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CAUSAL (ANTICAUSAL) BESSEL DERIVATIVE AND THE ULTRAHYPERBOLIC BESSEL OPERATOR

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Let B_C^α and B_A^α be ultrahyperbolic Bessel operator causal (anticausal) of the order α defined by $B_C^\alpha f = G_\alpha(P + i0, m, n) * f$, $B_A^\alpha f = G_\alpha(P - i0, m, n) * f$ and let D_C^α and D_A^α be generalized causal (anticausal) Bessel derivative of order α defined by $D_C^\alpha f = G_{-\alpha}(P - i0, m, n) * f$, $D_A^\alpha f = G_{-\alpha}(P + i0, m, n) * f$. In this note we give a sense to several relations of type:

$$B_C^\alpha(B_A^\beta f) + B_A^\alpha(B_C^\beta f),$$
$$D_C^\alpha(D_A^\beta f) + D_A^\alpha(D_C^\beta f),$$
$$e^{\frac{1}{2}(\alpha-\beta)\pi i} B_A^\alpha(B_C^\beta f) + e^{-\frac{1}{2}(\alpha-\beta)\pi i} B_C^\alpha(B_A^\beta f),$$
$$e^{\frac{1}{2}(\alpha-\beta)\pi i} D_A^\alpha(D_C^\beta f) + e^{-\frac{1}{2}(\alpha-\beta)\pi i} D_C^\alpha(D_A^\beta f).$$

I.1. Introduction.

Let $t = (t_1, t_2, \dots, t_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$ and let $P = P(t)$ be a non degenerate quadratic form in n-variables of the form

$$P = P(t) = t_1^2 + \dots + t_p^2 - t_{p+1}^2 - \dots - t_{p+q}^2, \qquad \text{(I;1.1)}$$

where $p + q = n$.

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Let $G_\alpha(P \pm i0, m, n)$ be the causal (anticausal) distribution defined by

(I;1.2) $$G_\alpha(P \pm i0, m, n) = H_\alpha(m, n)(P \pm i0)^{\frac{1}{2}(\frac{\alpha-n}{2})} K_{\frac{n-\alpha}{2}}\left(\sqrt{m^2(P \pm i0)}\right),$$

where m is a positive real number, α a complex number, K_ν designates the modified Bessel function of third kind

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi\nu},$$

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(\frac{z}{2})^{\nu+2m}}{m!\Gamma(m+\nu+1)}$$

and

$$H_\alpha(m, n) = \frac{2^{1-\frac{\alpha+n}{2}}(m^2)^{\frac{1}{2}(\frac{n-\alpha}{2})} e^{\frac{\pi}{2}qi}}{\pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})},$$

$(P \pm i0)^\lambda$ is the distribution defined by

(I;1.3) $$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0}(P \pm i\varepsilon|t|^2)^\lambda;$$

where $\varepsilon > 0$, $|t|^2 = t_1^2 + \cdots + t_n^2$; $\lambda \in \mathbb{C}$ and $P = P(t)$ is the quadratic form defined in (I;1.1) (cf. [7], p. 275).

The following formula is valid (cf. [11], p. 35)

(I;1.4) $$\mathcal{F}[G_\alpha(P \pm i0, m, n)] = \frac{e^{i\frac{\pi}{2}\alpha}}{(2\pi)^{\frac{n}{2}}}(m^2 + Q \mp i0)^{-\frac{\alpha}{2}},$$

where $(m^2 + Q \pm i0)^\lambda$ is the distribution defined by

(I;1.5) $$(m^2 + Q \pm i0)^\lambda = \lim_{\varepsilon \to 0}(m^2 + Q \pm i\varepsilon|t|^2)^\lambda;$$

m is a positive real number, $\lambda \in \mathbb{C}$ and $\mathcal{F}$ denotes the Fourier transform.

