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THE UPPER CHROMATIC NUMBER OF QUASI-INTERVAL CO-HYPERGRAPHS

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Mixed hypergraph consists of two families of subsets: the edges and the co-edges. In any colouring, every edge has at least two vertices of different colours, and every co-edge has at least two vertices of the same colour.

The upper chromatic number $\bar{\chi}$ is the maximum number of colours for which there exists a colouring of a mixed hypergraph using all the colours. If the mixed hypergraph contains the co-edges only then it is called a co-hypergraph.

We investigate the structural and colouring properties of clique hypergraphs of interval graphs called the quasi-interval hypergraphs. We find the conditions when they are interval hypergraphs. The upper chromatic number for the clique co-hypergraphs of interval graphs is found. It is shown that for the quasi-interval co-hypergraph $H_{\mathcal{A}} = (X, \mathcal{A})$, $\bar{\chi}(H_{\mathcal{A}}) = |X| - s(H_{\mathcal{A}})$, where $s(H_{\mathcal{A}})$ is the so called sieve number introduced in [7]. That coincides with the value of the upper chromatic number found in [7] for the mixed interval hypergraphs.

The co-stability number $\alpha_{\mathcal{A}}(H)$ of a mixed hypergraph H is the maximum number of vertices which contain no co-edges. H is called co-perfect if $\bar{\chi}(H') = \alpha_{\mathcal{A}}(H')$ for every its induced sybhypergraph H'. Using colouring properties of acyclic co-hypergraph partitions onto co-bistars, we prove that quasi-interval co-hypergraph is co-perfect if and only if it does not contain co-monostars. This shows that Voloshin's co-perfectness conjecture [6] for quasi-interval co-hypergraphs is true.

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1. Basic notions.

The colouring problem is one of the most important problems which was the stimulant of the graph theory development.

The existence of various generalizations of this problem on hypergraphs generated the necessity to find the common approach to the colouring problem in graph and hypergraph theory. It was an approach developed in [6] by V. Voloshin. The realization of such an approach is the really important and promising problem. The problem is to find not only the minimal possible number of colours required for the hypergraph colouring, it is to find the maximal possible number as well when we consider this colouring method. For a number of hypergraph classes such a number, named the upper chromatic number, was found in particular in [2] and [6]. In the present paper there were analysed the colouring and structural properties of the more extended hypergraph class.

Throughout this paper we use the terminology of [6]. Recall in details some basic notions given in [6]: let $X = \{x_1, x_2, \ldots, x_n\}$, $n \geq 1$, be a finite set, $\mathcal{S} = \{S_1, S_2, \ldots, S_t\}$, $t \geq 1$, be a family of subsets of X. The couple $H = (X, \mathcal{S})$ is called a hypergraph with the vertex set X and a family of subsets $\mathcal{S}$ if $\bigcup_{i=1}^{t} S_i \subseteq X$. In general we consider the hypergraphs $H = (X, \mathcal{S}')$, $|X| = n$ such that $\mathcal{S}' = \mathcal{A} \cup \mathcal{E}$ where $\mathcal{A}$ and $\mathcal{E}$ are two subfamilies of $\mathcal{S}'$. If $\mathcal{A} \neq \emptyset$ and $\mathcal{E} \neq \emptyset$ then arrange that $\mathcal{A} = \{A_1, \ldots, A_k\}$, $I = 1, \ldots, k$, $\mathcal{E} = \{E_1, \ldots, E_m\}$, $J = 1, \ldots, m$ and $|A_i| \geq 2$, $i \in I$; $|E_j| \geq 2$, $j \in J$. When discussing the colourings we call every E_i, $j \in J$ and edge, and every A_i, $i \in I$ a co-edge. In special cases if $\mathcal{A} = \emptyset$ then $H = (X, \mathcal{E}) = H_{\mathcal{E}}$ will be called simply a hypergraph, if $\mathcal{E} = \emptyset$, then $H = (X, \mathcal{A}) = H_{\mathcal{A}}$ will be called a co-hypergraph. In general case, if $\mathcal{E} \neq \emptyset$ and $\mathcal{A} \neq \emptyset$ then $H = (X, \mathcal{A} \cup \mathcal{E})$ will be called a mixed hypergraph.

Let us have $t \geq 1$ colours. A strict colouring of a mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$ with t colours is the colouring of its vertices X in such a way that the following four conditions hold:

1) any co-edge A_i, $i \in I$, has at least two vertices coloured with the same colour;
2) any edge E_j, $j \in J$, has at least two vertices coloured differently;
3) the number of used colours is exactly t;
4) all the vertices are coloured.

The maximal (respective minimal) t for which there exists a strict colouring of a mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$ with t colours is called the upper (respective lower) chromatic number of H and denoted by $\bar{\chi}(H)$ (respective $\chi(H)$).

Let $r(H)$ be the number of strict colourings of a mixed hypergraph H with $t \geq 1$ colours. For each such hypergraph we associate the vector $R(H) = (r_1, r_2, \ldots, r_n) \in \mathcal{R}^n$ and call it chromatic spectrum of H. The value $\chi_m(H) = \frac{(\bar{\chi}(H)+\chi(H))}{2}$ is called the middle chromatic number of a hypergraph H and the value $b(H) = \bar{\chi}(H) - \chi(H) + 1$ is called the breadth of chromatic spectrum of H.

A set $T \subseteq X$ is called a bitransversal of a hypergraph $H = (X, \mathcal{E})$ if $|T \cap E_j| \geq 2$ for any $j \in J$. The minimal cardinality of a bitransversal is denoted by $\tau_2(H)$. If $\mathcal{E}$ does not contain the elements of cardinality ≥ 2 then we put $\tau_2(H) = 0$. Bitransversal of a co-hypergraph $H_{\mathcal{A}}$ is called a co-bitransversal. By $\tau(H)$ we denote the transversal hypergraph number [1]. A mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$, $\mathcal{A} \neq \emptyset$ is called a co-monostar if the following conditions hold:

1) $\tau(H_{\mathcal{A}}) = 1$; 2) $\tau_2(H_{\mathcal{A}}) \geq 3$.

A mixed hypergraph is called a co-bistar if there exists a co-bitransversal of cardinality 2, that does not constitute an edge.

A mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$ is called a mixed interval hypergraph, if there exists a linear ordering of its vertex set X such that each edge E_j, $j \in J$ represents an interval and each co-edge A_i, $i \in I$, represents an interval [2].

In a mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$ the set of indices $I_1 \in I$ and a respective subfamily A_i, $i \in I_1$ of co-edges is called a sieve if for any $x, y \in X$ and $j, k \in I_1$ the following implication holds:

$$(x, y) \in A_j \cap A_k \Longrightarrow (x, y) = E_l \in \mathcal{E}$$

for some $l \in J$. A maximum cardinality of a sieve in a hypergraph H is called the sieve-number of H and denoted by $s(H)$ [2].

An undirected graph $G = (X, E)$ is called an interval graph if its vertices can be put into one-to-one correspondence with a set of intervals $\mathcal{I}$ of a linearly ordered set (like the real line) such that two vertices are connected by an edge of G if and only if their corresponding intervals have nonempty intersection [3].

