CAUSAL (ANTICAUSAL) BESSSEL DERIVATIVE
AND THE ULTRAHYPERBOLIC BESSSEL OPERATOR

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Let $B_C^\alpha$ and $B_A^\alpha$ be ultrahyperbolic Bessel operator causal (anticausal) of the order $\alpha$ defined by $B_C^\alpha f = G_\alpha(P + i0, m, n) * f$, $B_A^\alpha f = G_\alpha(P - i0, m, n) * f$ and let $D_C^\alpha$ and $D_A^\alpha$ be generalized causal (anticausal) Bessel derivative of order $\alpha$ defined by $D_C^\alpha f = G_{-\alpha}(P - i0, m, n) * f$, $D_A^\alpha f = G_{-\alpha}(P + i0, m, n) * f$. In this note we give a sense to several relations of type:

\[
\begin{align*}
B_C^\alpha(B_A^\beta f) + B_A^\alpha(B_C^\beta f), \\
D_C^\alpha(D_A^\beta f) + D_A^\alpha(D_C^\beta f), \\
e^{i(\alpha-\beta)\pi i}B_C^\alpha(B_C^\beta f) + e^{-i(\alpha-\beta)\pi i}B_A^\alpha(B_A^\beta f), \\
e^{i(\alpha-\beta)\pi i}D_C^\alpha(D_C^\beta f) + e^{-i(\alpha-\beta)\pi i}D_A^\alpha(D_A^\beta f).
\end{align*}
\]

I.1. Introduction.

Let $t = (t_1, t_2, \ldots, t_n)$ be a point of the $n$-dimensional Euclidean space $\mathbb{R}^n$ and let $P = P(t)$ be a non degenerate quadratic form in $n$-variables of the form

\[(I;1.1) \quad P = P(t) = t_1^2 + \ldots + t_p^2 - t_{p+1}^2 - \ldots - t_{p+q}^2,\]

where $p + q = n$.

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Let $G_\alpha(P \pm i0, m, n)$ be the causal (anticausal) distribution defined by

(I.1.2) \[ G_\alpha(P \pm i0, m, n) = H_\alpha(m, n)(P \pm i0)^{1/2}(\mp i/2)K_{\mp\alpha}(\sqrt{m^2(P \pm i0)}), \]

where $m$ is a positive real number, $\alpha$ a complex number, $K_\nu$ designates the modified Bessel function of third kind

\[ K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi \nu}, \]

\[ I_\nu(z) = \sum_{m=0}^{\infty} \frac{(z^2/4)^m}{m! \Gamma(m + \nu + 1)} \]

and

\[ H_\alpha(m, n) = \frac{2^{1 - \alpha/2} (m^2)^{1/4} e^{\frac{\pi}{4} \alpha i}}{\pi^{\frac{3}{2}} \Gamma(\frac{\alpha}{2})}, \]

$(P \pm i0)^\lambda$ is the distribution defined by

(I.1.3) \[ (P \pm i0)^\lambda = \lim_{\epsilon \to 0}(P \pm i\epsilon |t|^2)^\lambda; \]

where $\epsilon > 0$, $|t|^2 = t_1^2 + \cdots + t_n^2$, $\lambda \in \mathbb{C}$ and $P = P(t)$ is the quadratic form defined in (I.1.1) (cf. [7], p. 275).

The following formula is valid (cf. [11], p. 35)

(I.1.4) \[ \mathcal{F}[G_\alpha(P \pm i0, m, n)] = \frac{e^{\frac{i\pi}{4}}}{(2\pi)^{\frac{3}{2}}} (m^2 + Q \mp i0)^{-\frac{\lambda}{2}}, \]

where $(m^2 + Q \pm i0)^\lambda$ is the distribution defined by

(I.1.5) \[ (m^2 + Q \pm i0)^\lambda = \lim_{\epsilon \to 0}(m^2 + Q \pm i\epsilon |t|^2)^\lambda; \]

$m$ is a positive real number, $\lambda \in \mathbb{C}$ and $\mathcal{F}$ denotes the Fourier transform.

