LE MATEMATICHE
Vol. LII (1997) – Fasc. II, pp. 387–392

SEMILINEAR EQUATIONS WITH STRONGLY MONOTONE NONLINEARITY

CRISTINEL MORTICI

It is presented a method to solve semilinear equations in real Hilbert spaces. Some applications to differential equations are given.

1. Introduction.

In [1] it is studied semilinear equations of the form

$$Au = F(u) \tag{1}$$

in a real Hilbert space H, where $A : D(A) \subset H \to H$ is a self-adjoint linear operator with the resolvent set $\rho(A)$ and $F : H \to H$ is a Gateaux differentiable gradient operator. In particular it is known that equation (1) possesses multiple solutions if the nonlinearity F interacts suitably with the spectrum of A. In [2] it is presented the following existence and uniqueness theorem, as a corollary to some general considerations on saddle points:

Theorem 1 (Amann)**.** *Suppose that there exist real numbers $\nu < \mu$ such that $[\nu, \mu] \subset \rho(A)$ and*

$$\nu \leq \frac{< F(u) - F(v), u - v >}{|u - v|^2} \leq \mu, \quad \forall u, v \in H,\ u \neq v. \tag{2}$$

Then the equation $Au = F(u)$ possesses exactly one solution.

Entrato in Redazione il 9 ottobre 1997.

In this paper we consider the equation (1) of the form

$$Au + F(u) = 0. \tag{3}$$

We establish an existence and uniqueness result for (3) asking a condition of type (2) for F and maximal monotony for A, but giving up from self-adjointness of A and Gateaux differentiability of F. The condition of maximal monotony for A is not very restrictive because the most known differential equations have this property.

2. The main result.

We give the following

Theorem 2. *Assume that* $A : D(A) \subset H \to H$ *is maximal monotone and there exist* $m, M > 0$ *such that*

(i) $< F(u) - F(v), u - v > \geq m \cdot |u - v|^2, \ \forall u, v \in H;$

(ii) $|F(u) - F(v)| \leq M \cdot |u - v|, \ \forall u, v \in H.$

Then equation (3) *has an unique solution.*

Proof. We shall use the following known result

Lemma. *Suppose that* $F : H \to H$ *satisfy (i) and (ii). Then there exists* $\lambda > 0$ *such that* $S_\lambda : H \to H$, $S_\lambda(u) := u - \lambda F(u)$ *is a contraction.*

Proof. Indeed,

$$|S_\lambda(u) - S_\lambda(v)|^2 = |u - v|^2 - 2\lambda\langle F(u) - F(v), u - v\rangle + \lambda^2 |F(u) - F(v)|^2 \leq$$
$$\leq (1 - 2\lambda m + \lambda^2 M^2) |u - v|^2,$$

thus

$$|S_\lambda(u) - S_\lambda(v)| \leq c \cdot |u - v|, \tag{4}$$

with $c := \sqrt{1 - 2\lambda m + \lambda^2 M^2} < 1$, if $\lambda \in (0, \frac{2m}{M^2})$.

Now equation (3) can be written as

$$(I + \lambda A)u - (u - \lambda F(u)) = 0, \tag{5}$$

or

$$(I + \lambda A)u = S_\lambda(u), \tag{6}$$

where $\lambda > 0$ is taken from the lemma. Using the fact that $(I + \lambda A)$ is inversable and $\left|(I + \lambda A)^{-1}\right| \leq 1$ for each $\lambda > 0$ (because A is maximal monotone, e.g. [3], p 101) the equation (6) is equivalent with

$$u = (I + \lambda A)^{-1} S_\lambda(u). \tag{7}$$

We have

$$\left|(I + \lambda A)^{-1} S_\lambda(u) - (I + \lambda A)^{-1} S_\lambda(v)\right| = \left|(I + \lambda A)^{-1}(S_\lambda(u) - S_\lambda(v))\right| \leq$$
$$\leq \left|(I + \lambda A)^{-1}\right| \cdot |S_\lambda(u) - S_\lambda(v)| \leq c \cdot |u - v|\,, \ \forall u, v \in H.$$

Therefore, $u \mapsto (I + \lambda A)^{-1} S_\lambda(u)$ is a contraction having an unique fixed point, thus (7) and consequently (3) has an unique solution. □

A similar result can be proved in the next case.