Definition 1. The hypergraph $H = (X, \mathcal{C})$ is called the *clique hypergraph of an interval graph (the quasi-interval hypergraph)* if there exists an interval graph $G = (X, E)$ with the clique family $\mathcal{C}(G)$ such that for every edge of hypergraph there exists the corresponding maximal clique in graph G and vice-versa.

It is known [6] that all the properties of arbitrary mixed hypergraph colourings may be reduced to the respective properties of reduced hypergraphs, i.e. the hypergraphs with $|A_i| \geq 3$, $i \in I$, and $|E_j| \geq 2$, $j \in J$, and without included edges and co-edges. So, let us consider further the reduced hypergraphs.

2. The structural properties of quasi-interval hypergraphs.

Remind at first some basic notions of the hypergraph theory we will use below.

The *2-section* $(H)_2$ of a hypergraph $H = (X, \mathcal{S})$ is defined to be a graph $G = (X, E)$ formed by X and the set $E = \{E_l : x_i x_j = E_l \iff \exists S_k \in \mathcal{S} : x_i \in S_k, x_j \in S_k\}$.

Let $H = (E, X_1, \ldots, X_n)$ be a hypergraph with n edges. The representative graph of H is defined to be a simple graph $L(H)$ of order n whose vertices $x_1, \ldots x_n$ respectively represent the edges $X_1, \ldots, X_n$ of H and the vertices x_i and x_j are joined by an edge if and only if $X_i \cap X_j \neq \emptyset$. [1] The Gilmore and Hoffman Theorem says that for undirected graph $G = (X, E)$ the following three statements are equivalent:

1) G is an interval graph;
2) G contains no chordless 4-cycle and its complement $\overline{G}$ is a comparability graph;
3) the maximal cliques of G can be linearly ordered such that, for every vertex x of G the maximal cliques containing x occur consecutively [3].

Theorem 1. *If $H = (X, \mathcal{S})$ is an interval hypergraph then $(H)_2$ is an interval graph.*

Proof. Let $H = (X, \mathcal{S})$ be an arbitrary interval hypergraph. Construct $(H)_2$ and enumerate the vertices of $(H)_2$ in the same way in which they have been numbered in $H = (X, \mathcal{S})$. After that we order the edges of hypergraph by the left end. This is possible because each vertex may represent the left end of the only one hyperedge (otherwise we should have included edges). In the ordering we constructed for any vertex the edges containing it appear consecutively. Further we number the cliques of $(H)_2$ in the same way in which have been numbered in $H = (X, \mathcal{S})$ the corresponding hyperedges. So, we obtain the order in $(H)_2$ such that for any vertex x the cliques C_i containing x have the consecutive numbers. Since we have the third condition of the Gilmore and Hoffman Theorem satisfied, $(H)_2$ is an interval graph. □

Corollary 1. *If $H = (X, \mathcal{S})$ is an arbitrary interval hypergraph and $H^* = (X^*, \mathcal{S}^*)$ is its dual hypergraph then $L(H^*)$ is an interval graph.*

Proof. Since $L(H^*) = (H)_2$ [1] the corollary evidently follows. □

Theorem 2. *If $H = (X, \mathcal{S})$ is an arbitrary interval hypergraph and $H^* = (X^*, \mathcal{S}^*)$ is its dual hypergraph then $(H^*)_2$ is an interval graph.*

Proof. It follows from the definition of interval hypergraph that if $H = (X, \mathcal{S})$ is an interval hypergraph then $L(H)$ is an interval graph. But since $L(H) = (H^*)_2$ we obtain that $(H^*)_2$ is an interval graph. □

Theorem 3. *Let $G = (X, E)$ be an arbitrary interval graph and $H(G) = (X, \mathcal{C})$ be its clique hypergraph. Then the dual hypergraph $H^*(G) = (X^*, \mathcal{C}^*)$ is an interval hypergraph.*

Proof. The Fullkerson-Gross Theorem [3] says that an undirected graph $G = (X, E)$ is an interval graph if and only if its clique matrix $\mathcal{M}(G)$ has the consecutive one's property for columns. Since $G = (X, E)$ is an interval graph $\mathcal{M}(G)$ has the mentioned property

$$\mathcal{M}(G) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 1 \end{pmatrix},$$

But the clique matrix of G is simultaneously the incidence matrix of the clique hypergraph $H(G)$.

$$\mathcal{M}(G) = \boldsymbol{A}(H(G)).$$

Now let us consider the incidence matrix of the dual hypergraph $H^*(G)$. It is known [1] that $\boldsymbol{A}(H^*(G)) = \boldsymbol{A}^T(H(G))$. Hence $\boldsymbol{A}(H^*(G))$ has the consecutive one's property for rows:

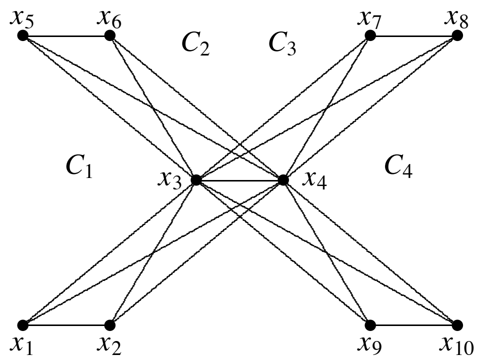
$$\boldsymbol{A}(H(G)) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

i.e. each hyperedge in $H^*(G)$ represents an interval. So, the proof is complete. □

Now let us show that not for any interval graph $G = (X, E)$ the clique hypergraph $H(G)$ is an interval hypergraph.

Example 1. Let us consider the graph $G = (X, E)$:

$G = (X, E)$

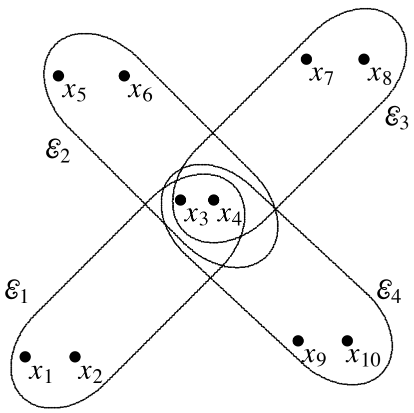


$$\mathcal{M}(G) = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

As one can see, $\mathcal{M}(G)$ has the consecutive one's property for columns. Hence, by the Fullkerson-Gross criterion $G = (X, E)$ is the interval graph indeed.

But it is evidently that $H(G)$ is not an interval hypergraph:

$H(G)$



Thus the interval hypergraphs are not the self-dual hypergraphs.

Now, let us clarify the conditions in which the quasi-interval hypergraphs are the interval hypergraphs in fact.

Theorem 4. *Let* $\{I_i\}$, $i = 1, \dots, n$, *be the system of intervals on the real line without coinciding intervals and* $G = (X, E)$ *the corresponding interval graph. Then the clique hypergraph* $H(G)$ *is an interval hypergraph.*

Proof. Let us number the intervals from $\{I_i\}$ from left to right in ascending order. Thus $\{I_i\}$, $i = 1, \dots, n$, does not contain the coinciding intervals, then, obviously each clique in $G = (X, E)$ will represent a numerical interval, hence $H(G)$ will be an interval hypergraph. □

Definition 2. Given graph $G = (X, E)$. For any $x \in X$ the number of cliques containing x we call *the power* of x, denoted by $p(x)$.