From [3], p. 566, we have,

(I.1.6) \[ (m^2 + Q \pm i0)^\lambda = (m^2 + Q)^\lambda_\pm + e^{\pm i\pi} (m^2 + Q)^\lambda_\mp, \]

where

(I.1.7) \[ (m^2 + Q)^\lambda_\pm = \begin{cases} (m^2 + Q)^\lambda & \text{if } m^2 + Q \geq 0 \\ 0 & \text{if } m^2 + Q < 0 \end{cases} \]
and
\[
(m^2 + Q_+^\lambda)^\pm = \begin{cases} 0 & \text{if } m^2 + Q > 0 \\ (-m^2 - Q)^\lambda & \text{if } m^2 + Q \leq 0 \end{cases}.
\]

From (I;1.6) and considering the formula
\[
(I;1.9) \quad \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin z\pi}
\]
we have
\[
(I;1.10) \quad (m^2 + Q)\_\lambda = \Gamma(\lambda)\Gamma(1 - \lambda)(2\pi i)^{-1}[(m^2 + Q + i0)^\lambda - (m^2 + Q - i0)^\lambda]
\]
and
\[
(I;1.11) \quad (m^2 + Q)^+\lambda = \Gamma(\lambda)\Gamma(1 - \lambda)(2\pi i)^{-1} \cdot \left[e^{\lambda\pi i}(m^2 + Q - i0)^\lambda - e^{-\lambda\pi i}(m^2 + Q + i0)^\lambda\right]
\]

On the other hand, S.E. Trione ([11], p. 23, formula (I.3.6)) proves the validity of multiplicative product:
\[
(I;1.12) \quad (m^2 + Q \pm i0)^\lambda \cdot (m^2 + Q \pm i0)^\mu = (m^2 + Q \pm i0)^{\lambda + \mu}
\]
for } m > 0}, where
\[
(I;1.13) \quad m^2 + Q = m^2 + y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2
\]
and \((m^2 + Q \pm i0)^\lambda\) is defined by (I;1.5).

Let } B^\alpha f } be the ultrahyperbolic Bessel operator defined by the formula
\[
(I;1.14) \quad B^\alpha f = G_\alpha * f, \quad f \in S,
\]
(cf. [10], p. 75).

We consider an auxiliary weight function } \lambda_\alpha(P \pm i0, m, n) \text{ defined as follows
\[
(I;1.15) \quad \lambda_\alpha(P \pm i0, m, n) = \frac{e^{i\frac{\pi}{2}q}2^{\frac{q}{4}}(m^2)^{\frac{1}{4}}(\frac{m^2}{\sqrt{2\pi}})}{\Gamma(\frac{m^2}{\sqrt{2\pi}})}(P \pm i0)^{\frac{q}{2}} K_{\frac{q}{2}}\left(\sqrt{m^2(P \pm i0)}\right).
\]
(cf. [6]), and

\[
\lambda_\alpha(|t|^2, m = 1) = \frac{1}{\Gamma\left(\frac{\alpha+\eta}{2}\right)} \int_0^\infty \eta^\frac{\alpha+\eta}{2} e^{-\eta^2} \frac{\partial}{\partial \eta} d\eta,
\]

\[\lambda_\alpha(P \pm i0, m, n)\] is the causal (anticausal) analogue of the weight function introduced by Rubin (cf. [8]).

Let \( f \) be a function belong to \( S \), \( S \) the Schwartz class of infinitely differentiable functions on \( \mathbb{R}^n \) decreasing at infinity faster than \( |x|^{-1} \). The weighted difference of order \( \ell \) of a function \( f \) at the point \( x \), with interval \( t \) and weight \( \lambda_\alpha \), is defined by

\[
(\Delta_\ell^t f)(x, \lambda_\alpha) = \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \lambda_\alpha(k(P \pm i0)) f(x - kt).
\]

Let the following causal hypersingular operator on weighted differences defined by

\[
(T_\ell^\alpha f)(x) = \\
= \int_{\mathbb{R}^n} (P + i0)^{-\frac{\alpha+\eta}{2}} \left\{ \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \lambda_\alpha(k(P + i0)) f(x - kt) \right\} dt,
\]

(cf. [5], p. 72).

