LE MATEMATICHE
Vol. LII (1997) – Fasc. II, pp. 337–343

MISCELLANEOUS IDENTITIES OF GENERALIZED HERMITE POLYNOMIALS

G. DATTOLI - S. LORENZUTTA - A. TORRE

We extend a number of identities valid for the ordinary case to generalized Hermite polynomials with two indices and two variables. These identities, new to the authors knowledge, are obtained by using an operatorial procedure based on the properties of the Weyl group.

1. Introduction.

Two variable two index Hermite polynomials are denoted by $H_{m,n}(x, y)$ and are specified by the generating function [1]

$$\exp\left[\underline{w}^T \widehat{M}\underline{z} - \frac{1}{2}\underline{w}^T \widehat{M}\underline{w}\right] = \sum_{m,n} \frac{t^m}{m!}\frac{u^n}{n!} H_{m,n}(x, y)\,, \tag{1}$$

where

$$\begin{gathered} \underline{w} = \begin{pmatrix} t \\ u \end{pmatrix},\ \underline{z} = \begin{pmatrix} x \\ y \end{pmatrix},\ \widehat{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (a, c) > 0 \\ t \neq u,\ (|t|, |u|) < \infty\ \left(\Delta_M = ac - b^2 > 0\right) \end{gathered} \tag{2}$$

and the superscript “T” denotes transpose. The above quoted polynomials are exploited in many fields of pure and applied mathematics (see [3] and references therein).

Entrato in Redazione il 9 giugno 1997.

In particular they have been exploited for the quantum treatment of coupled harmonic oscillator and to describe the associated coherent states. Furthermore within the context of classical electromagnetics they have been employed to treat the emission by relativistic electrons propagating in multicomponent magnetic undulators.

In the following we will derive a number of identities regarding the $H_{m,n}(x, y)$. We will exploit a technique involving operational rule valid for the Weyl group; namely, if $\widehat{A}$ and $\widehat{B}$ are two operators such that their commutator is a c-number k (or any operator commuting either with $\widehat{A}$ and $\widehat{B}$), the following relations hold

$$\begin{aligned} &e^{\hat{A}+\hat{B}} = e^{-k/2} e^{\hat{A}} e^{\hat{B}}, \\ &e^{\hat{A}^m} f(\hat{B}) = f(\hat{B} + mk\hat{A}^{m-1}) e^{\hat{A}^m}. \end{aligned} \tag{3}$$

In the forthcoming sections we will show that many of the properties of the $H_{m,n}$ are just a consequence of (3).

2. Generalized Burchnall identities.

The first identity we derive is an extension to the two-dimensional case of the so called Burchnall identity [2]. We consider, indeed, the following operator

$$\widehat{I}_{m,n} = \Big(ax + by - \frac{\partial}{\partial x}\Big)^m \cdot \Big(bx + cy - \frac{\partial}{\partial y}\Big)^n. \tag{4}$$

By multiplying both sides of (4) by $(t^m/m!)\cdot(u^n/n!)$ and then by summing up, we obtain

$$\sum_{(m,n)=0}^{\infty} \frac{t^m}{m!}\frac{u^n}{n!}\widehat{I}_{m,n} = e^{t(ax+by-\frac{\partial}{\partial x})} \cdot e^{u(bx+cy-\frac{\partial}{\partial y})}. \tag{5}$$

By exploiting Eqs. (3) we find

$$\begin{aligned} \sum_{(m,n)=0}^{\infty} \frac{t^m}{m!}\frac{u^n}{n!}\widehat{I}_{m,n} = \exp\Big[\underline{w}^T \widehat{M}\underline{z} - \frac{1}{2}\underline{w}^T \widehat{M}\underline{w}\Big] \cdot \\ \cdot \exp\Big(-t\frac{\partial}{\partial y}\Big)\exp\Big(-u\frac{\partial}{\partial y}\Big). \end{aligned} \tag{6}$$

By expanding the r.h.s. of (6), by rearranging the sums and by equating the like (t, u) power coefficients we end up with

$$\widehat{I}_{m,n} = \sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}(-1)^{r+s}H_{m-r,n-s}(x, y)\cdot\frac{\partial^{r+s}}{\partial x^r\partial y^s}. \tag{7}$$

The ordinary Burchnall identity can be derived as a particular case of (7), by setting $a = 1$ and $y = 0$ we find, indeed ([1])

$$\widehat{I}_m = \sum_{r=0}^{m}\binom{m}{r}(-1)^r H_{m-r}(x)\frac{\partial^r}{\partial x^r}. \tag{8}$$