In fact, $p(x)$ is equal to the degree of x in the clique hypergraph $H(G)$.

Theorem 5. *Let* $G = (X, E)$ *be an arbitrary interval graph and* $\mathcal{C} = \{C_1, \dots, C_m\}$ *be the set of its maximal cliques. The clique hypergraph* $H(G) = (X, \mathcal{C})$ *will be an interval hypergraph if and only if the following two conditions hold:*

1) $C_i \cap C_j \neq C_i \cap C_k$, $\forall i, j, k = \overline{1, m}$
2) *If* C_{i-1}, C_i *and* C_{i+1} *are the first three or the last three cliques in* G *and* $C_{i-1} \cap C_i \cap C_{i+1} \neq \emptyset$,

then there is no vertex $x_i \in C_i$ *such that* $p(x_i) < 2$.

Proof. $\Longrightarrow$ Let $H(G) = (X, \mathcal{C})$ be an interval hypergraph. Then by Theorem 1, $(H)_2$ is an interval graph. The first condition obviously follows from prohibition of the existence of included edges.

Now let us prove the second condition. Let $C_{i-1} = \{x_{k_{i-1}}\}$, $C_i = \{x_{j_i}\}$ and $C_{i+1} = \{x_{l_{i+1}}\}$, $i = 2, \dots, m-1$, be the edges of the interval hypergraph $H = (X, \mathcal{C})$. Since every hyperedge represents an interval, the following precedence relation for the vertices of H holds:

$$x_{k_{i-1}} \preceq x_{j_i} \preceq x_{l_{i+1}}.$$

It means that for the indices of the vertices the following inequality is true:

$$k_{i-1} \leq j_i \leq l_{i+1},$$

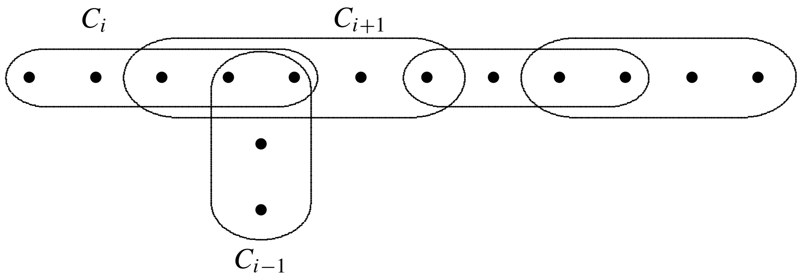
where the equality achieves only for the vertices of degree more than one. If there exists the vertex $x_{h_i} \in C_i$ such that $p(x_{h_i}) = 1$ then

$$k_{i-1} < h_i < l_{i+1}, \quad \forall k, l = 1, \dots, n,$$

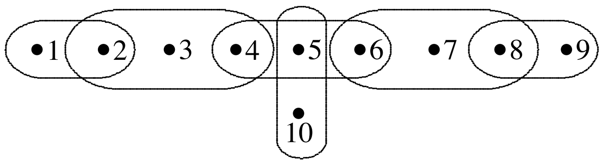
i.e. this vertex must be settled after the vertices from C_{i-1} and before the vertices from the C_{i+1}, what is impossible because of

$$C_{i-1} \cap C_i \cap C_{i+1} \neq \emptyset.$$

$\Longleftarrow$ Note at first that in given conditions the hypergraph $H(G)$ can be always represented as a chain, since the second condition excludes the presence of branches of the following mode:



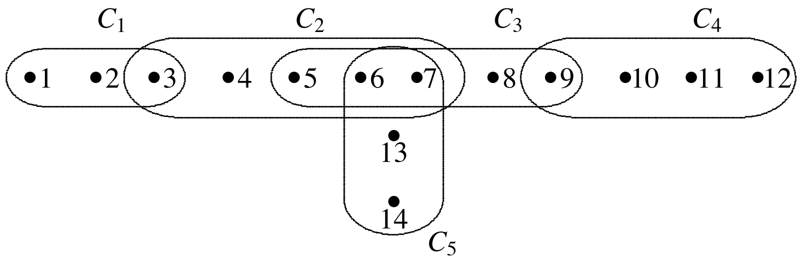
Other models of branches are excluded by the properties of interval graphs. Indeed, the cliques of interval graph can not be intersected in such that a way:



because the clique matrix $\mathcal{M}(G)$ has not the consecutive one's property for columns.

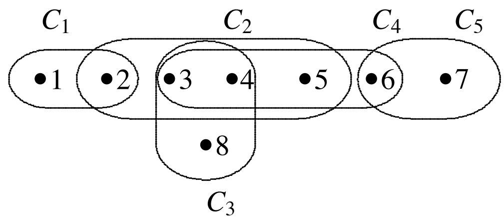
$$\mathcal{M}(G) = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{array} \begin{array}{c} \begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

and in such a way:



$$\mathcal{M}(G) = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{array} \begin{array}{c} \begin{array}{cccccccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} \end{array} \\ \left(\begin{array}{cccccccccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \end{array}$$

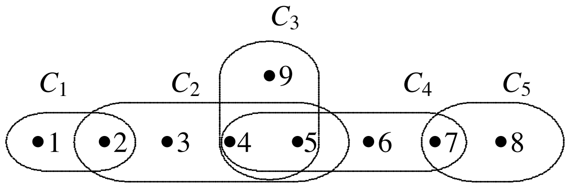
or in such a way:



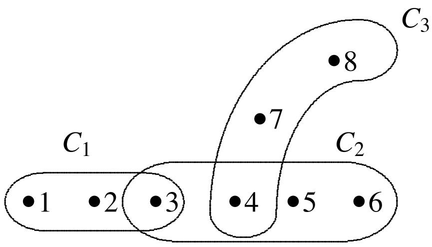
$$\mathcal{M}(G) = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{array} \\ \left(\begin{array}{cccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right) \end{array}$$

by the same reason.

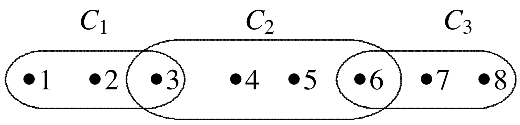
The branches of this mode:



are forbidden by the condition 1) of Theorem, and, the last, in the case of such clique intersection:

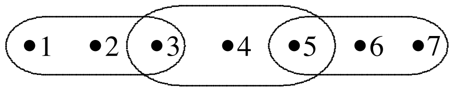


we can always renumber our intervals and corresponding vertices of $G = (X, E)$ in such a way:



Now let us begin the proof of Theorem 5.

Let $G = (X, E)$ be an arbitrary interval graph which satisfies the two conditions mentioned above. We number the vertices of X in the following way: from all the simplicial vertices of G (remind that the vertex x is called the simplicial if its neighbourhood forms the clique and the simplicial vertex in triangulated graph necessarily exists [4]) we choose one, to which corresponds the most left of the intervals, corresponding to the simplicial vertices. (It is done with the aim to choose from the simplicial vertices 1, 2 , 4, 6, 7 (see the picture) the vertex number one).



