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GROUPS WITH THE RÉDEI PROPERTY

SÁNDOR SZABÓ

Let G be a finite abelian group of type $(4, 4, 2)$ or (p^2, p), where p is a prime. Assume that AB is a direct product giving G, where A and B are subsets of G both containing the identity element of G. Then A or B lies in a proper subgroup of G.

1. Introduction.

Let G be a finite abelian group written multiplicatively with identity element e. If A and B are subsets of G such that $G = AB$ and the product AB is direct we will say that G is factored by its subsets A and B. In other words $G = AB$ is a factorization of G if each g in G is uniquely expressible in the form $g = ab$, where $a \in A$ and $b \in B$. If $e \in A$ and $e \in B$ we say that the subsets A and B are normed or normalized and also we call the factorization $G = AB$ in this case a normed or normalized factorization. The smallest subgroup of G which contains a subset A of G, that is, the span of A we will denote by $\langle A \rangle$. By the fundamental theorem of finite abelian groups each finite abelian group is a direct product of cyclic groups. The fact that G is a direct product of cyclic groups of orders $t_1, \ldots, t_r$ we will express shortly saying that G is of type $(t_1, \ldots, t_r)$. If from each factorization $G = AB$ it follows that $\langle A \rangle \neq G$ or $\langle B \rangle \neq G$ we will say that G has the Rédei property. Equivalently, G has the

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Rédei property if whenever $G = AB$ is a normed factorization of G then one of the factors A, B must be contained by a proper subgroup of G.

A.D. Sands [6] showed that groups of type (p^α, q^β), where p and q are distinct primes, have the Rédei property and asked if each finite abelian group has the Rédei property. Using result from coding theory O. Fraser and B. Gordon [2] proved that if p is a prime $p \geq 5$, then groups of type $(p, \ldots, p)(p + 1$ p's) do not have the Rédei property. By a construction of S. Szabó [7] groups of type $(t_1, \ldots, t_r)$ do not have the Rédei property if $r \geq 3$ and $t_i/p_i \geq 4$, where p_i is the least prime divisor of t_i. We would like to point out that groups of type (4, 4, 4) do not posses the Rédei property. This does not follow from [7] directly. However, [7] contains a relevant construction and an inspection shows that none of the factors spans the whole group. L. Rédei [4] conjectured that groups of type (p, p, p) have the Rédei property, where p is a prime. S. Szabó and C. Ward [8] verified this conjecture in the $p \leq 11$ special case.

A subset A of a finite abelian group G is called periodic if there is an element $g \in G \setminus \{e\}$ such that $Ag = A$. Each element g with this property is called a periodic of A. All the periods of A together with the identity element form a subgroup H of G. In addition there is a subset C of G such that $A = HC$, where the product is direct. If from each factorization $G = AB$ it follows that either A or B is periodic, then we say that the finite abelian group G has the Hajós property. By A.D. Sands [5] the classification of finite abelain groups with Hajós property is complete. Namely, a finite abelian group has the Hajós property if it is one of the following types or a subgroup of such a group.

$$(p^\alpha, q), \quad (p^2, q^2), \quad (p^2, q, r), \quad (p, q, r, s),$$
$$(p^3, 2, 2), \quad (p^2, 2, 2, 2), \quad (p, 2^2, 2), \quad (p, 2, 2, 2, 2),$$
$$(p, q, 2, 2), \quad (p, 3, 3), \quad (3^2, 3), \quad (2^\alpha, 2),$$
$$(2^2, 2^2), \quad (p, p),$$

where p, q, r, s are distinct primes; the $p = 2$ and $p = 3$ cases are included; $\alpha \geq 3$ is an integer.

We will see that the Hajós property implies the Rédei property and so the groups described above have the Rédei property. The purpose of this paper is to show that groups of type (p^2, p), and (4, 4, 2) have the Rédei property. The proofs of these results represent two different approaches. The first one uses a standard technique, replacing factors, from the factorization theory of finite abelian groups. The second is based on an ad hoc combinatorial argument.

2. p-groups.

First we verify our claim about the connection between the Hajós and Rédei properties.
Then we will consider p-groups of types (p^α, p^β), $(p^\alpha, p^\beta, p^\gamma)$, where p is a prime.