The following formula is valid (cf. [5])

\[
\mathcal{F}[T_\ell^\alpha f](\xi) = d_{n,\ell}(\alpha)(m^2 + Q - i0)^{\frac{\alpha}{2}} \mathcal{F}[f](\xi),
\]

where

\[
d_{n,\ell}(\alpha) = \left\{ \begin{array}{ll}
\frac{\pi^{\frac{\alpha+1}{2}} e^{\frac{\alpha}{2} \eta} A_{\ell}(\alpha)}{2\alpha \Gamma\left(\frac{\alpha+\eta}{2}\right) \Gamma\left(\frac{\alpha}{2} + 1\right) \sin \frac{\pi \eta}{2}} & \text{for } \alpha \neq 2, 4, \ldots \\
\frac{(-1)^{\frac{\alpha}{2}} \pi^{\frac{\alpha+1}{2}} e^{\frac{\alpha}{2} \eta} 2^{1-\alpha}}{\Gamma\left(\frac{\alpha}{2} + 1\right) \Gamma\left(\frac{\alpha+\eta}{2}\right)} A_{\ell}(\alpha) & \text{for } \alpha = 2, 4, \ldots 
\end{array} \right.
\]

here \( A_{\ell}(\alpha) \) is defined by

\[
A_{\ell}(\alpha) = \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k k^\alpha.
\]
For $\alpha \neq 1, 3, 5, \ldots$ we define the generalized causal Bessel derivative as follows

(I; 1.22) \[ D_C^\alpha f(x) = \frac{1}{d_{n,\ell}(\alpha)}(T_\ell^\alpha f)(x) \]

(cf. [5]).

Then in virtue of (I; 1.11), we have

(I; 1.23) \[ \mathscr{F}[D^\alpha f](\xi) = (m^2 + Q - i0)^2 \mathscr{F}[f](\xi). \]

We define (cf. [5]) the ultrahyperbolic causal Bessel operator by the formula

(I; 1.24) \[ B_C^\alpha f = G_\alpha(P + i0, m, n) * f \]

and the ultrahyperbolic anticausal Bessel operator as

(I; 1.25) \[ B_A^\alpha f = G_\alpha(P - i0, m, n) * f. \]

The generalized anticausal Bessel operator is the anticausal analogue to the define in (I; 1.14), and in the same way we have the generalized anticausal Bessel derivative

(I; 1.26) \[ D_A^\alpha f = \frac{1}{d_{n,\ell}(\alpha)}(T_\ell^\alpha f)(x). \]

We had prove (cf. [5]) that

(I; 1.27) \[ \mathcal{D}_C^\alpha B_C^\alpha f = f \]

and, analogously

(I; 1.28) \[ \mathcal{D}_A^\alpha B_A^\alpha f = f. \]

Also we had prove that (cf. [5])

(I; 1.29) \[ \mathcal{D}_C^\alpha f = G_{-\alpha}(P - i0, m, n) * f. \]
1.2. In this paragraph we will consign some elementary properties of the ultrahyperbolic causal Bessel kernel. We begin by observe that the distributional function $G_\alpha(P \pm i0, m, n)$ is the causal (anticausal) analogue of the kernel due to A. Calderón and Aronszajn-Smith (cf. [4] and [2]) and share many properties with the Bessel kernel, like the following:

a) for all $\alpha, \beta$ complex numbers

(I;2.1) \[ \mathcal{F}[G_\alpha \ast G_\beta] = \mathcal{F}[G_\alpha] \cdot \mathcal{F}[G_\beta] \]

and

(I;2.2) \[ G_\alpha \ast G_\beta = G_{\alpha + \beta} \]

(cf. [11], p. 37).

b)

(I;2.3) \[ G_{-2k}(P \pm i0, m, n) = K^k(\delta) \]

and when $k = 0$

(I;2.4) \[ G_0(P \pm i0, m, n) = \delta, \]

where $K^k$ designate the $n$-dimensional ultrahyperbolic Klein-Gordon operator iterated $k$-times

(I;2.5) \[ K^k = \left\{ \frac{\partial^2}{\partial t_1^2} + \cdots + \frac{\partial^2}{\partial t_p^2} - \frac{\partial^2}{\partial t_{p+1}^2} - \cdots - \frac{\partial^2}{\partial t_{p+q}^2} + m^2 \right\}^k, \]

where $m$ is a positive real number.

c) The distributional functions $G_{2k}(P \pm i0, m, n)$ where $n$ is an integer $\geq 2$; $k = 1, 2, \ldots$, are elementary causal (anticausal) solutions of the ultrahyperbolic Klein-Gordon operator iterated $k$ times.