The chosen vertex is numbered as x_1 and the clique containing x_1 as C_1. Remind that x_1 can represent the left end of the only one clique since otherwise we deal with the included cliques what contradicts the maximality of the cliques. If C_1 still contains vertices x_i with the $p(x_i) = 1$ we arrange them next and number them in ascending order. After we have exhausted all the vertices of power one, we find the vertex x_j of power 2. If $G = (X, E)$ is not a complete graph such a vertex necessarily exists because C_1 can not be intersected with the two cliques by the same vertex set. We assign to x_j the following number and call C_2 the second clique containing x_j. After x_j we arrange all the vertices of power 2 belonging to C_2 as well as to C_1. After that if the clique C_1 has not yet been exhausted we place the vertices from C_1 of power 3 and the clique containing them we name C_3 and so on. But if the clique C_1 has been settled, we arrange further the vertices of power 1 belonging to the last clique we numbered C_i and further we operate with C_i in the same way as we did with the C_1. If we can not found the vertices of power 1 in C_i then we arrange the vertices of power 2, number them in ascending order, and the new clique containing them name C_{i+1} and so on. After we have numbered all the vertices in this way, we obtain the enumeration the vertices of G under which any clique represents an interval, i.e. $H(G) = (X, \mathcal{C})$ is an interval hypergraph. □

Definition 3. We shall call the hyperedge C_i of the hypergraph $H = (X, \mathcal{C})$ *the terminal* if the following two conditions hold:

1) There exists a vertex x_i such that $x_i \in C_i$ and $p(x_i) = 1$.
2) The strong elimination of x_i does not break the connectivity of H.

Definition 4. The clique C_i of a graph $G = (X, E)$ is said to be *final* if the corresponding hyperedge C_i in a clique hypergraph $H = (X, \mathcal{C})$ is final.

Theorem 6. *Let $G = (X, E)$ be an arbitrary interval graph with the clique family $\{C_1, \dots, C_m\}$. The following conditions are equivalent:*

1) *a) $C_i \cap C_j \neq C_i \cap C_k \ \ \forall i, j, k \in J$.*
   b) If C_{i-1}, C_i, C_{i+1} are the first three or the last three cliques of G and $C_{i-1} \cap C_i \cap C_{i+1} \neq \emptyset$ then C_i does not contain vertices of power one.
2) *$G = (X, E)$ does not contain more than two final cliques.*

Proof. Note at first that if $G = (X, E)$ is the complete graph then it has the only one clique and we have both of our conditions satisfied automatically, so let us consider further G to be an arbitrary interval graph different from the complete.

1) $\Longrightarrow$ 2)

It follows from Theorem 5 that if the condition 1) holds then $H(G)$ is an interval hypergraph, but each interval hypergraph (different from the complete) has exactly two final hyperedges: the first and the last.

2) $\Longrightarrow$ 1)

Let $G = (X, E)$ be an arbitrary interval hypergraph having two final cliques. The proof is performed by induction on k, where k is the number of maximal cliques of G. Let G contains three cliques. We arrange them in following way: the first we arrange one of final cliques and number its vertices: at first we put all the vertices of power one and number them in ascending order. After that we find the vertex of power two. It necessarily exists because otherwise either one of our cliques contain another in contradiction with the cliques maximality or we have three final cliques what contradicts to our condition. Hence, no more than two cliques can be intersected on the same set of vertices. Moreover, if G has only two final cliques and $C_1 \cap C_2 \cap C_3 \neq \emptyset$, then C_2 obviously can not contain the vertices of power one since otherwise C_2 will be the third final clique. Let $G = (X, E)$ be the interval graph containing n maximal cliques. Its arbitrary subgraph $G_1 = (X_1, E_1)$ containing $(n - 1)$ maximal cliques is an interval graph too [3]. We assume 1) to be true for G_1. Then by Theorem 5, $H(G_1)$ is an interval hypergraph. Hence the vertices of X_1 can be numbered in such a way that every clique C_i, $i = 1, \ldots, n-1$, represents an interval. Now let us see what happens with the graph G. The vertices of C_n contained in the other $(n - 1)$ cliques number with the indexes assigned them earlier. As one can easy see, the beginning of C_n can not coincide with the beginning of some other clique as well as the end of C_n can not coincide with the end of other clique, otherwise either one of the cliques will contain another or the number of final cliques will increase. On the same reason if $C_{n-2} \cap C_{n-1} \cap C_n \neq \emptyset$, C_{n-1} can not contain the vertices of power one. □

Let us remember some definitions given in [1] and [8]:

In a hypergraph $H = (X, \mathcal{E})$ a chain of length q is defined to be a sequence $(x_1, E_1, x_2, E_2, \ldots, E_q, x_{q+1})$ such that

(1) $x_1, \ldots, x_q$ are all distinct vertices of H.
(2) $E_1, \ldots, E_q$ are all distinct edges of H.
(3) $x_k, x_{k+1} \in E_k$ for $k = 1, 2, \ldots, q$.

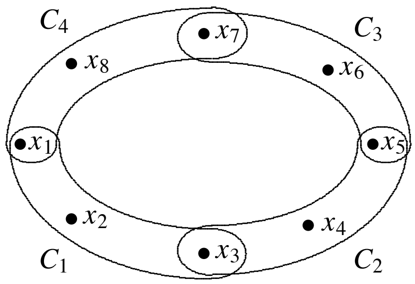
If $q \geq 1$ and $x_{q+1} = x_1$ then this chain is called a cycle of length q.

Any edge E containing two non-consecutive vertices of a cycle is called a chord.

Theorem 7. *Let $G = (X, E)$ be an arbitrary interval graph and $H(G) = (X, \mathcal{C})$ be the clique hypergraph of G. Then $H(G)$ contains no chordless cycles.*

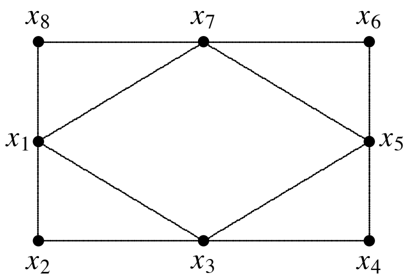
Proof. By contradiction. Let us suppose that $H(G)$ contain the cycle $\boldsymbol{C}_n$, $n \geq 4$, for example C_1, C_2, C_3, C_4, C_1.

$H(G)$



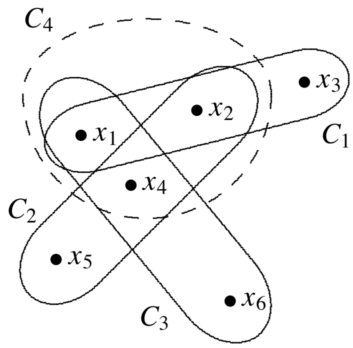
Since the vertices x_1, x_2, x_3 are contained in the same hyperedge C_1, then in graph G they are representing the clique, i.e. there exists an edge $E_3 = \{x_1, x_3\}$. By the same reason there exist the edges $E_6 = \{x_3, x_5\}$, $E_9 = \{x_5, x_7\}$ and $E_{12} = \{x_7, x_1\}$. So we obtain the chordless cycle $\boldsymbol{C} = \{x_1, x_3, x_5, x_7, x_1\}$ in G, what is in contradiction with the triangulated property of interval graphs [3].