Lemma 1. *The Hajós property implies the Rédei property.*

Proof. Let G be a finite abelian group with the Hajós property. We show that if $G = AB$ is a normed factorization of G, then either $\langle A\rangle \neq G$ or $\langle B\rangle \neq G$. We proceed by induction on n, the number of not necessarily distinct prime factors of $|G|$. If $n = 1$, then since $|G| = |A||B|$ it follows that either $A = G$, $B = \{e\}$ or $A = \{e\}$, $B = G$. Hence either $\langle A\rangle \neq G$ or $\langle B\rangle \neq G$. Suppose that $n \geq 2$. As G has the Hajós property, either A or B is periodic. Assume that B is periodic, that is, $B = HC$, where the product is direct H is a proper subgroup and C is a subset of G. From the factorization $G = AB = A(HC)$ we get the factorization $\overline{G} = \overline{A}\overline{B}$ of the factor group $\overline{G} = G/H$, where $\overline{A} = (AH)/H$ and $\overline{B} = (HC)/H$. Note that $|\overline{G}| < |G|$ and that $\overline{G}$ also have the Hajós property because of its type so by the inductive assumption either $\langle\overline{A}\rangle \neq \overline{G}$ or $\langle\overline{B}\rangle \neq \overline{G}$. From this it follows that either $\langle A\rangle \neq G$ or $\langle B\rangle \neq G$. This completes the proof.

Let A and A' be subsets of a finite abelian group G. We say that A can be replaced by A' if $G = A'B$ is a factorization of G whenever $G = AB$ is a factorization of G.

Lemma 2. *Let G be a finite abelian p-group, where p is a prime. Let $G = AB$ be a normed factorization of G. If $|A| = p$, then either $\langle A\rangle$ is an elementary p-group or B is periodic.*

Proof. If each $a \in A \setminus \{e\}$ is of order p, then $\langle A\rangle$ is of type $(p, \dots, p)$, that is, $\langle A\rangle$ is an elementary group and so there is nothing to prove. We may assume that there is an $a \in A \setminus \{e\}$ with $|a| \geq p^2$. By L. Rédei [3] A can be replaced by $A' = \{e, a, a^2, \dots, a^{p-1}\}$ for each $a \in A \setminus \{e\}$. The factorization $G = A'B$ is equivalent to that the subsets

$$B, aB, a^2B, \dots, a^{p-1}B$$

form a partition of G. Multiply the factorization $G = A'B$ by a to obtain the factorization $G = Ga = (aA')B$. This means that the subsets

$$aB, a^2B, a^3B, \dots, a^pB$$

form a partition of G. Comparing the two partitions of G gives that $B = a^pB$. As $a^p \neq e$, B is periodic. This completes the proof.

Theorem 1. *Let G be a group of type (p^α, p^β), where p is a prime. If $G = AB$ is a normed factorization of G such that $|A| = p$, then either $\langle A\rangle \neq G$ or $\langle B\rangle \neq G$.*

Proof. We may assume that $\alpha \geq \beta$. We proceed by induction on $n = \alpha + \beta$. If $n = 1$, then $|A| = p$ and $|B| = 1$. Hence $B = \{e\}$ and so $\langle B\rangle \neq G$. If $n = 2$, then $|A| = |B| = p$ so by the main result of L. Rédei [3] either A or B is a subgroup of G and so either $\langle A\rangle \neq G$ or $\langle B\rangle \neq G$. For the remaining part of the proof suppose that $n \geq 3$. Now $\alpha \geq 2$ and consequently G is not an elementary group. By Lemma 2 either $\langle A\rangle$ is an elementary group or B is periodic. If $\langle A\rangle$ is an elementary group, then $\langle A\rangle$ cannot be equal to G and so we are done. Assume that B is periodic, that is, $B = HC$, where the product is direct H is a proper subgroup and C is a subset of G. From the factorization $G = AB = A(HC)$ we get the factorization $\overline{G} = \overline{A}\overline{B}$ of the factor group $\overline{G} = G/H$, where $\overline{A} = (AH)/H$ and $\overline{B} = (HC)/H$. Since $|\overline{G}| < |G|$ and since the type of $\overline{G}$ has not charged by the inductive hypothesis either $\langle\overline{A}\rangle \neq \overline{G}$ or $\langle\overline{B}\rangle \neq \overline{G}$. From this it follows that either $\langle A\rangle \neq G$ or $\langle B\rangle \neq G$. This completes the proof.

Corollary 1. *Groups of type (p^2, p) have the Rédey property for each prime p.*

Proof. Let p be a prime and let $G = AB$ be a normed factorization, where G is a group of type (p^2, p). As $|G| = |A||B|$ by relabelling we may assume that $|A| = p$ and $|B| = p^2$. By Theorem 1 either $\langle A\rangle \neq G$ or $\langle B\rangle \neq G$. This completes the proof.