(I;2.6) \[ K^k[G_{2k}] = \delta. \]
I.3. Along this paper we shall need the following Lemmas.

Lemma 1. Let $\lambda$ and $\mu$ be complex numbers such that $\lambda, \mu$ and $\lambda + \mu \neq -k$, $k = 1, 2, \ldots$; then the following formulae are valid

\[(1;3.1)\hspace{1cm} e^{(\lambda - \mu)\pi i} (m^2 + Q - i0)^\lambda \cdot (m^2 + Q + i0)^\mu +
\]
\[+ e^{-(\lambda - \mu)\pi i} (m^2 + Q + i0)^\lambda \cdot (m^2 + Q - i0)^\mu =
\]
\[= [1 - C(\lambda, \mu)] e^{(\lambda + \mu)\pi i} (m^2 + Q + i0)^{\lambda + \mu} +
\]
\[+ [1 + C(\lambda, \mu)] e^{-(\lambda + \mu)\pi i} (m^2 + Q - i0)^{\lambda + \mu},
\]

and

\[(1;3.2)\hspace{1cm} (m^2 + Q + i0)^\lambda \cdot (m^2 + Q - i0)^\mu + (m^2 + Q - i0)^\lambda \cdot (m^2 + Q + i0)^\mu =
\]
\[= [1 + C(\lambda, \mu)] (m^2 + Q - i0)^{\lambda + \mu} +
\]
\[+ [1 - C(\lambda, \mu)] (m^2 + Q + i0)^{\lambda + \mu},
\]

where

\[(1;3.3)\hspace{1cm} C(\lambda, \mu) = 2i \sin \lambda \pi \cdot \sin \mu \pi \csc(\lambda + \mu)\pi =
\]
\[= \frac{2\pi i \Gamma(\lambda + \mu) \Gamma(1 - \lambda - \mu)}{\Gamma(\lambda) \Gamma(1 - \lambda) \Gamma(\mu) \Gamma(1 - \mu)}
\]

and $(m^2 + Q \pm i0)^\lambda$ is defined by $(1;1.5)$.

Proof. It results from $(1;1.10)$, $(1;1.11)$ and considering the multiplicative product $(1;1.12)$ (see [11], p. 39).

In the next we will write

\[(1;3.4)\hspace{1cm} G^a_c = G_a(P + i0, m, n)
\]

and

\[(1;3.5)\hspace{1cm} G^a_A = G_a(P - i0, m, n),
\]

where $G_a(P \pm i0, m, n)$ is defined by $(1;1.2)$.

On the other hand, from [11], p. 37, we have

\[(1;3.6)\hspace{1cm} G_{-2k}(P \pm i0, m, n) \in O'_C
\]

and

\[(1;3.7)\hspace{1cm} G_a(P \pm i0, m, n) \in S' \text{ for all } \alpha \in C,
\]

where $O'_C$ designates the space of rapidly decreasing distribution ([9], p. 244) and $S'$ designates the dual of $S$ and $S$ is the Schwartz set of functions ([9], p. 268).
Lemma 2. Let $\alpha$ and $\beta$ be complex numbers, then the following formula is valid

(I;3.8) \[ e^{\frac{1}{2}(\alpha-\beta)\pi i} (G_A^\alpha \ast G_C^\beta) + e^{-\frac{1}{2}(\alpha-\beta)\pi i} G_A^\alpha \ast G_C^\beta = \]
\[ = \left[ 1 - C \left( -\frac{\alpha}{2}, -\frac{\beta}{2} \right) \right] e^{-\frac{1}{2}(\alpha+\beta)\pi i} G_A^{\alpha+\beta} + \]
\[ + \left[ 1 + C \left( -\frac{\alpha}{2}, -\frac{\beta}{2} \right) \right] e^{\frac{1}{2}(\alpha+\beta)\pi i} G_C^{\alpha+\beta} \]
and