$G = (X, E)$



Furthermore H can not contain the cycles $\boldsymbol{C}_n$, $n = 3$. In fact, let C_1, C_2, C_3, C_1 form the cycle in H. Since the hyperedges C_1, C_2 and C_3 are pairwise intersected, there exists an edge C_4 containing the vertices x_1, x_2, x_4 [1].

H(G)



Now let us examine the incidence matrix of H:

$$A(H) = \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ C_1 & 1 & 1 & 1 & 0 & 0 & 0 \\ C_2 & 0 & 1 & 0 & 1 & 1 & 0 \\ C_3 & 1 & 0 & 0 & 1 & 0 & 1 \\ C_4 & 1 & 1 & 0 & 1 & 0 & 0 \end{matrix}$$

As one can easy see the matrix $A(H)$ has not the consecutive one's property for columns. Since the incidence matrix of H coincides with the clique matrix of G we obtain that G is not an interval graph. So, we have the contradiction, i.e. H can not contain the assumed cycle. The proof is complete. □

Further we will deal with the co-hypergraphs $H = (X, \mathcal{A})$, i.e. when we have an interval graph $G = (X, E)$ with the clique family $\{C_1, \dots, C_m\}$ each clique C_i will represent a co-edge A_i in the clique co-hypergraph $H = (X, \mathcal{A})$.

3. The colouring properties of quasi-interval co-hypergraphs.

Thus in conditions of Theorems 4, 5 and 6 the clique hypergraph of an interval graph is the interval hypergraph and the upper chromatic number for such the hypergraphs is already known [2], let us investigate the colouring properties of the quasi-interval hypergraphs which are not the interval hypergraphs.

Theorem 8. *Let $H_{\mathcal{A}} = (X, \mathcal{A})$ be an acyclic co-hypergraph such that*

$$|A_i \cap A_j| \leq 1\,, \quad \forall i, j \in J.$$

Then

$$\bar{\chi}(H_{\mathcal{A}}) = |X| - |\mathcal{A}|. \tag{1}$$

Proof. Remember at first that the hypergraph is called acyclic if it contains no cycles. The proof is performed by induction on the number of co-edges. If $H^1_{\mathcal{A}} = (X^1, \mathcal{A}^1)$ is such that $\mathcal{A}^1 = A_1$, then $\bar{\chi}(H^1_{\mathcal{A}})$ is obviously equal to

$$|A_1| - 1 = |X^1| - \mathcal{A}^1.$$

Let $H^2_{\mathcal{A}} = (X^2, \mathcal{A}^2)$, $\mathcal{A}^2 = \{A_1, A_2\}$. Then on the one hand we can colour $H^2_{\mathcal{A}}$ with $|X^2| - 2$ colours, hence $\bar{\chi}(H^2_{\mathcal{A}}) \geq |X|^2 - 2$. On the other hand: $\bar{\chi}(H^2_{\mathcal{A}}) \leq \bar{\chi}(H^1_{\mathcal{A}}) + (|A_2| - 1) - 1 = (|A_1| - 1) + (|A_2| - 1) - 1$. Since $|A_1| + |A_2| - 1 = |X^2|$ we obtain $\bar{\chi}(H^2_{\mathcal{A}}) \leq |X^2| - 2$. Hence,

$$\bar{\chi}(H^2_{\mathcal{A}}) = |X^2| - 2.$$

Let $H = (X, A)$ contain m co-edges. We assume (1) to be true for its arbitrary subhypergraph $H^{(m-1)}_{\mathcal{A}}$ containing $(m-1)$ co-edges: $H^{(m-1)}_{\mathcal{A}} = (X^{(m-1)}, \mathcal{A}^{(m-1)})$, $\mathcal{A}^{(m-1)} = \{A_1, \ldots, A_{m-1}\}$, $\bar{\chi}(H^{(m-1)}_{\mathcal{A}}) = |X^{(m-1)}| - (m-1)$. Let us find $\bar{\chi}(H^m_{\mathcal{A}})$, $H^m_{\mathcal{A}} = (X^m, \mathcal{A}^m)$, where $\mathcal{A}^m = \{A_1, \ldots, A_m\}$. On the one hand we can colour $H^m_{\mathcal{A}}$ by the $|X^m| - m$ colours, hence

$$\bar{\chi}(H^m_{\mathcal{A}}) \geq |X^m| - m.$$

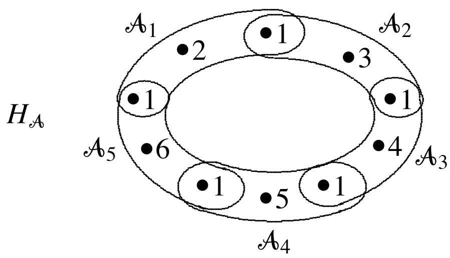
On the other hand:

$$\bar{\chi}(H^m_{\mathcal{A}}) \leq \bar{\chi}(H^{(m-1)}_{\mathcal{A}}) + (|A^m| - 1) - 1 =$$
$$= |X^{(m-1)}| - (m-1) + |A^m| - 1 - 1 = |X^m| - m.$$

Hence,

$$\bar{\chi}(H^m_{\mathcal{A}}) = |X^m| - m. \qquad \square$$

Remark. Note that the prohibition on the existence of cycles in a co-hypergraph is really important since the cycle in $H_{\mathcal{A}}$ could considerably increase the value of $\bar{\chi}$ and the formula (1) would be only the lower bound of $\bar{\chi}(H_{\mathcal{A}})$. For example:



As one can see $|X| = 10$, $|\mathcal{A}| = 5$, but $\bar{\chi}(H_{\mathcal{A}}) = 6$.

Corollary 2. *If $H_{\mathcal{A}} = (X, \mathcal{A})$ is an acyclic co-hypergraph such that*

$$|A_i \cap A_j| \leq 1\,, \quad \forall i, j \in J,$$

then

$$\bar{\chi}(H_{\mathcal{A}}) = |X| - s(H_{\mathcal{A}}),$$

where $s(H_{\mathcal{A}})$ is the sieve number of $H_{\mathcal{A}}$.