From [3], p. 566, we have,

(I;1.6) $$(m^2 + Q \pm i0)^\lambda = (m^2 + Q)_+^\lambda + e^{\pm\lambda\pi i}(m^2 + Q)_-^\lambda,$$

where

(I;1.7) $$(m^2 + Q)_+^\lambda = \begin{cases} (m^2 + Q)^\lambda & \text{if} \quad m^2 + Q \geq 0 \\ 0 & \text{if} \quad m^2 + Q < 0 \end{cases}$$

and

(I;1.8) $$(m^2+Q)^{\lambda}_{-} = \begin{cases} 0 & \text{if} \quad m^2+Q>0 \\ (-m^2-Q)^{\lambda} & \text{if} \quad m^2+Q \le 0 \end{cases}.$$

From (I;1.6) and considering the formula

(I;1.9) $$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}$$

we have

(I;1.10) $$(m^2+Q)^{\lambda}_{-} = \Gamma(\lambda)\Gamma(1-\lambda)(2\pi i)^{-1}[(m^2+Q+i0)^{\lambda} - (m^2+Q-i0)^{\lambda}]$$

and

(I;1.11) $$(m^2+Q)^{\lambda}_{+} = \Gamma(\lambda)\Gamma(1-\lambda)(2\pi i)^{-1} \cdot \cdot [e^{\lambda\pi i}(m^2+Q-i0)^{\lambda} - e^{-\lambda\pi i}(m^2+Q+i0)^{\lambda}]$$

On the other hand, S.E. Trione ([11], p. 23, formula (I,3.6)) proves the validity of multiplicative product:

(I;1.12) $$(m^2+Q \pm i0)^{\lambda} \cdot (m^2+Q \pm i0)^{\mu} = (m^2+Q \pm i0)^{\lambda+\mu}$$

for $m > 0$, where

(I;1.13) $$m^2+Q = m^2 + y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2$$

and $(m^2+Q \pm i0)^{\lambda}$ is defined by (I;1.5).

Let $B^{\alpha} f$ be the ultrahyperbolic Bessel operator defined by the formula

(I;1.14) $$B^{\alpha} f = G_{\alpha} * f, \quad f \in S,$$

(cf. [10], p. 75).

We consider an auxiliary weight function $\lambda_{\alpha}(P \pm i0, m, n)$ defined as follows

(I;1.15) $$\lambda_{\alpha}(P \pm i0, m, n) = = \frac{e^{i\frac{\pi}{2}q} 2^{1-\frac{n+\alpha}{2}} (m^2)^{\frac{1}{2}(\frac{n+\alpha}{2})}}{\Gamma(\frac{n+\alpha}{2})} (P \pm i0)^{\frac{n+\alpha}{2}} K_{\frac{n+\alpha}{2}}\left(\sqrt{m^2(P \pm i0)}\right),$$

(cf. [6]), and

$$\text{(I;1.16)} \qquad \lambda_\alpha(|t|^2, m = 1) = \frac{1}{\Gamma(\frac{n+\alpha}{2})} \int_0^\infty \eta^{\frac{n+\alpha}{2}-1} e^{-\eta - \frac{|t|^2}{4\eta}} d\eta,$$

$\lambda_\alpha(P \pm i0, m, n)$ is the causal (anticausal) analogue of the weight function introduced by Rubin (cf. [8]).

Let f be a function belong to S, S the Schwartz class of infinitely differentiable functions on $\mathbb{R}^n$ decreasing at infinity fasten than $|x|^{-1}$. The weighted difference of order ℓ of a function f at the point x, with interval t and weight λ_α, is defined by

$$\text{(I;1.17)} \qquad (\Delta_t^\ell f)(x, \lambda_\alpha) = \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \lambda_\alpha(k(P \pm i0)) f(x - kt).$$

Let the following causal hypersingular operator on weighted differences defined by

$$\text{(I;1.18)} \qquad (T_\ell^\alpha f)(x) =$$

$$= \int_{\mathbb{R}^n} (P + i0)^{-\frac{n+\alpha}{2}} \left\{ \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \lambda_\alpha(k(P + i0)) f(x - kt) \right\} dt,$$

(cf. [5], p. 72).