Theorem 3. *Suppose that F satisfy (i), (ii) and $A : D(A) \subset H \to H$ is linear, compact and monotone. Then equation* (3) *has an unique solution.*

Proof. Equation (3) can be equivalently written as

$$(\lambda I + A)u = T_\lambda(u), \tag{8}$$

where $T_\lambda(u) := \lambda u - F(u)$, $\lambda > 0$. We have

$$|T_\lambda(u) - T_\lambda(v)|^2 = \lambda^2 |u - v|^2 - 2\lambda \langle F(u) - F(v), u - v\rangle +$$
$$+ |F(u) - F(v)|^2 \leq (\lambda^2 - 2\lambda m + M^2)|u - v|^2,$$

therefore

$$|T_\lambda(u) - T_\lambda(v)| \leq \sqrt{\lambda^2 - 2\lambda m + M^2} \cdot |u - v|. \tag{9}$$

Let us choose $\lambda > \max\{\|A\|\,, \frac{M^2}{2m}\}$. In particular, $\lambda > \|A\|$ imply that $\lambda I + A$ is inversable because $\sigma(A) \subset [-\|A\|\,, \|A\|]$. Moreover,

$$|(\lambda I + A)u|^2 = \lambda^2 |u|^2 + 2\lambda (Au, u) + |Au|^2 \geq \lambda^2 |u|^2, \tag{10}$$

(because A is monotone), or

$$|(\lambda I + A)u| \geq \lambda |u|,$$

hence $|(\lambda I + A)^{-1}| \leq \frac{1}{\lambda}$. Equation (8) is equivalent with

$$u = (\lambda I + A)^{-1} T_\lambda(u). \tag{11}$$

We have

$$\begin{aligned} &\left|(\lambda I + A)^{-1} T_\lambda(u) - (\lambda I + A)^{-1} T_\lambda(v)\right| = \\ &= \left|(\lambda I + A)^{-1}(T_\lambda(u) - T_\lambda(v))\right| \leq \left|(\lambda I + A)^{-1}\right| \cdot |T_\lambda(u) - T_\lambda(v)| \leq \\ &\leq \frac{1}{\lambda}\sqrt{\lambda^2 - 2\lambda m + M^2} \cdot |u - v|, \quad \forall u, v \in H. \end{aligned}$$

Because $\lambda > \frac{M^2}{2m}$, it results that $\gamma := \frac{1}{\lambda}\sqrt{\lambda^2 - 2\lambda m + M^2} < 1$, therefore $u \longmapsto (\lambda I + A)^{-1} T_\lambda(u)$ is a contraction. Now equation (11) and consequently (3) has an unique solution. □

Remark. Compactness and boundedness of A was used to choose a number $\lambda > 0$ such that $\lambda I + A$ is inversable. This is possible in weaker hypothesis. Indeed, the condition A *compact and bounded* can be replaced with *spectrum of A is bounded.*

We can state the more general result

Theorem 4. *Let $F : H \to H$ satisfy (i), (ii) and $A : D(A) \subset H \to H$ be monotone and the spectrum $\sigma(A)$ is bounded from below. Then equation* (3) *has an unique solution.*

Indeed, it can be repeated the proof from Theorem 3 taking $\lambda > \frac{M^2}{2m}$ such that $-\lambda \in \rho(A)$.