(I;3.9) \[ G_A^\alpha \ast G_C^\beta + G_A^\alpha \ast G_C^\beta = \]
\[ = \left[ 1 + C \left( -\frac{\alpha}{2}, -\frac{\beta}{2} \right) \right] G_C^{\alpha+\beta} + \left[ 1 - C \left( -\frac{\alpha}{2}, -\frac{\beta}{2} \right) \right] G_A^{\alpha+\beta}. \]

Proof. From (I;2.14) we conclude, by appealing to the theorem of Schwartz (cf. [9], p. 268), that the following formula is valid

(I;3.10) \[ \mathcal{F}\{e^{\frac{1}{2}(\alpha-\beta)\pi i} (G_A^\alpha \ast G_C^\beta) + e^{-\frac{1}{2}(\alpha-\beta)\pi i} (G_C^\alpha \ast G_A^\beta)\} = \]
\[ = e^{\frac{1}{2}(\alpha-\beta)\pi i} \mathcal{F}\{G_A^\alpha\} \cdot \mathcal{F}\{G_C^\beta\} + e^{-\frac{1}{2}(\alpha-\beta)\pi i} \mathcal{F}\{G_C^\alpha\} \cdot \mathcal{F}\{G_A^\beta\} \]
and

(I;3.11) \[ \mathcal{F}\{G_A^\alpha \ast G_C^\beta + G_C^\alpha \ast G_A^\beta\} = \mathcal{F}\{G_A^\alpha\} \cdot \mathcal{F}\{G_C^\beta\} + \mathcal{F}\{G_C^\alpha\} \cdot \mathcal{F}\{G_A^\beta\}. \]

Substituting (I;1.4) into (I;3.10) and (I;3.11) we have

(I;3.12) \[ \mathcal{F}\{e^{\frac{1}{2}(\alpha-\beta)\pi i} (G_A^\alpha \ast G_C^\beta) + e^{-\frac{1}{2}(\alpha-\beta)\pi i} (G_C^\alpha \ast G_A^\beta)\} = \]
\[ = e^{\frac{1}{2}(\alpha+\beta)\pi i} \frac{1}{2} \cdot (m^2 + Q + i0)^{-\frac{\alpha}{2}} (m^2 + Q - i0)^{-\frac{\beta}{2}} ](2\pi)^{-n} + \]
\[ + e^{-\frac{1}{2}(\alpha-\beta)\pi i} \frac{1}{2} \cdot (m^2 + Q - i0)^{-\frac{\alpha}{2}} (m^2 + Q + i0)^{-\frac{\beta}{2}} ](2\pi)^{-n} \]
and

(I;3.13) \[ \mathcal{F}\{G_A^\alpha \ast G_C^\beta + G_C^\alpha \ast G_A^\beta\} = \]
\[ = e^{(\alpha+\beta)\pi i} \frac{1}{2} \cdot (m^2 + Q + i0)^{-\frac{\alpha}{2}} (m^2 + Q - i0)^{-\frac{\beta}{2}} ](2\pi)^{-n} + \]
\[ + e^{(\alpha+\beta)\pi i} \frac{1}{2} \cdot (m^2 + Q - i0)^{-\frac{\alpha}{2}} (m^2 + Q + i0)^{-\frac{\beta}{2}} ](2\pi)^{-n}. \]
Putting \( \lambda = -\frac{\alpha}{2} \) and \( \mu = -\frac{\beta}{2} \) in (I;3.1) and (I;3.2) and substituting in the right hand member of (I;3.12) and (I;3.13) we have