An important consequence of (7) is the generalization of the Rainville sum rule [5]. To this aim we remind that [1]

$$\widehat{I}_{r,s}H_{m,n}(x, y) = H_{m+r,n+s}(x, y). \tag{9}$$

Accordingly we find

$$\sum_{(r,s)=0}^{\infty}\frac{t^r}{r!}\frac{u^s}{s!}H_{m+r,n+s}(x, y) = \sum_{(r,s)=0}^{\infty}\frac{t^r}{r!}\frac{u^s}{s!}\widehat{I}_{r,s}H_{m,n}(x, y) \tag{10}$$

which after using (6) and (3) yields ([2])

$$\sum_{(r,s)=0}^{\infty}\frac{t^r}{r!}\frac{u^s}{s!}H_{m+r,n+s}(x, y) = \tag{11}$$

$$= \exp\left[\underline{w}^T\widehat{M}\underline{z} - \frac{1}{2}\underline{w}^T\widehat{M}\underline{w}\right]H_{m,n}(x - t, y - u).$$

([1]) Strictly speaking the Hermite polynomials appearing in (8) should be denoted by $He_m(x)$ being generated by $\exp(xt - t^2/2)$.

([2]) Note that the strict application of (3) would give for r.h.s. of (11)

$$\exp\left[\underline{w}^T\widehat{M}\underline{z} - \frac{1}{2}\underline{w}^T\widehat{M}\underline{w}\right]H_{m,n}(x - t, y - u)\cdot\exp\left(-t\frac{\partial}{\partial x}\right)\exp\left(-u\frac{\partial}{\partial y}\right).$$

Since the r.h.s. of (11) is not an operator, i.e. the operator $\widehat{I}_{r,s}$ acts on $H_{m,n}$ only, the exponential operators can be replaced by 1.

We complete this section by proving a further important identity, but we premise the following relation ([3])

$$\text{(12)}\qquad \exp\left[-\frac{1}{2}(\partial_x\partial_y)\widehat{M}^{-1}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}\right](ax+by)^m(bx+cy)^n = $$
$$= \widehat{I}_{m,n}\exp\left[-\frac{1}{2}(\partial_x\partial_y)\widehat{M}^{-1}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}\right]$$

which is a consequence of the second of the Eqs. (3). The use of (7) along with the assumption that the exponential operator is acting on its r.h.s. only, yields the identity

$$\text{(13)}\qquad \exp\left[-\frac{1}{2}(\partial_x\partial_y)\widehat{M}^{-1}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}\right]\left[(ax+by)^m(bx+cy)^n\right] = H_{m,n}(x,y)$$

whose importance will be discussed in the concluding section.

3. Concluding remarks.

The identities of the previous section can be complemented with analogous relations relevant to the dual of the $H_{m,n}(x,y)$ usually denoted by $G_{m,n}(x,y)$ [1]. To recover such identities we start from the relation

$$\text{(14)}\qquad e^{-\frac{1}{2}(\partial_x\partial_y)\widehat{M}^{-1}\binom{\partial_x}{\partial_y}}x^m y^n =$$
$$=\left[x+\frac{1}{\Delta_M}(b\partial_y-c\partial_x)\right]^m\left[y+\frac{1}{\Delta_M}(b\partial_x-a\partial_y)\right]^n\exp\left[-\frac{1}{2}(\partial_x\partial_y)\widehat{M}^{-1}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}\right].$$

The r.h.s. part of (14) not containing the exponential operator is the dual of $\widehat{I}_{m,n}$ defined by Eq. (4). Accordingly we get

$$\text{(15)}\qquad \widehat{\mathfrak{I}}_{m,n}=\left[x+\frac{1}{\Delta_M}(b\partial_y-c\partial_x)\right]^m\left[y+\frac{1}{\Delta_M}(b\partial_x-c\partial_y)\right]^n$$

and by following the procedure leading to Eq. (6) we obtain

$$\text{(16)}\qquad \sum_{(m,n)=0}^{\infty}\frac{t^m}{m!}\frac{u^n}{n!}\widehat{\mathfrak{I}}_{m,n}=$$
$$=\exp\left[tx+uy-\frac{1}{2}\underline{w}^T M^{-1}\underline{w}\right]\exp\left[\frac{t}{\Delta_M}(b\partial_y-c\partial_x)+\frac{u}{\Delta_M}(b\partial_x-a\partial_y)\right].$$

([3]) Note that $(\partial_x\partial_y)\widehat{M}^{-1}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}$ is an unorthodox notation for the generalized Laplacian $\frac{1}{\Delta_M}[c\partial_x^2-2b\partial_{x,y}+a\partial_y^2]$.