Proof. It is evidently that if $|A_i \cap A_j| \leq 1, \forall i, j \in J$, then in such a hypergraph $s(H_{\mathcal{A}}) = |\mathcal{A}|$. □

Corollary 3. *Let $H_{\mathcal{A}} = (X, \mathcal{A})$ be a quasi-interval co-hypergraph such that*

$$|A_i \cap A_j| \leq 1\,, \quad \forall i, j \in J.$$

Then

$$\bar{\chi}(H_{\mathcal{A}}) = |X| - |\mathcal{A}| = |X| - s(H_{\mathcal{A}}).$$

Proof. It follows from Theorem 7 that the quasi-interval co-hypergraph contains no cycles. Then by Theorem 8 and Corollary 2

$$\bar{\chi}(H_{\mathcal{A}}) = |X| - |\mathcal{A}| = |X| - s(H_{\mathcal{A}}). \qquad \square$$

Corollary 4. *Let $H_{\mathcal{A}} = (X, \mathcal{A})$ be an acyclic co-hypergraph such that*

$$|A_i \cap A_j| \leq 1\,, \quad \forall i, j \in J\,.$$

Then

$$\chi_m(H_{\mathcal{A}}) = \frac{|X| - |\mathcal{A}| + 1}{2}$$

and

$$b(H_{\mathcal{A}}) = |X| - |\mathcal{A}|.$$

Definition 5. Let $(H_{\mathcal{A}}) = (X, \mathcal{A})$ be an arbitrary co-hypergraph. Then set of co-bistars $k_1 = (X_1, \mathcal{A}_1), \ldots, k_m = (X_m, \mathcal{A}_m)$ will be called *the partition of $H_{\mathcal{A}} = (X, \mathcal{A})$ onto co-bistars* if $\bigcup_{i=1}^{m} X_i = X$ and $\bigcup_{j=1}^{m} \mathcal{A}_j = \mathcal{A}$.

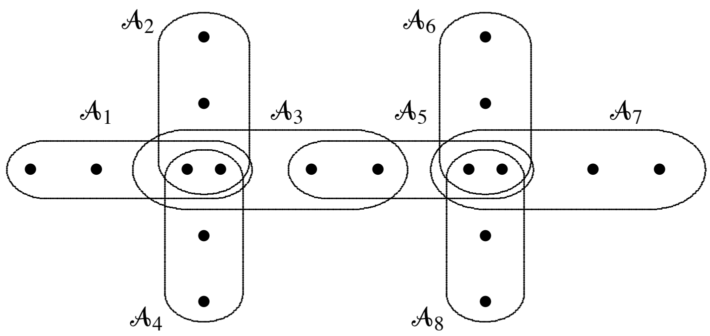
The number of co-bistars in the partition of $H_{\mathcal{A}}$ will be called *the cardinality* of the partition.

Definition 6. The partition of a co-hypergraph $H_{\mathcal{A}} = (X, \mathcal{A})$ onto co-bistars $k_1 = (X_1, \mathcal{A}_1), \dots, k_m = (X_m, \mathcal{A}_m)$ will be called *the minimal* if the number of co-bistars in it is as small as possible.

It is clear that in such a partition every co-bistar $k_i = (X_i, \mathcal{A}_i)$ has at least one co-edge A_i such that:

$$A_i \in \mathcal{A}_i \quad \text{and} \quad A_i \notin \mathcal{A}_j\ , \ \ \forall j = 1, \dots, m,\ i \neq j\ .$$

Example 2. The partition of $H_{\mathcal{A}} : k_1 = (X_1, \mathcal{A}_1)$, $\mathcal{A}_1 = \{A_1, A_2, A_3, A_4\}$; $k_2 = (X_2, \mathcal{A}_2)$, $\mathcal{A}_2 = \{A_3, A_5\}$; $k_3 = (X_3, \mathcal{A}_3)$, $\mathcal{A}_3 = \{A_5, A_6, A_7, A_8\}$. The cardinality of this partition is equal to 3.



The minimal partition of $H_{\mathcal{A}} : k_1 = (X_1, \mathcal{A}_1)$, $\mathcal{A}_1 = A_1, A_2, A_3, A_4$; $k_2 = (X_2, \mathcal{A}_2)$, $\mathcal{A}_2 = A_5, A_6, A_7, A_8$. The cardinality of the minimal partition is 2.

Theorem 9. *Let $H_{\mathcal{A}} = (X, \mathcal{A})$ be an arbitrary co-hypergraph and $k_1, \dots, k_l$ be the minimal partition of $H_{\mathcal{A}}$ onto co-bistars. Then*

$$\bar{\chi}(H_{\mathcal{A}}) \geq |X| - l.$$

Proof. Let us show that we can colour $H_{\mathcal{A}}$ with $|X| - l$ colours. Consider three possible cases:

1. Assume that the minimal partition of $H_{\mathcal{A}}$ onto co-bistars $k_1, \dots, k_l$ contains no co-bistars k_i, k_j with the co-bitransversals b_i, b_j such that $b_i \cap b_j \neq \emptyset$, $\forall i, j = \overline{1, l}$, $i \neq j$. Then we colour any two vertices from b_1 with the first

colour, two vertices from b_2 with the second colour and so on. Since the co-bitransversals are not intersecting we have coloured $2 \times l$ vertices with l colours. The remainder of vertices we colour overall differently. Since $\bigcup_{i=1}^{l} \mathcal{A}_i = \mathcal{A}$, every co-edge A_j has two vertices of the same colour, i.e. we obtain the correct colouring of $H_{\mathcal{A}}$ and we have used $|X| - 2 \times l + l = |X| - l$ colours, i.e.

$$\bar{\chi}(H_{\mathcal{A}}) \geq |X| - l.$$

2. Now let us presuppose that the partition $k_1, \ldots, k_l$ contains a number of co-bistars k_i such that $\forall k_i$ there exists at least one k_j such that $b_i \cap b_j \neq \emptyset$, $i \neq j$, $i, j = \overline{1, l_1}$, where $1 \leq l_1 \leq l$ and a number of co-bistars k_m such that $b_m \cap b_s = \emptyset$, $\forall m, s, m \neq s$, $m = \overline{l_1, l}$, $s = \overline{1, l}$ and $H_{\mathcal{A}}$ contains no cycles. Let us see how we can colour such a hypergraph with l_1 co-bistars with intersecting co-bitransversals and $(l - l_1)$ co-bistars with non-intersecting co-bitransversals.

At first we colour two vertices from each co-bitransversal b_i, $i = \overline{1, l_1}$, with the first colour. Since $|b_i \cap b_j| \leq 1$, $\forall i, j = \overline{1, l_1}$, $i \neq j$ (otherwise we could consider k_i and k_j as a single co-bistar and the cardinality of minimal partition would be $(l - 1)$) and $H_{\mathcal{A}}$ contains no cycles we have coloured $2 \times l_1 - l_1 + 1 = l_1 + 1$ vertices with the first colour.