The following formula is valid (cf. [5])

$$\text{(I;1.19)} \qquad \mathcal{F}[T_\ell^\alpha f](\xi) = d_{n,\ell}(\alpha)(m^2 + Q - i0)^{\frac{\alpha}{2}} \mathcal{F}[f](\xi),$$

where

$$\text{(I;1.20)} \qquad d_{n,\ell}(\alpha) = \begin{cases} \dfrac{\pi^{\frac{n}{2}+1} e^{i\frac{\pi}{2}q} \mathcal{A}_\ell(\alpha)}{2^\alpha \Gamma(\frac{n+\alpha}{2}) \Gamma(\frac{\alpha}{2}+1) \sin\frac{\pi\alpha}{2}} & \text{for } \alpha \neq 2, 4, \dots \\ \dfrac{(-1)^{\frac{\alpha}{2}} \pi^{\frac{n}{2}} e^{i\frac{\pi}{2}q} 2^{1-\alpha}}{\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{n+\alpha}{2})} \dfrac{d}{d\alpha} \mathcal{A}_\ell(\alpha) & \text{for } \alpha = 2, 4, \dots \end{cases}$$

here $\mathcal{A}_\ell(\alpha)$ is defined by

$$\text{(I;1.21)} \qquad \mathcal{A}_\ell(\alpha) = \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k k^\alpha.$$

For $\alpha \neq 1, 3, 5, \ldots$ we define the generalized causal Bessel derivative as follows

(I;1.22) $$D_C^{\alpha} f(x) = \frac{1}{d_{n,\ell}(\alpha)} (T_{\ell}^{\alpha} f)(x)$$

(cf. [5]).

Then in virtue of (I;1.11), we have

(I;1.23) $$\mathcal{F}[D^{\alpha} f](\xi) = (m^2 + Q - i0)^{\frac{\alpha}{2}} \mathcal{F}[f](\xi).$$

We define (cf. [5]) the ultrahyperbolic causal Bessel operator by the formula

(I;1.24) $$B_C^{\alpha} f = G_{\alpha}(P + i0, m, n) * f$$

and the ultrahyperbolic anticausal Bessel operator as

(I;1.25) $$B_A^{\alpha} f = G_{\alpha}(P - i0, m, n) * f.$$

The generalized anticausal Bessel operator is the anticausal analogue to the define in (I;1.14), and in the same way we have the generalized anticausal Bessel derivative

(I;1.26) $$D_A^{\alpha} f = \frac{1}{d_{n,\ell}(\alpha)} (T_{\alpha}^{\ell} f)(x).$$

We had prove (cf. [5]) that

(I;1.27) $$\mathcal{D}_C^{\alpha} B_C^{\alpha} f = f$$

and, analogously

(I;1.28) $$\mathcal{D}_A^{\alpha} B_A^{\alpha} f = f\,.$$

Also we had prove that (cf. [5])

(I;1.29) $$\mathcal{D}_C^{\alpha} f = G_{-\alpha}(P - i0, m, n) * f.$$

I.2. In this paragraph we will consignate some elementary properties of the ultrahyperbolic causal Bessel kernel. We begin by observe that the distributional function $G_\alpha(P \pm i0, m, n)$ is the causal (anticausal) analogue of the kernel due to A. Calderón and Aronszajn-Smith (cf. [4] and [2]) and share many properties with the Bessel kernel, like the following:

a) for all α, β complex numbers

$$\mathcal{F}[G_\alpha * G_\beta] = \mathcal{F}[G_\alpha] \cdot \mathcal{F}[G_\beta] \tag{I;2.1}$$

and

$$G_\alpha * G_\beta = G_{\alpha+\beta} \tag{I;2.2}$$

(cf. [11], p. 37).

b)

$$G_{-2k}(P \pm i0, m, n) = K^k(\delta) \tag{I;2.3}$$

and when $k = 0$

$$G_0(P \pm i0, m, n) = \delta, \tag{I;2.4}$$

where K^k designate the n-dimensional ultrahyperbolic Klein-Gordon operator iterated k-times

$$K^k = \left\{ \frac{\partial^2}{\partial t_1^2} + \cdots + \frac{\partial^2}{\partial t_p^2} - \frac{\partial^2}{\partial t_{p+1}^2} - \cdots - \frac{\partial^2}{\partial t_{p+q}^2} + m^2 \right\}^k, \tag{I;2.5}$$

where m is a positive real number.

