le h - m \\ s + i + \rho_i - h & \text{for } h - m < i \le h \end{cases}$$

where ρ_i is the integer defined as follows:

$$\rho_i = min\{q \mid i \le h - m + \sum_{j=0}^{q} B_{s+j}\}$$

in case $B_{j-1} > \alpha^0_{j-h}$ for some j, cannot give a curve with the minimal number of generators in degree j.

The above results can be applied in a natural way both when C_0 is a minimal curve in its biliaison class $\mathscr{L}_M$ and to give a Dubreil type bound to the number of generators of a curve C in terms of the numbers A_i, B_i and α^0_i. Of course we must restrict ourselves to the subset $\mathscr{L}'_M \subset \mathscr{L}_M$ consisting of the curves which can be reached from a minimal curve by strong domination.

We obtain a particularly meaningful application for Buchsbaum curves, since in this case the graded Betti numbers of a minimal curve only depend on the length n_i of the graded pieces of the Rao module. Hence for Buchsbaum curves we obtain bounds for all the graded Betti numbers just in terms of its Hilbert function and the ranks n_i.

The following result is an immediate consequence of Theorem 2.1.

Corollary 2.4. *If C_0, C are two curves such that C strongly dominates C_0 at height h, then*

$$\nu(I_0) + \sum(-\alpha^0_{i-h} - A_i) \le \nu(I) \le s - s_0 + \nu(I_0) \tag{3}$$

where $I = I(C)$, $I_0 = I(C_0)$, $s = s(C)$, $s_0 = s(C_0)$ and the sum is taken on all i such that $\alpha^0_{i-h} + A_i \le 0$.

Proof. Just make the sum of the bounds for α_i in Theorem 2.1. □

It is worth noting that the upper bound of Corollary 2.4 could be deduced directly from Lemma 1.2: a curve C bilinked to C_0 by $(r, 1)$ and surface F has one more generator if one takes $r > s_0$ and F is not a minimal generator for C_0. Note that, as in the Cohen-Macaulay case, the only invariant of C in the upper bound is s, but the lower bound depends on the entire Hilbert function H.

When C_0 is a minimal curve in its biliaison class $\mathscr{L}_M$ then it has a resolution of type (E):

$$0 \to \mathscr{E}_0 \to \mathscr{F}_0 \to \mathscr{I}_{C_0} \to 0$$

where $\mathscr{F}_0$ is a decomposed sheaf and any other curve C in $\mathscr{L}_M$ has resolution:

$$0 \to \mathscr{E}_0(-h) \oplus \mathscr{L} \to \mathscr{F} \to \mathscr{I}_C \to 0$$

for some $h \ge 0$ and $\mathscr{L}, \mathscr{F}$ are decomposed sheaves. Taking a minimal free resolution of $\mathscr{E}_0(-h) \oplus \mathscr{L}$:

$$0 \to \mathscr{H} \to \mathscr{G} \to \mathscr{E}_0(-h) \oplus \mathscr{L} \to 0$$

one gets a resolution for $\mathscr{I}_C$. Hence $h^0(\mathscr{E}_0(i-h) \oplus \mathscr{L}(i)) = \beta_i - \gamma_i \ge h^0(\mathscr{E}_0(i-h)) = \beta^0_{i-h} - \gamma^0_{i-h}$. Now, since $\gamma_i = \gamma^0_{i-h}$, it follows $\beta_i \ge \beta^0_{i-h}$ so we find :

$$\alpha_i = \beta_i - \gamma_i - \Delta^4 H(i) = \beta_i - \Delta^4 H(i) - (-\Delta^4 H_0(i-h) + \beta^0_{i-h} - \alpha^0_{i-h}) =$$
$$= A_i + \alpha^0_{i-h} + \beta_i - \beta^0_{i-h}.$$

Finally we get that $\beta_i \ge \beta^0_{i-h}$ implies $\alpha_i \ge A_i + \alpha^0_{i-h}$, i.e. the left bound of the theorem holds for all $C \in \mathscr{L}_M$ when C_0 is a minimal curve in the class; in particular the left bound in (3) is true for every curve in $\mathscr{L}_M$. It would be interesting and very useful to know if the same is true for the upper bound of (3).