After that we colour two vertices from b_{l_1+1} with the second colour, two vertices from b_{l_1+2} with the third colour and so on. So we have coloured $(l_1 + 1) + 2 \times (l - l_1)$ vertices using $(l - l_1 + 1)$ colours and each co-edge A_i already has two vertices coloured equally. The remainder of the vertices we colour overall differently. Finally we obtain the correct colouring of $H_{\mathcal{A}}$ with

$$|X| - (2 \times (l - l_1) + (l_1 + 1)) + l - l_1 + 1 =$$
$$= |X| - 2 \times l + 2 \times l_1 - l_1 - 1 + l - l_1 + 1 = |X| - l$$

colours, i.e.

$$\bar{\chi}(H_{\mathcal{A}}) \geq |X| - l.$$

3. Now let the partition $k_1, \ldots, k_l$ contains a number of co-bistars k_i such that $\forall k_i$ there exists at least one k_j such that $b_i \cap b_j \neq \emptyset$, $i \neq j$, $i, j = \overline{1, l_1}$, where $1 \leq l_1 \leq l$ and a number of co-bistars k_m such that $b_m \cap b_s = \emptyset$, $\forall m, s, m \neq s$, $m = \overline{l_1, l}$, $s = \overline{1, l}$ and $H_{\mathcal{A}}$ contains an arbitrary cycle. If $H_{\mathcal{A}}$ contains a cycle then it could happen that the co-bitransversal b_i has no vertices x such that $x \in b_i$ and $x \notin b_j$, $i, j = \overline{1, l_1}$, i.e. the common number of co-bitransversal vertices of the co-bistars $k_1, \ldots, k_{l_1}$ is less than $l_1 + 1$. Let this number be equal to y. So we colour y vertices with the first colour, $2 \times (l - l_1)$ vertices with $(l - l_1)$ colours and every co-edge already has two vertices coloured equally.

The remained vertices we colour overall differently and obtain the correct colouring with

$$|X| - (y + 2 \times (l - l_1)) + (l - l_1) + 1 = |X| - (y + (l - l_1)) + 1 \geq$$
$$\geq |X| - (l_1 + 1 + (l - l_1)) + 1 = |X| - l$$

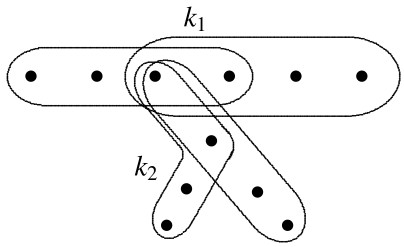
colours, i.e.

$$\bar{\chi}(H_{\mathcal{A}}) \geq |X| - l. \qquad \square$$

Theorem 10. *Let $H_{\mathcal{A}} = (X, \mathcal{A})$ be an arbitrary acyclic co-hypergraph and $k_1, \ldots, k_l$ be its minimal partition onto co-bistars. Then*

$$\bar{\chi}(H_{\mathcal{A}}) = |X| - l.$$

Proof. It follows from Theorem 9 that $\bar{\chi}(H_{\mathcal{A}}) \geq |X| - l$. Let us prove that $\bar{\chi}(H_{\mathcal{A}}) \leq |X| - l$. The proof is performed by induction on the number of co-bistars. It is known [6] that $\bar{\chi}(H_{\mathcal{A}}) = |X| - 2$ if $H_{\mathcal{A}}$ is either a union of two co-bistars with non-intersecting co-bitransversals, or a hole. It means that if we have the co-hypergraph $H_{\mathcal{A}}$ with the minimal partition onto co-bistars k_1, k_2 we can assume $\bar{\chi}(H_{\mathcal{A}}) \leq |X| - 2 = |X| - l$. Really, if $b_1 \cap b_2 \neq \emptyset$ then $H_{\mathcal{A}}$ represents a hole.



In fact, we have a minimal co-bitransversal of power 3 that is not an edge and no edge of cardinality 2 coincides with any two of this vertices.

Now let us have the co-hypergraph $H = (X, A)$ with the minimal partition onto co-bistars $k_1, k_2, \ldots, k_l$. Assume the hypothesis is true for its subhypergraph $H^1_{\mathcal{A}} = (X^1, \mathcal{A}^1)$, where $k_1, \ldots, k_{l-1}$ is the minimal partition of $H^1_{\mathcal{A}}$ onto co-bistars. It is evident that $\nexists b_i$, $i = \overline{1, l-1}$, such that $b_i = H^1_{\mathcal{A}} \cap b_l$ otherwise the cardinality of the minimal partition would be $(l - 1)$. Then, obviously:

$$\bar{\chi}(H_{\mathcal{A}}) \leq \bar{\chi}(H^1_{\mathcal{A}}) + |X_l| - |X^1 \cap X_l| - 1 =$$
$$= |X^1| - (l - 1) + |X_l| - |X^1 \cap X_l| - 1 = |X| - l. \qquad \square$$

Corollary 5. *Let $H_{\mathcal{A}} = (X, \mathcal{A})$ be an arbitrary quasi-interval co-hypergraph. Then*

$$\bar{\chi}(H_{\mathcal{A}}) = |X| - l,$$

where l is the cardinality of the minimal partition of $H_{\mathcal{A}}$ onto co-bistars.

Corollary 6. *Let $H_{\mathcal{A}} = (X, \mathcal{A})$ be an arbitrary acyclic co-hypergraph. Then*

$$\bar{\chi}(H_{\mathcal{A}}) = |X| - s(H_{\mathcal{A}}),$$

where $s(H_{\mathcal{A}})$ is the maximal sieve of $H_{\mathcal{A}}$.

Proof. It is evidently that if we have an acyclic co-hypergraph $H_{\mathcal{A}}$ with the minimal partition onto co-bistars $k_1 = (X_1, \mathcal{A}_1), \ldots, k_l = (X_l, \mathcal{A}_l)$, then for every co-edge $A_{j_i} \in \mathcal{A}_i$, $i = \overline{1, l}$, $|A_{j_i} \cap A_{s_i}| \geq 2$ and $|A_{j_i} \cap A_{m_k}| \leq 1$, $k = \overline{1, l}$, $k \neq i$, i.e. we can take the only one co-edge from any co-bistar k_i, $i = \overline{1, l}$, to form the maximal sieve of $H_{\mathcal{A}}$. So, we obtain $s(H_{\mathcal{A}}) = l$. □

Since the quasi-interval co-hypergraphs contain no cycles then evidently follows:

Corollary 7. *Let $H_{\mathcal{A}} = (X, \mathcal{A})$ be an arbitrary quasi-interval co-hypergraph. Then*

$$\bar{\chi}(H_{\mathcal{A}}) = |X| - s(H_{\mathcal{A}}),$$

where $s(H_{\mathcal{A}})$ is the sieve number of $H_{\mathcal{A}}$.

Thus although the class of quasi-interval co-hypergraphs is wider than the class of interval co-hypergraphs we obtain that the value of the upper chromatic number of a quasi-interval co-hypergraphs coincides with the value of $\bar{\chi}$ for the mixed interval hypergraph.

Now let us specify some additional conditions:

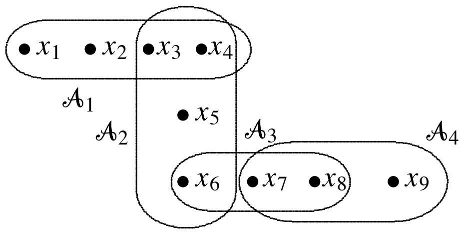
The number of edges in a bi-or a monostar k_i is called the cardinality of a star and denoted by $card(k_i)$.

Definition 7. A co-hypergraph $H_{\mathcal{A}} = (X, \mathcal{A})$ is said to be *divisible onto co-bistars* if the minimal partition of $H_{\mathcal{A}}$ does not contain the co-bistars of cardinality less than 2.

Otherwise the co-hypergraph is called undivisible and the single edges are not considered like a co-bistars (as like as the co-monostars).

Example 3.

