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A NOTE ON LINE GRAPHS

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The aim of this note is to give several sufficient conditions, for some classes of line graphs, to be Hamiltonian.

Introduction.

Graphs, considered here, are finite, undirected and simple (without loops or multiple edges, [1], [2] being followed for terminology and notation. Let $G = (V, E)$ be a graph, with V the set of vertices and E the set of edges. Suppose that W is a nonempty subset of V. The subgraph of G, whose vertex set is W and whose edge set is the set of those edges of G that have both ends in W, is called the subgraph of G *induced* by W and is denoted by $G[W]$. For any vertex v in V, the *neighbour set* of v is the set of all vertices adjacent to v. This set is denoted by $N(v)$. For a graph $G = (V, E)$, we shall denote

$$\delta(G) = \min_{v \in V} |N(v)| \quad \text{and} \quad \Delta(G) = \max_{v \in V} |N(v)|.$$

Following [3], a graph $G = (V, E)$ is *locally connected*, if for each vertex v the graph $G[N(v)]$ is connected. With every graph G, having at least one edge, there exists associated a graph $L(G)$, called the *line graph* of G, whose vertices

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can be put in a one-to-one correspondence with the edges of G, in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. It was first shown by Sedlacek [6] that if G is Hamiltonian, then $L(G)$ is Hamiltonian. Since then, many Hamiltonian results, involving the line graphs, have been obtained. Several of these are given in [5].

The main results.

In their paper, Chartrand and Pippert [3] have shown that every connected and locally connected graph G, on $n \geq 3$ vertices and having $\Delta(G) \leq 4$, is either Hamiltonian or the graph $K_{1,1,3}$. In this note, we shall prove the following fundamental

Theorem. *If G is a connected and locally connected graph, on $n \geq 3$ vertices, which does not contain an induced $K_{1,3}$, then G is Hamiltonian.*

Proof. Suppose that the Theorem is false and let G be a connected and locally connected graph on at least three vertices, which does not contain an induced $K_{1,3}$, but which is not Hamiltonian. Clearly, G contains a cycle. Let C be a largest cycle in G. Then, C does not span G and, since G is connected, there exists a vertex v, not on C, which is adjacent to a vertex u, lying on C. Let u_1 and u_2 be the vertices neighbouring u, on the cycle C. Since G is locally connected, there exists a path P, in $G[N(u)]$, from v to the one of u_1 or u_2, which does not include the other. Without loss of generality, we shall suppose that P is a path from v to u_1 and that $u_2 \notin P$.
Now, if $P \cap C = \{u_1\}$, then, by attaching P to C at u_1 and v, we could obtain a cycle larger than C. Hence, we may assume that $P \cap C$ contains vertices other than u_1. Also, we cannot have v adjacent to either u_1 or u_2, without producing a cycle larger than C. Thus, since $\{u, u_1, u_2, v\}$ cannot induce a $K_{1,3}$ in G, then it must be that u_1u_2 is an edge of G.
For the purpose of this proof, we shall define a *singular* vertex to be a vertex $w \in P \cap C - \{u_1\}$, such that neither of the vertices, neighbouring w in C, belongs to $N(u)$. We shall consider two cases:

Case 1. Every vertex in $P \cap C - \{u_1\}$ is singular. Then, for any vertex $w \in P \cap C - \{u_1\}$, w is adjacent to u, but neither of the vertices w_1 and w_2, neighbouring w on C, belongs to $N(u)$. Thus, since $\{w, w_1, w_2, u\}$ cannot induce a $K_{1,3}$ in G, then it must be that w_1w_2 is an edge of G.
Now, traverse C, starting at u_2 and moving away from u and for each vertex $w \in P \cap C - \{u_1\}$, by-pass w, by taking the edge w_1w_2. Continue, until the vertex u_1 is reached. Then, follow P from u_1 to v then to u and finish at u_2.

Then, we have passed through each vertex of $C \cup P$, exactly once, and have thus constructed a cycle larger than C.

Case 2. $P \cap C - \{u_1\}$ contains nonsingular vertices. Then, follow P from v toward u_1, until the first nonsingular vertex w is reached. Let w_1 and w_2 be the vertices neighbouring w along C. Then, at least one of w_1 and w_2 is adjacent to u. Without loss of generality, suppose that w_1 is adjacent to u. Now, form a new cycle C', containing exactly the same vertices as C, as follows. Delete the edges ww_1, uu_1 and uu_2 and add the edges wu, w_1u and u_1u_2. Note that if w is a neighbour of u_1 or u_2, then not all of these edges may be distinct (e.g., if $w_1 = u_1$, then $uu_1 = uw_1$). But now, the vertices neighbouring u in C' are w and w_1, and the subpath P' of P, from w to v, does not include w_1 (as else w_1, being a nonsingular vertex, would have been chosen earlier, instead of w). Moreover, from the choice of w, it follows that P' cannot contain any nonsingular vertex with respect to C' and w (in the place of u_1). Thus, relative to P' and C', we are back to the Case 1. Hence, in any case, C cannot have been a largest cycle and, with this contradiction, the Theorem is proved. □

Remark. The above Theorem does not provide a necessary condition. For example, let us consider the graph $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_6\}$ and $E = \{v_1v_2,\ v_1v_6,\ v_2v_3,\ v_2v_4,\ v_2v_5,\ v_2v_6,\ v_3v_4,\ v_3v_6,\ v_4v_5,\ v_4v_6,\ v_5v_6\}$. Obviously, this graph is connected, locally connected, Hamiltonian, but $G[\{v_1, v_2, v_3, v_5\}]$ is isomorphic to $K_{1,3}$.

If $L(G)$ is the line graph of a graph G, then it is well known that $L(G)$ cannot contain $K_{1,3}$ as an induced subgraph. Thus, we have the following

Corollary 1. *Every connected and locally connected line graph, on $n \geq 3$ vertices, is Hamiltonian.*

Corollary 2. *If every edge of a connected graph G lies in a triangle, then $L(G)$ is Hamiltonian.*

Proof. If every edge of G lies in a triangle, then $L(G)$ is locally connected and, by Corollary 1, $L(G)$ is Hamiltonian, since, according to [4], G and $L(G)$ have the same connectivity. □

Corollary 3. *If G is a connected and locally connected graph, on $n \geq 3$ vertices, then $L(G)$ is Hamiltonian.*

Proof. If G is connected and locally connected, on at least three vertices, then every edge of G must lie in a triangle and, hence, the result follows from Corollary 2. □

Corollary 4. *If G is a connected graph with $\delta(G) \geq 3$, then $L(L(G))$ is Hamiltonian.*

Proof. If $\delta(G) \geq 3$, then every edge in $L(G)$ lies in a triangle. Hence, by Corollary 2, $L(L(G))$ is Hamiltonian. $\square$

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