Remark 2.5. There are biliaison classes in which 'every' curve strongly dominates a minimal one; precisely, if in $\mathcal{L}_M$ every curve satisfies the upper bounds of Theorem 2.1 with respect to a minimal curve then, by Theorem 2.2 one gets that every curve can be obtained from a minimal one by a sequence of basic double linkages and a deformation with constant graded Betti numbers. In fact, if $C \in \mathcal{L}_M$ has $\alpha_i(C)$ satisfying the bounds of Theorem 2.1, using Theorem 2.2 one can construct a curve C' with $H(C') = H(C)$, C, C' dominate C_0 at the same height h and $\alpha_i(C') = \alpha_i(C)$. Then C and C' have (minimal) E-type resolutions:

$$0 \to \mathcal{E} \xrightarrow{\phi} \mathcal{F} \to \mathcal{I}_C \to 0$$
$$0 \to \mathcal{E} \xrightarrow{\phi'} \mathcal{F} \to \mathcal{I}_{C'} \to 0.$$

With the same arguments as in [14], Proposition 2.5.6, we can find a flat family of curves C_t by $t\phi + (1-t)\phi'$ with constant graded Betti numbers. Indeed, if

$$0 \to \mathcal{H} \to \mathcal{G} \xrightarrow{\pi} \mathcal{E} \to 0$$

is a minimal free resolution of $\mathcal{E}$ then C and C' have minimal free resolutions:

$$0 \to \mathcal{H} \to \mathcal{G} \xrightarrow{A} \mathcal{F} \to \mathcal{I}_C \to 0$$
$$0 \to \mathcal{H} \to \mathcal{G} \xrightarrow{B} \mathcal{F} \to \mathcal{I}_{C'} \to 0$$

where $A = \phi \circ \pi$, $B = \phi' \circ \pi$ are matrices whose entries are polynomials of positive degree or zero. So $(t\phi + (1-t)\phi') \circ \pi = tA + (1-t)B$ and the conclusion follows. Therefore in these classes the strong L-R property holds.

The Buchsbaum biliaison classes do have this property. In fact, from the proof of Theorem 2.10 in [6] one has for any curve C in the class:

$$\alpha_k \leq -\Delta^2(\Gamma, k) - h^1(\mathcal{I}_C(k-1)) + 2h^1(\mathcal{I}_C(k-2))$$

where Γ is the generic plane section of C. So, if $C \in \mathcal{L}^h_{n_1 \ldots n_d}$ one can easily check that this is the upper bound in the next Theorem 2.6. Hence in any Buchsbaum biliaison class the strong L-R property holds. Of course in these classes the upper bound of (3) is Amasaki's bound.

We observe that our results can be applied also to ACM curves. For this we take as minimal curves C_0 in the ACM class all the lines. If C is an ACM curve with Hilbert function $H = H(C)$, define $h + 1 = max\{i \mid \Delta^3 H(i) < 0\}$ and observe that H dominates $H_0 = H(C_0)$ of height h. One sees that $B_{1+h} = -\Delta^3 H(1+h) - 1$, $B_i = -\Delta^3 H(i)$ for $i \neq h+1$, $A_{1+h} = -\Delta^4 H(1+h) - 2, A_{h+2} = -\Delta^4 H(h+2) + 1$, $A_i = -\Delta^4 H(i)$ for $i \neq h+1, h+2$; hence, computing the bounds of Theorem 2.1 one gets the well known bounds mentioned in the introduction. In this case the upper bound of (3) becomes the Dubreil number $s+1$.

Note that any ACM curve with minimal generators of degree, say, $a_1 \leq a_2 \leq \ldots \leq a_n$, can be obtained from a line by a sequence of $a_1 + a_2 - n$ basic double linkages: $a_1 - n + 1$ of type $(1, 1)$ using a minimal generator of degree 1, followed by $a_2 - n + 1$ of type $(a_1 - n + 2, 1)$ using the minimal generator of degree $a_1 - n + 2$ and the last $n - 2$ of type $(a_i - n + i, 1)$, $i = 3, \ldots, n$, using surfaces which are not minimal generators.

The biliaison classes of Buchsbaum curves have a nice property: all the invariants of a minimal curve in the class (Hilbert function and graded Betti numbers) depend only on the ranks of the graded pieces of the Hartshorne-Rao module (see Lemma 1.3). This property, coupled with the results of this section, allow us to give bounds for the Betti numbers of a curve $C \in L^h_{n_1 \ldots n_d}$ in terms of its Hilbert function and the integers n_i.

Theorem 2.6. *Let $C \in L^h_{n_1 \ldots n_d}$, be a Buchsbaum curve, $N = \sum n_i$; then for every $i > s(C)$:*

a) $\alpha_s = -\Delta^4 H(s)$
$\max\{0, -\Delta^4 H(i) + 4n_{i-2N-h} - n_{i-2N-h-1}\} \leq \alpha_i \leq -\Delta^3 H(i) + n_{i-2N-h}$

b) $\max\{4n_{i-2N-h}, n_{i-2N-h-1} + \Delta^4 H(i)\} \leq \beta_i \leq -\Delta^3 H(i-1) + n_{i-2N-h} + n_{i-2N-h-1}$

c) $\gamma_i = n_{i-2N-h-1}$

(convention $n_j = 0$ for $j < 0$, $j > d$).

Proof. It is enough to use Theorem 2.1, Lemma 1.3 and straight computations. □

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