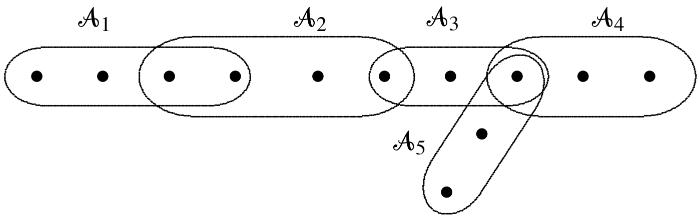
1) The divisible co-hypergraph:



The minimal partition of $H_{\mathcal{A}} : k_1 = (X_1, \mathcal{A}_1)$; $k_2 = (X_2, \mathcal{A}_2)$, where $X_1 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $\mathcal{A}_1 = \{A_1, A_2\}$; $X_2 = \{x_6, x_7, x_8, x_9\}$, $\mathcal{A}_2 = \{A_3, A_4\}$.

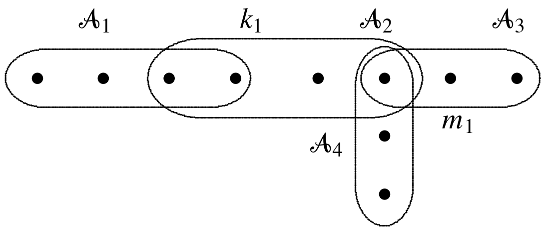
2) The undivisible co-hypergraph:

The edges A_3, A_4, A_5 do not constitute the co-bistars.



So, further when we speak about the co-bistars we mean the co-bistars of power 2 or more. And now we introduce some change in notation. Since dealing with the co-bistars we are interested in co-edges belonging to them and for the convenience of notations instead of $k_1 = (X_1, \mathcal{A}_1)$, $X_1 = \{x_1, \dots, x_i\}$, $\mathcal{A}_1 = \{A_1, \dots, A_j\}$ we will write further $k_1 = (A_1, \dots, A_j)$. And one more supplementary condition: if the co-hypergraph $H_{\mathcal{A}}$ contain the co-monostar $m_z = (A_1, A_2, \dots, A_z)$ and the co-bistar $k_n = (A_z, A_{z+1}, \dots, A_n)$ then we consider the edge A_z belonging to the co-bistar and not belonging to the co-monostar and the cardinality of co-monostar is $z - 1$ (instead of z).

Example 4.



The co-hypergraph $H_{\mathcal{A}}$ contains the co-bistar $k_1 = (A_1, A_2)$ and the co-monostar $m_1 = (A_3, A_4)$. The edge A_2 we consider belonging to the co-bistar and not to the co-monostar.

In given conditions the Theorem 10 may be rephrased in the following way:

Theorem 11. *Let $H_{\mathcal{A}} = (X, \mathcal{A})$ be an acyclic co-hypergraph, l be the cardinality of its minimal partition onto co-bistars and k be the number of co-edges not belonging to co-bistars. Then*

$$\bar{\chi}(H_{\mathcal{A}}) = |X| - (k + l).$$

Corollary 8. *Let $H_{\mathcal{A}} = (X, \mathcal{A})$ be an arbitrary quasi-interval co-hypergraph, l be the cardinality of its minimal partition onto co-bistars and k be the number of co-edges not belonging to co-bistars. Then*

$$\bar{\chi}(H_{\mathcal{A}}) = |X| - (k + l).$$

Theorem 12. *Let $H_{\mathcal{A}} = (X, \mathcal{A})$ be an arbitrary acyclic co-hypergraph. Then*

$$s(H_{\mathcal{A}}) = k + l,$$

where $s(H_{\mathcal{A}})$ is the sieve number of $H_{\mathcal{A}}$, l is the cardinality of the minimal partition of $H_{\mathcal{A}}$ onto co-bistars and k is the number of co-edges not belonging to co-bistars.

Proof. Since by definition of co-bistar every single co-edge could be considered as the co-bistar (we may assume any two vertices to be the co-bitransversal), $(k + l)$ would be the cardinality of the minimal partition of $H_{\mathcal{A}}$ onto co-bistars. In fact, we may consider the edges to be the co-bistars of power 1. Since l is the smallest number of co-bistars of power 2 or more, contained in $H_{\mathcal{A}}$ and k is the number of co-edges of power 1 and it obviously could not be decreased, $(l + k)$ would be the smallest number of co-bistars of $H_{\mathcal{A}}$. By Corollary 6 we obtain $s(H_{\mathcal{A}}) = (k + l)$. □

For a mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$ a set $P \subseteq X$ is said to be *co-stable* if it does not contain any co-edge A_i, $i \in I$. The maximal cardinality of a co-stable set is called the *co-stability number* and denoted by $\alpha_{\mathcal{A}}(H)$ [6].

Let $H = (X, \mathcal{A} \cup \mathcal{E})$. $H' = (X', \mathcal{A}' \cup \mathcal{E}')$ is called induced subhypergraph of H if $X' \subseteq X$, $\mathcal{A}'$ and $\mathcal{E}'$ contain all elements of $\mathcal{A}$ and $\mathcal{E}$ respectively, which are wholly (entirely) contained in X'.

A mixed hypergarph H is called a co-perfect hypergraph if for any its subhypergraph H' the following equality holds ([7]):

$$\bar{\chi}(H') = \alpha_{\mathcal{A}}(H').$$

Theorem 13. *Let $H_{\mathcal{A}} = (X, \mathcal{A})$ be an arbitrary acyclic co-hypergraph. Then $H_{\mathcal{A}}$ is co-perfect if and only if it does not contain co-monostars as a subhypergraphs.*

Proof. $\Longrightarrow$ It is obviously because the co-monostars are not the co-perfect hypergraphs.

$\Longleftarrow$ It is clear that if $H_{\mathcal{A}} = (X, \mathcal{A})$ is an acyclic co-hypergraph then any its subhypergraph $H'_{\mathcal{A}} = (X', \mathcal{A}')$ will be also an acyclic co-hypergraph. So, it is sufficient only to prove that $\bar{\chi}(H_{\mathcal{A}}) = \alpha_{\mathcal{A}}(H_{\mathcal{A}})$. Let $H_{\mathcal{A}}$ does not contain the co-monostars. Then, obviously, $\alpha_{\mathcal{A}}(H_{\mathcal{A}}) = |X| - l$, where l is the cardinality of the minimal partition of $H_{\mathcal{A}}$ onto co-bistars. But by Theorem 10, $\bar{\chi}(H_{\mathcal{A}}) = |X| - l$. Since we have $\alpha_{\mathcal{A}}(H_{\mathcal{A}}) = \bar{\chi}(H_{\mathcal{A}})$, $H_{\mathcal{A}}$ is a co-perfect co-hypergraph. □

4. Conclusion.

The result of this paper is the complete examination of the structural properties of quasi-interval hypergraphs and the colouring properties of quasi-interval co-hypergraphs.

Although the class of quasi-interval hypergraphs is wider than the class of interval co-hypergraphs it was found that the basic results for colourings of co-hypergraphs from this class coincide with the respective results for the mixed interval hypergraphs introduced in [2].

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