3. Applications.

(A1). SEMILINEAR ELLIPTIC BOUNDARY PROBLEMS

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $a_{ij} \in C^1(\overline{\Omega})$, $1 \leq i, j \leq N$, having the ellipticity property

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha |\xi|^2, \quad \forall \xi \in \mathbb{R}^N,$$

for some $\alpha > 0$. Let us consider the following elliptic problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + g(x, u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{12}$$

where the nonlinearity is given by the real valued function $f \in L^2(\Omega)$. The particular case when $g(x, u) = a_0(x)u$, with $a_0 \in C(\overline{\Omega})$, $a_0 > p > 0$ is studied in [3], p. 177, using Lax-Milgram Theorem and in [1], p. 165, using the above Theorem 1. Now we suppose that $g(x, u)$ has partial derivative in u of the first order and

$$m \leq \frac{\partial g}{\partial u} \leq M \quad \text{in } \Omega, \ (m, M > 0). \tag{13}$$

Under these hypotheses, problem (12) has an unique solution in weak sense, for every $f \in L^2(\Omega)$. Indeed, we can apply Theorem 2 for the following functional background:

$$H = L^2(\Omega), \quad Au := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right), \quad D(A) := H^2(\Omega) \cap H_0^1(\Omega),$$

$F(u) := g(\cdot, u) - f$. A is monotone:

$$(Au, u) = \int_\Omega \sum_{i,j=1}^{N} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i} \geq 0$$

and $I + A$ is surjective ([3], p. 177), thus A is maximal monotone. The conditions (i) and (ii) follows from (13).

(A2). In [5] is studied the perturbed Laplace problem

$$\begin{cases} -\Delta u + Pu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{14}$$

using the variational theorem of Langenbach. We can apply theorem, asking that $P : L^2(\Omega) \to L^2(\Omega)$ satisfy (i) and (ii). In particular, if P is Gateaux differentiable with

$$m \cdot |h|^2 \leq \langle (DP)(u)h, h\rangle \leq M \cdot |h|^2, \quad (m, M > 0)$$

then (14) has an unique solution, because $Au := -\Delta u$, $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$ is maximal monotone.

(A3). PERIODIC SOLUTIONS OF SEMILINEAR WAVE EQUATION

Let V be a Hilbert space. Suppose that $L : D(L) \subset V \to V$ is maximal monotone and $F \in C(\mathbb{R} \times V, V)$ such that, for some $T > 0$,

$$F(t + T, \cdot) = F(t, \cdot), \quad \forall\ t \in \mathbb{R}.$$

Then we are interested in the existence of $T-$periodic solutions for the semilinear abstract equation:

$$\begin{cases} -u'' + Lu + F(t, u) = 0, & t \in \mathbb{R} \\ u(0) = u(T), u'(0) = u'(T). \end{cases} \tag{15}$$

Let now

$$H := L^2((0, T); V),\ A : D(A) \subset H \to H,\ Au := -u'' + Lu,$$

with $D(A) := \{u \in C^2([0, T]; V) \cap L^2((0, T), D(L)) \mid u(0) = u(T), u'(0) = u'(T)\}$. A is maximal monotone and if F satisfy (i) and (ii), in particular, a condition of type (13), then problem (15) has exactly one periodic solution.

Acknowledgements: I want to express my gratitude to the referee for the useful remarks. I also thank to my professor dr. Silviu Sburlan for the indications in writing of this paper.

REFERENCES

[1] H. Amann, *On the unique solvability of the semilinear operator equations in Hilbert spaces,* J. Math. Pures Appl., 61 (1982), pp. 149–175.

[2] H. Amann, *Saddle points and multiple solutions of differential equations,* Math. Zeitschr., 169 (1979), pp. 127–166.

[3] H. Brezis, *Analyse fonctionnelle - Théorie et applications,* Masson Editeur, Paris, 1983.

[4] P. Mazilu - S. Sburlan, *Metode functionale in rezolvarea ecuatiilor teoriei elasticitatii,* Ed. Acad., Bucuresti, 1973.

[5] C. Mortici, *Variational methods for the perturbed Laplace problem,* A VI-a Conf. de analiza neliniara, Constantza,1997.

[6] D. Pascali - S. Sburlan, *Nonlinear mappings of monotone type,* Sijthoff & Noordhoff, Intern. Publ., Alphen aan den Rijn, 1978.

Department of Mathematics,
"Ovidius" University,
Bd. Mamaia 124,
8700 Constantza (ROMANIA)