\[
\mathcal{F}\{e^{\frac{1}{2}(\alpha - \beta)\pi i}(G_A^\alpha * G_C^\beta) + e^{-\frac{1}{2}(\alpha - \beta)\pi i}(G_C^\alpha * G_A^\beta)\} = \\
= (2\pi)^{-\alpha} e^{(\alpha + \beta)\frac{1}{2}} \left\{ (1 - C(-\frac{\alpha}{2}, -\frac{\beta}{2})) e^{-\frac{1}{2}(\alpha + \beta)\pi i}(m^2 + Q + i0)^{-\frac{\alpha + \beta}{2}} + \\
+ \left(1 + C(-\frac{\alpha}{2}, -\frac{\beta}{2})\right) e^{\frac{1}{2}(\alpha + \beta)\pi i}(m^2 + Q - i0)^{-\frac{\alpha + \beta}{2}} \right\}
\]

and

\[
\mathcal{F}\{G_A^\alpha * G_C^\beta + G_C^\alpha * G_A^\beta\} = \\
= (2\pi)^{-\alpha} e^{(\alpha + \beta)\frac{1}{2}} \left\{ (1 + C(-\frac{\alpha}{2}, -\frac{\beta}{2}))(m^2 + Q - i0)^{-\frac{\alpha + \beta}{2}} + \\
+ \left(1 - C(-\frac{\alpha}{2}, -\frac{\beta}{2})\right)(m^2 + Q + i0)^{-\frac{\alpha + \beta}{2}} \right\}
\]

By substituting (I;1.4) in (I;3.14), (I;3.15) and taking into account the uniqueness theorem of the Fourier transform, we conclude

\[
e^{\frac{1}{2}(\alpha - \beta)\pi i}(G_A^\alpha * G_C^\beta) + e^{-\frac{1}{2}(\alpha - \beta)\pi i}(G_C^\alpha * G_A^\beta) = \\
= \left(1 - C(-\frac{\alpha}{2}, -\frac{\beta}{2})\right) e^{-\frac{1}{2}(\alpha + \beta)\pi i}G_A^{\alpha + \beta} + \\
+ \left(1 + C(-\frac{\alpha}{2}, -\frac{\beta}{2})\right) e^{\frac{1}{2}(\alpha + \beta)\pi i}G_C^{\alpha + \beta}
\]

and

\[
G_A^\alpha * G_C^\beta + G_C^\alpha * G_A^\beta = \\
= \left(1 + C(-\frac{\alpha}{2}, -\frac{\beta}{2})\right)G_C^{\alpha + \beta} + \left(1 - C(-\frac{\alpha}{2}, -\frac{\beta}{2})\right)G_A^{\alpha + \beta}
\]

which is the thesis of Lemma 2.

II.1.

**Theorem 1.** Let \( \alpha, \beta \) be complex numbers and let \( B_C^\gamma \) (resp. \( B_A^\gamma \)) be the ultrahyperbolic causal (anticausal) Bessel operator of order \( \lambda \) of a function \( f \) belong to \( S \). Then

\[
(B_C^\gamma f)^\alpha + (B_A^\gamma f)^\beta = [1 + C(-\frac{\alpha}{2}, -\frac{\beta}{2})]B_C^{\alpha + \beta} f + [1 - C(-\frac{\alpha}{2}, -\frac{\beta}{2})]B_A^{\alpha + \beta} f.
\]
Proof. It results from the definitions (I;1.16), (I;1.17) and from (I;3.9).

**Theorem 2.** Let \(\alpha, \beta\) be complex numbers and let \(D^\lambda_C\) (resp. \(D^\lambda_A\)) be the generalized causal (anticausal) Bessel derivative of order \(\lambda\) of a function \(f\) belong to \(S\). Then

\[
(\text{II;1.2}) \quad D^\alpha_C(D^\beta_A f) + D^\alpha_A(D^\beta_C f) = [1 + C(-\frac{\alpha}{2}, -\frac{\beta}{2})]D^{\alpha+\beta}_C f + [1 - C(-\frac{\alpha}{2}, -\frac{\beta}{2})]D^{\alpha+\beta}_A f.
\]

Proof. It results from (I;1.14), (I;1.16) and (I;3.9).