c) The distributional functions $G_{2k}(P \pm i0, m, n)$ where n is an integer ≥ 2; $k = 1, 2, \ldots$, are elementary causal (anticausal) solutions of the ultrahyperbolic Klein-Gordon operator iterated k times.

$$K^k\{G_{2k}\} = \delta. \tag{I;2.6}$$

I.3. Along this paper we shall need the following Lemmas.

Lemma 1. *Let* λ *and* μ *be complex numbers such that* λ, μ *and* $\lambda + \mu \neq -k$, $k = 1, 2, \ldots$; *then the following formulae are valid*

(I;3.1)
$$e^{(\lambda-\mu)\pi i}(m^2 + Q - i0)^{\lambda} \cdot (m^2 + Q + i0)^{\mu} + \\ + e^{-(\lambda-\mu)\pi i}(m^2 + Q + i0)^{\lambda} \cdot (m^2 + Q - i0)^{\mu} = \\ = [1 - C(\lambda, \mu)]e^{(\lambda+\mu)\pi i}(m^2 + Q + i0)^{\lambda+\mu} + \\ + [1 + C(\lambda, \mu)]e^{-(\lambda+\mu)\pi i}(m^2 + Q - i0)^{\lambda+\mu}$$

and

(I;3.2)
$$(m^2 + Q + i0)^{\lambda} \cdot (m^2 + Q - i0)^{\mu} + (m^2 + Q - i0)^{\lambda} \cdot \\ \cdot (m^2 + Q + i0)^{\mu} = [1 + C(\lambda, \mu)](m^2 + Q - i0)^{\lambda+\mu} + \\ + [1 - C(\lambda, \mu)](m^2 + Q + i0)^{\lambda+\mu},$$

where

(I;3.3)
$$C(\lambda, \mu) = 2i \sin \lambda\pi \cdot \sin \mu\pi \csc(\lambda + \mu)\pi = \\ = \frac{2\pi i \Gamma(\lambda + \mu)\Gamma(1 - \lambda - \mu)}{\Gamma(\lambda)\Gamma(1 - \lambda)\Gamma(\mu)\Gamma(1 - \mu)}$$

and $(m^2 + Q \pm i0)^{\lambda}$ *is defined by* (I;1.5).

Proof. It results from (I;1.10), (I;1.11) and considering the multiplicative product (I;1.12) (see [11], p. 39).

In the next we will write

(I;3.4)
$$G_C^{\alpha} = G_{\alpha}(P + i0, m, n)$$

and

(I;3.5)
$$G_A^{\alpha} = G_{\alpha}(P - i0, m, n),$$

where $G_{\alpha}(P \pm i0, m, n)$ is defined by (I;1.2).

On the other hand, from [11], p. 37, we have

(I;3.6)
$$G_{-2k}(P \pm i0, m, n) \in O_C'$$

and

(I;3.7)
$$G_{\alpha}(P \pm i0, m, n) \in S' \quad \text{for all } \alpha \in C,$$

where O_C' designates the space of rapidly decreasing distribution ([9], p. 244) and S' designates the dual of S and S is the Schwartz set of functions ([9], p. 268).

Lemma 2. *Let α and β be complex numbers, then the following formula is valid*

(I;3.8)
$$e^{\frac{1}{2}(\alpha-\beta)\pi i}(G_A^\alpha * G_C^\beta) + e^{-\frac{1}{2}(\alpha-\beta)\pi i} G_C^\alpha * G_A^\beta) = = \Big[1 - C\Big(-\frac{\alpha}{2}, -\frac{\beta}{2}\Big)\Big] e^{-\frac{1}{2}(\alpha+\beta)\pi i} G_A^{\alpha+\beta} + + \Big[1 + C\Big(-\frac{\alpha}{2}, -\frac{\beta}{2}\Big)\Big] e^{\frac{1}{2}(\alpha+\beta)\pi i} G_C^{\alpha+\beta}$$

and

(I;3.9)
$$G_A^\alpha * G_C^\beta + G_C^\alpha * G_A^\beta = = \Big[1 + C\Big(-\frac{\alpha}{2}, -\frac{\beta}{2}\Big)\Big] G_C^{\alpha+\beta} + \Big[1 - C\Big(-\frac{\alpha}{2}, -\frac{\beta}{2}\Big)\Big] G_A^{\alpha+\beta}.$$