Theorem 2. *Let G be a group of type $(p^\alpha, p^\beta, p^\gamma)$, where p is a prime $p \leq 11$. If $G = AB$ is a normed factorization of G such that $|A| = p$, then either $\langle A\rangle \neq G$ or $\langle B\rangle \neq G$.*

Proof. We may assume that $\alpha \geq \beta \geq \gamma$. The $\gamma = 0$ case is covered by Theorem 1 so we assume that $\gamma \geq 1$. We proceed by induction on $n = \alpha+\beta+\gamma$. If $n = 3$, then $|A| = p$, $|B| = p^2$ and as $\gamma \geq 1$, G is of type (p, p, p). Hence by S. Szabó and C. Ward [8] either $\langle A\rangle \neq G$ or $\langle B\rangle \neq G$. In the remaining part of the proof we suppose that $n \geq 4$. Now G cannot be an elementary group. By Lemma 2 either $\langle A\rangle$ is an elementary group or B is periodic. If $\langle A\rangle$ is an elementary group, then $\langle A\rangle \neq G$ and we are done. So we may assume that B is periodic, that is, $B = HC$, where the product is direct H is a proper subgroup and C is a subset of G. From the factorization $\overline{G} = \overline{A}\overline{B}$ of the factor group $\overline{G} = G/H$, where $\overline{A} = (AH)/H$ and $\overline{B} = (HC)/H$. In the way we have we seen in the proof of Theorem 1, it follows that either $\langle A\rangle \neq G$ or $\langle B\rangle \neq G$. This completes the proof.

3. 2-groups.

In this section we will deal with 2-groups of type $(2^{\alpha_1}, \ldots, 2^{\alpha_s})$, where $1 \leq \alpha_i \leq 2$ for each i, $1 \leq i \leq s$. First we prove two lemmas essentially about groups of types $(4, 4, 4)$ and $(4, 4, 2)$.

Lemma 3. *If $G = AB$ is a normed factorization of the finite abelian group G, where $|A| = 4$ and $\langle A\rangle$ is of type $(4, 4, 4)$, then B is periodic.*

Proof. Let $b \in B$. Multiplying the factorization $G = AB$ by b^{-1} we get the factorization $G = Gb^{-1} = A(Bb^{-1})$. Let $H = \langle A\rangle$. Restricting the factorization $G = A(Bb^{-1})$ to H we get the factorization $H = G \cap H = A(Bb^{-1} \cap H) = AC$ of H. Here $C = (Bb^{-1} \cap H)$. Set $A = \{e, x, y, z\}$. By the hypothesis of the lemma x, y, z form a basis of H, that is, $H = \langle x\rangle \times \langle y\rangle \times \langle z\rangle$, where $|x| = |y| = |z| = 4$.

We show that C is periodic with period $(xyz)^2$. This gives that $(xyz)^2 \in C = Bb^{-1} \cap H$ and so

$$(xyz)^2 \in \bigcap_{b \in B} Bb^{-1}.$$

From this by Lemma 4 of [1] it follows that $(xyz)^2$ is a period of B.

Consider the subgroup $T = \langle x\rangle \times \langle y\rangle$ of H. Let n_0, n_1, n_2, n_3 be the number of elements of C contained by the cosets T, Tz, Tz^2, Tz^3 respectively. As $H = AC$ is a factorization we get the following system of equations.

$$\begin{aligned} 3n_0 \qquad\qquad\qquad\quad + n_3 &= 16 \\ n_0 + 3n_1 \qquad\qquad\qquad &= 16 \\ n_1 + 3n_2 \qquad\quad &= 16 \\ n_2 + 3n_3 &= 16 \end{aligned}$$

This gives $n_0 = n_1 = n_2 = n_3 = 4$, that is, each coset contains 4 elements from C.

The following tables show that there are only 4 possibilities for $C \cap T$. Namely, it can only be

$$\langle xy\rangle, \quad \langle x^2, y^2\rangle, \quad \langle x^2 y\rangle, \quad \langle xy^2\rangle.$$

The entries marked by [ ] belong to C; and the entries marked by ( ) do not belong to $C \cap T$.