**Theorem 3.** Let \(\alpha, \beta\) be complex numbers and let \(B^\alpha_C\) (resp \(B^\alpha_A\)) be the ultra-hyperbolic causal (anticausal) Bessel operator of order \(\lambda\) of a function \(f\) belong to \(S\). Then the following formula is valid

\[
(\text{II;1.3}) \quad e^{\frac{1}{2}(\alpha-\beta)^\pi i} B^\alpha_A(B^\beta_C f) + e^{-\frac{1}{2}(\alpha-\beta)^\pi i} B^\alpha_C(B^\beta_A f) = \left[1 - C(-\frac{\alpha}{2}, -\frac{\beta}{2})\right]e^{-\frac{1}{2}(\alpha+\beta)^\pi i} B^\alpha_A B^\beta_C f + \left[1 + C(-\frac{\alpha}{2}, -\frac{\beta}{2})\right]e^{\frac{1}{2}(\alpha+\beta)^\pi i} B^\alpha_C B^\beta_A f.
\]

Proof. (II;1.3) it results from (I;1.16), (I;1.17) and (I;3.8).

**Theorem 4.** Let \(\alpha, \beta\) be complex numbers and let \(D^\lambda_C\) (resp. \(D^\lambda_A\)) be the generalized causal (anticausal) Bessel derivative of order \(\lambda\) of a function \(f\) belong to \(S\). Then is valid

\[
(\text{II;1.4}) \quad e^{\frac{1}{2}(\alpha-\beta)^\pi i} D^\alpha_C(D^\beta_A f) + e^{-\frac{1}{2}(\alpha-\beta)^\pi i} D^\alpha_A(D^\beta_C f) = \left[1 - C(-\frac{\alpha}{2}, -\frac{\beta}{2})\right]e^{-\frac{1}{2}(\alpha+\beta)^\pi i} D^\alpha_A D^\beta_C f + \left[1 + C(-\frac{\alpha}{2}, -\frac{\beta}{2})\right]e^{\frac{1}{2}(\alpha+\beta)^\pi i} D^\alpha_C D^\beta_A f.
\]

Proof. (II;1.4) it results from (I;1.14), (I;1.18) and (I;3.8).

Otherwise we can observe that the following formulae are valid for \(\alpha\) and \(\beta\) complex numbers

(II;1.5) \(D_C^\beta B_C^\alpha f = D_C^{\beta+\alpha} f.\)

(II;1.6) \(D_A^\beta B_A^\alpha f = D_A^{\beta+\alpha} f.\)

(II;1.7) \(B_C^\beta D_C^\alpha f = B_C^{\beta+\alpha} f.\)

(II;1.8) \(B_A^\beta D_A^\alpha f = B_A^{\beta+\alpha} f.\)
Theorem 5. Let $\alpha, \beta$ be complex numbers and let $B^\lambda_\alpha(B^\lambda_\beta)$ be the ultrahyperbolic causal (anticausal) Bessel operator and let $D^\mu_C(D^\mu_A)$ be the generalized causal (anticausal) Bessel derivative of order $\mu$ of a function $f$ belong to $S$. Then the following formula is valid

$$B^\alpha_C(D^\beta_A f) + B^\alpha_A(D^\beta_C f) =$$

$$= (1 + C(-\frac{\alpha}{2}, -\frac{\beta}{2}))B^{\alpha-\beta}_C + (1 - C(-\frac{\alpha}{2}, -\frac{\beta}{2}))B^{\alpha-\beta}_A.$$

Proof. (II:1.9) results from (I;1.16), (I;1.17) and (I;2.8).

Theorem 6. Let $\alpha, \beta$ be complex numbers and let $B^\lambda_\alpha(B^\lambda_\beta)$ be the ultrahyperbolic causal (anticausal) Bessel operator of order $\lambda$ and let $D^\mu_C$ (resp $D^\mu_A$) be the generalized causal (anticausal) Bessel derivative of order $\mu$ of a function $f$ belong to $S$. Then is valid

$$D^\alpha_C(B^\beta_A f) + D^\alpha_A(B^\beta_C f) =$$

$$= (1 + C(\frac{\alpha}{2}, -\frac{\beta}{2}))D^{\alpha+\beta}_C f + (1 - C(\frac{\alpha}{2}, -\frac{\beta}{2}))D^{\alpha+\beta}_A f.$$

Proof. From (I;1.14), (I;1.15) and (I;2.13) it results (II;1.10).

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