Furthermore by noting that

$$\exp\Big[tx + uy - \frac{1}{2}\underline{w}^T \widehat{M}^{-1}\underline{w}\Big]\sum_{m,n}\frac{t^m}{m!}\frac{u^n}{n!}G_{m,n}(x, y) \tag{17}$$

and

$$\begin{aligned}\partial_\xi &= \frac{c}{\Delta_M}\partial_x - \frac{b}{\Delta_M}\partial_y\\ \partial_\eta &= \frac{a}{\Delta_M}\partial_y - \frac{b}{\Delta_M}\partial_x\end{aligned} \tag{18a}$$

if

$$\xi = ax + by, \qquad \eta = bx + cy \tag{18b}$$

we end up with the identity

$$\widehat{\Im}_{m,n} = \sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}(-1)^{r+s}G_{m-r,n-s}(x, y)\partial^{r+s}_{\xi^r\eta^s} \tag{19}$$

which is the extension of (8) to the $G_{m,n}$ polynomials.

The use of Eq. (16), along the relation

$$\widehat{\Im}_{r,s}G_{m,n}(x, y) = G_{m+r,n+s}(x, y) \tag{20}$$

yields the further relation

$$\sum_{r,s}\frac{t^r}{r!}\frac{u^s}{s!}G_{m+r,n+s}(x, y) = \tag{21}$$

$$= \exp\Big[rt + yu - \frac{1}{2}\underline{w}^T \widehat{M}^{-1}\underline{w}\Big]G_{m,n}\Big(x + \frac{1}{\Delta_M}(bu - ct), y + \frac{1}{\Delta_M}(bt - au)\Big).$$

It is also evident that Eq. (14) can be exploited to conclude that

$$\exp\Big[-\frac{1}{2}(\partial_x\partial_y)\widehat{M}^{-1}\binom{\partial_x}{\partial_y}\Big]x^m y^n = G_{m,n}(x, y) \tag{22}$$

which is the analogous of Eq. (13) and holds under the same conditions.

We want to emphasize that either Eqs. (13) and (22) indicate that both polynomials $H_{m,n}(x, y)$ and $G_{m,n}(x, y)$ are solutions of the p.d.e.

$$\frac{\partial}{\partial \tau} S_{m,n}(x, y; \tau) = -\frac{1}{2}(\partial_x \partial_y)\widehat{M}^{-1}\begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix} S_{m,n}(x, y; \tau) \tag{23a}$$

satisfying the following conditions at $\tau = 0$

$$S_{m,n}(x, y; 0) = \begin{cases} \xi^m \eta^n \\ x^m y^n \end{cases} \tag{23b}$$

respectively.

According to the above considerations it is possible to introduce two new classes of polynomials generalizing the $H_{m,n}(x, y)$ and $G_{m,n}(x, y)$. We can indeed use the generating functions

$$\begin{aligned} \exp\left(\underline{w}^T \widehat{M}\underline{z} - \frac{1}{2}\tau \underline{w}^T \widehat{M}\underline{w}\right) &= \sum_{m,n} \frac{t^m}{m!}\frac{u^n}{n!} H_{m,n}(x, y; \tau) \\ \exp\left(\underline{w}^T \underline{z} - \frac{1}{2}\tau \underline{w}^T \widehat{M}^{-1}\underline{w}\right) &= \sum_{m,n} \frac{t^m}{m!}\frac{u^n}{n!} G_{m,n}(x, y; \tau) \end{aligned} \tag{24}$$

to define the three variable polynomials $H_{m,n}(x, y; \tau)$ and $G_{m,n}(x, y; \tau)$ which can be viewed as an extension of the Kampé de Fériet polynomials defined through [1]

$$\exp(xt + yt^2) = \sum_n \frac{t^n}{n!} H_n(x, y). \tag{25}$$

Before concluding this paper it is worth commenting on the generalization of the Nielsen formula [4]. By reminding indeed that [1]

$$\begin{aligned} \partial_\xi H_{m,n}(x, y) &= m H_{m-1,n}(x, y) \\ \partial_\eta H_{m,n}(x, y) &= n H_{m,n-1}(x, y). \end{aligned} \tag{26}$$

We obtain

$$\begin{aligned} H_{m+r,n+s}(x, y) = \sum_{u=0}^{r}\sum_{v=0}^{s} \binom{r}{u}\binom{s}{v}(-1)^{u+v} H_{r-u,s-v}(x, y)\cdot \\ \cdot \sum_{p=0}^{u}\sum_{q=0}^{v}\binom{u}{p}\binom{v}{q} b^p c^q a^{u-p} b^{v-q} \frac{m!}{(m-u+p-v+q)!}\frac{n!}{(n-p-q)!}\cdot \\ \cdot H_{m-(u-p+v-q),n-(p+q)}(x, y). \end{aligned} \tag{27}$$

An analogous formula holds for the $G_{m,n}(x, y)$ too and is omitted for the sake of conciseness.

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G. Dattoli - A. Torre,
ENEA, Dipartimento Innovazione,
Divisione Fisica Applicata,
Centro Ricerche Frascati,
C.P. 65,
00044 Frascati, Roma (ITALY)

S. Lorenzutta,
ENEA, Dipartimento Innovazione,
Divisione Fisica Applicata,
Centro Ricerche Bologna,
Bologna (ITALY)