Proof. From (I;2.14) we conclude, by appealing to the theorem of Schwartz (cf. [9], p. 268), that the following formula is valid

(I;3.10)
$$\mathcal{F}\{e^{\frac{1}{2}(\alpha-\beta)\pi i}(G_A^\alpha * G_C^\beta) + e^{-\frac{1}{2}(\alpha-\beta)\pi i}(G_C^\alpha * G_A^\beta)\} = = e^{\frac{1}{2}(\alpha-\beta)\pi i} \mathcal{F}\{G_A^\alpha\} \cdot \mathcal{F}\{G_C^\beta\} + e^{-\frac{1}{2}(\alpha-\beta)\pi i} \mathcal{F}\{G_C^\alpha\} \cdot \mathcal{F}\{G_A^\beta\}$$

and

(I;3.11)
$$\mathcal{F}\{G_A^\alpha * G_C^\beta + G_C^\alpha * G_A^\beta\} = \mathcal{F}\{G_A^\alpha\} \cdot \mathcal{F}\{G_C^\beta\} + \mathcal{F}\{G_C^\alpha\} \cdot \mathcal{F}\{G_A^\beta\}.$$

Substituting (I;1.4) into (I;3.10) and (I;3.11) we have

(I;3.12)
$$\mathcal{F}\{e^{\frac{1}{2}(\alpha-\beta)\pi i}(G_A^\alpha * G_C^\beta) + e^{-\frac{1}{2}(\alpha-\beta)\pi i}(G_C^\alpha * G_A^\beta)\} = = e^{\frac{1}{2}(\alpha-\beta)\pi i}[e^{(\alpha+\beta)\frac{\pi i}{2}} \cdot (m^2 + Q + i0)^{-\frac{\alpha}{2}} (m^2 + Q - i0)^{-\frac{\beta}{2}}](2\pi)^{-n} + + e^{-\frac{1}{2}(\alpha-\beta)\pi i}[e^{(\alpha+\beta)\frac{\pi i}{2}} (m^2 + Q - i0)^{-\frac{\alpha}{2}} \cdot (m^2 + Q + i0)^{-\frac{\beta}{2}}](2\pi)^{-n}$$

and

(I;3.13)
$$\mathcal{F}\{G_A^\alpha * G_C^\beta + G_C^\alpha * G_A^\beta\} = = e^{(\alpha+\beta)\frac{\pi i}{2}}[(m^2 + Q + i0)^{-\frac{\alpha}{2}} \cdot (m^2 + Q - i0)^{-\frac{\beta}{2}}](2\pi)^{-n} + + e^{(\alpha+\beta)\frac{\pi i}{2}}[(m^2 + Q - i0)^{-\frac{\alpha}{2}} \cdot (m^2 + Q + i0)^{-\frac{\beta}{2}}](2\pi)^{-n}.$$

Putting $\lambda = -\frac{\alpha}{2}$ and $\mu = -\frac{\beta}{2}$ in (I;3.1) and (I;3.2) and substituting in the right hand member of (I;3.12) and (I;3.13) we have