$[e]$	y	(y^2)	y^3
x	$[xy]$	xy^2	(xy^3)
(x^2)	x^2y	$[x^2y^2]$	x^2y^3
x^3	(x^3y)	x^3y^2	$[x^3y^3]$

$[e]$	y	$[y^2]$	y^3
x	(xy)	xy^2	(xy^3)
$[x^2]$	x^2y	$[x^2y^2]$	x^2y^3
x^3	(x^3y)	x^3y^2	(x^3y^3)

$[e]$	y	$[y^2]$	y^3		$[e]$	y	y^2	(y^3)
x	(xy)	xy^2	(xy^3)		x	(xy)	$[xy^2]$	xy^3
x^2	$[x^2y]$	x^2y^2	$[x^2y^3]$		$[x^2]$	x^2y	x^2y^2	(x^2y^3)
(x^3)	x^3y	(x^3y^2)	x^3y^3		x^3	(x^3y)	$[x^3y^2]$	x^3y^3

Note that $C \cap T$ determines $C \cap Tz^3$ uniquely; $C \cap Tz^3$ determines $C \cap Tz^2$; $C \cap Tz^2$ determines $C \cap Tz$. Thus C can only be one of the following subgroups of H

$$\langle xy \rangle \times \langle x^2z^3 \rangle, \quad \langle x^2, y^2 \rangle \times \langle xyz^3 \rangle, \quad \langle x^2y \rangle \times \langle xyz^3 \rangle, \quad \langle xy^2 \rangle \times \langle xyz^3 \rangle.$$

We can now verify that in each cases $(xyz)^2$ is a period of C. This completes the proof.

Lemma 4. *Let $G = AB$ be a normed factorization, where G is of type $(4, 4, 2)$, $|A| = 4$, $\langle A \rangle = G$. Then $\langle B \rangle \neq G$.*

Proof. As $\langle A \rangle = G$, there is a basis x, y, z of G such that $|x| = |y| = 4$, $|z| = 2$ and $A = \{e, x, y, az\}$, where $a \in \langle x, y \rangle = T$. Let n_0, n_1 be the number of elements of B contained by the cosets T, Tz respectively. The factorization $G = AB$ yields the following system of equations.

$$\begin{aligned} 3n_0 + n_1 &= 16 \\ n_0 + 3n_1 &= 16 \end{aligned}$$

The only solution of this system is $n_0 = 4$, $n_1 = 4$ and consequently the cosets T, Tz both contain 4 elements from B. In the way we have seen in the proof of Lemma 3 we get that

$$\langle xy \rangle, \quad \langle x^2, y^2 \rangle, \quad \langle x^2y \rangle, \quad \langle xy^2 \rangle$$

are the only possibilities for $B \cap T$. Observe that in each of the four cases $B \cap T$ determines $B \cap Tz$ and so the whole of B uniquely. Thus $B = K \cup Kb$, where $b \in G$ and K is one of the above four subgroups of G. Now $|\langle B \rangle| \leq |K| \cdot |b| \leq 4 \cdot 4 = 16$ and so $\langle B \rangle \neq G$. This completes the proof.

Theorem 3. *Let G be a group of type $(2^{\alpha_1}, \ldots, 2^{\alpha_s})$, where $1 \leq \alpha_i \leq 2$ for each i, $1 \leq i \leq s$. If $G = AB$ is a normed factorization of G, where $|A| = 2$ or $|A| = 4$, then either $\langle A \rangle \neq G$ or $\langle B \rangle \neq G$.*

Proof. We may assume that $\langle A \rangle = G$ since otherwise there is nothing to prove. If $|A| = 2$, then A contains only one nonidentity element. Hence G must be of type (2) or (4) by the assumption on the type of G. In these cases G has the Hajós and consequently the Rédei property.

Turn to the case when $|A| = 4$. A contains three nonidentity elements and so by the type of G it follows that the type of $\langle A \rangle = G$ is one of the following.

$$(2, 2, 2), \quad (4, 2, 2), \quad (4, 4, 2), \quad (4, 4, 4).$$

In the first two cases G has the Hajós property and so $\langle B \rangle \neq G$. In the third case Lemma 4 leads to this conclusion. In the last case by Lemma 3 B is periodic, that is, $B = HC$, where the product is direct and H is a proper subgroup of G. In this case considering the factorization $\overline{G} = \overline{A}\overline{B}$ of the factor group $\overline{G} = G/H$, where $\overline{A} = (AH)/H$ and $\overline{B} = (HC)/H$ completes the proof in the known way.

Corollary 2. *A group of type* $(4, 4, 2)$ *has the Rédei property.*

Proof. Let G be a group of type $(4, 4, 2)$ and let $G = AB$ be a normed factorization of G. We may assume that $|A| \leq |B|$ and so either $|A| = 2$ or $|A| = 4$. Further we may assume that $\langle A \rangle = G$ since otherwise there is nothing to prove. When $|A| = 2$ A has only one nonidentity element and in this case A cannot span the whole G. So $|A| = 4$. Now by Lemma 4 $\langle B \rangle \neq G$. This completes the proof.

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Department of General Sciences and Mathematics,
College of Health Sciences,
P.O. Box 12,
Manama (STATE of BAHRAIN)