(I;3.14) $$\mathcal{F}\{e^{\frac{1}{2}(\alpha-\beta)\pi i}(G_A^\alpha * G_C^\beta) + e^{-\frac{1}{2}(\alpha-\beta)\pi i}(G_C^\alpha * G_A^\beta)\} =$$
$$= (2\pi)^{-n} e^{(\alpha+\beta)\frac{\pi i}{2}}\Big\{\Big(1 - C\Big(-\frac{\alpha}{2}, -\frac{\beta}{2}\Big)\Big)e^{-\frac{1}{2}(\alpha+\beta)\pi i}(m^2 + Q + i0)^{-\frac{\alpha+\beta}{2}} +$$
$$+ \Big(1 + C\Big(-\frac{\alpha}{2}, -\frac{\beta}{2}\Big)\Big)e^{\frac{1}{2}(\alpha+\beta)\pi i}(m^2 + Q - i0)^{-\frac{\alpha+\beta}{2}}\Big\}$$

and

(I;3.15) $$\mathcal{F}\{G_A^\alpha * G_C^\beta + G_C^\alpha * G_A^\beta\} =$$
$$= (2\pi)^{-n} e^{(\alpha+\beta)\frac{\pi i}{2}}\Big\{\Big(1 + C\Big(-\frac{\alpha}{2}, -\frac{\beta}{2}\Big)\Big)(m^2 + Q - i0)^{-\frac{\alpha+\beta}{2}} +$$
$$+ \Big(1 - C\Big(-\frac{\alpha}{2}, -\frac{\beta}{2}\Big)\Big)(m^2 + Q + i0)^{-\frac{\alpha+\beta}{2}}\Big\}.$$

By substituting (I;1.4) in (I;3.14), (I;3.15) and taking into account the uniqueness theorem of the Fourier transform, we conclude

(I;3.16) $$e^{\frac{1}{2}(\alpha-\beta)\pi i}(G_A^\alpha * G_C^\beta) + e^{-\frac{1}{2}(\alpha-\beta)\pi i}(G_C^\alpha * G_A^\beta) =$$
$$= \Big(1 - C\Big(-\frac{\alpha}{2}, -\frac{\beta}{2}\Big)\Big)e^{-\frac{1}{2}(\alpha+\beta)\pi i}G_A^{\alpha+\beta} +$$
$$+ \Big(1 + C\Big(-\frac{\alpha}{2}, -\frac{\beta}{2}\Big)\Big)e^{\frac{1}{2}(\alpha+\beta)\pi i}G_C^{\alpha+\beta}$$

and

(I;3.17) $$G_A^\alpha * G_C^\beta + G_C^\alpha * G_A^\beta =$$
$$= \Big(1 + C\Big(-\frac{\alpha}{2}, -\frac{\beta}{2}\Big)\Big)G_C^{\alpha+\beta} + \Big(1 - C\Big(-\frac{\alpha}{2}, -\frac{\beta}{2}\Big)\Big)G_A^{\alpha+\beta}$$

which is the thesis of Lemma 2.

II.1.

Theorem 1. *Let α, β be complex numbers and let B_C^λ (resp. B_A^λ) be the ultrahyperbolic causal (anticausal) Bessel operator of order λ of a function f belong to S. Then*

(II;1.1) $$B_C^\alpha(B_A^\beta f) + B_A^\alpha(B_C^\beta f) =$$
$$= [1 + C(-\frac{\alpha}{2}, -\frac{\beta}{2})]B_C^{\alpha+\beta} f + [1 - C(-\frac{\alpha}{2}, -\frac{\beta}{2})]B_A^{\alpha+\beta} f.$$

Proof. It results from the definitions (I;1.16), (I;1.17) and from (I;3.9).

Theorem 2. *Let α, β be complex numbers and let D_C^λ (resp. D_A^λ) be the generalized causal (anticausal) Bessel derivative of order λ of a function f belong to S. Then*

(II;1.2)
$$D_C^\alpha(D_A^\beta f) + D_A^\alpha(D_C^\beta f) =$$
$$= [1 + C(-\frac{\alpha}{2}, -\frac{\beta}{2})]D_C^{\alpha+\beta} f + [1 - C(-\frac{\alpha}{2}, -\frac{\beta}{2})]D_A^{\alpha+\beta} f\ .$$

Proof. It results from (I;1.14), (I;1.16) and (I;3.9).

Theorem 3. *Let α, β be complex numbers and let B_C^λ (resp B_A^λ) be the ultrahyperbolic causal (anticausal) Bessel operator of order λ of a function f belong to S. Then the following formula is valid*

(II;1.3)
$$e^{\frac{1}{2}(\alpha-\beta)\pi i} B_A^\alpha(B_C^\beta f) + e^{-\frac{1}{2}(\alpha-\beta)\pi i} B_C^\alpha(B_A^\beta f) =$$
$$= [1 - C(-\frac{\alpha}{2}, -\frac{\beta}{2})]e^{-\frac{1}{2}(\alpha+\beta)\pi i} B_A^{\alpha+\beta} f +$$
$$+ [1 + C(-\frac{\alpha}{2}, -\frac{\beta}{2})]e^{\frac{1}{2}(\alpha+\beta)\pi i} B_C^{\alpha+\beta} f.$$

Proof. (II;1.3) it results from (I;1.16), (I;1.17) and (I;3.8).

Theorem 4. *Let α, β be complex numbers and let D_C^λ (resp. D_A^λ) be the generalized causal (anticausal) Bessel derivative of order λ of a function f belong to S. Then is valid*

(II;1.4)
$$e^{\frac{1}{2}(\alpha-\beta)\pi i} D_A^\alpha(D_C^\beta f) + e^{-\frac{1}{2}(\alpha-\beta)\pi i} D_C^\alpha(D_A^\beta f) =$$
$$= [1 - C(-\frac{\alpha}{2}, -\frac{\beta}{2})]e^{-\frac{1}{2}(\alpha+\beta)\pi i} D_A^{\alpha+\beta} f +$$
$$+ [1 + C(-\frac{\alpha}{2}, -\frac{\beta}{2})]e^{\frac{1}{2}(\alpha+\beta)\pi i} D_C^{\alpha+\beta} f.$$

Proof. (II;1.4) it results from (I;1.14), (I;1.18) and (I;3.8).

Otherwise we can observe that the following formulae are valid for α and β complex numbers

(II;1.5) $$D_C^\beta\, B_C^\alpha f = D_C^{\beta-\alpha} f.$$

(II;1.6) $$D_A^\beta\, B_A^\alpha f = D_A^{\beta-\alpha} f.$$

(II;1.7) $$B_C^\beta\, D_C^\alpha f = B_C^{\beta-\alpha} f.$$

(II;1.8) $$B_A^\beta\, D_A^\alpha f = B_A^{\beta-\alpha} f.$$

Theorem 5. *Let α, β be complex numbers and let $B_C^\lambda(B_A^\lambda)$ be the ultrahyperbolic causal (anticausal) Bessel operator and let $D_C^\mu(D_A^\mu)$ be the generalized causal (anticausal) Bessel derivative of order μ of a function f belong to S. Then the following formula is valid*

(II;1.9)
$$B_C^\alpha(D_A^\beta f) + B_A^\alpha(D_C^\beta f) = \\ = (1 + C(-\frac{\alpha}{2}, \frac{\beta}{2}))B_C^{\alpha-\beta} + (1 - C(-\frac{\alpha}{2}, \frac{\beta}{2}))B_A^{\alpha-\beta}.$$

Proof. (II;1.9) results from (I;1.16), (I;1.17) and (I;2.8).

Theorem 6. *Let α, β be complex numbers and let $B_C^\lambda(B_A^\lambda)$ be the ultrahyperbolic causal (anticausal) Bessel operator of order λ and let D_C^μ (resp D_A^μ) be the generalized causal (anticausal) Bessel derivative of order μ of a function f belong to S. Then is valid*

(II;1.10)
$$D_C^\alpha(B_A^\beta f) + D_A^\alpha(B_C^\beta f) = \\ = (1 + C(\frac{\alpha}{2}, -\frac{\beta}{2}))D_C^{\alpha+\beta} f + (1 - C(\frac{\alpha}{2}, -\frac{\beta}{2}))D_A^{\alpha+\beta} f.$$

Proof. From (I;1.14), (I;1.15) and (I;2.13) it results (II;1.10).

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