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NECESSARY AND SUFFICIENT CONDITIONS FOR HÖLDER CONTINUITY OF SOLUTIONS OF DEGENERATE SCHRÖDINGER OPERATORS

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In this paper it is studied the Hölder-continuity of solutions of a linear degenerate elliptic equation of the form

$$(*) \qquad -\sum_{i,j=1}^{n}(a_{ij}u_{x_i})_{x_j} + Vu = 0.$$

It is proved that the solutions of $(*)$ are Hölder-continuous if the coefficient V belongs to an appropriate "degenerate" Morrey space. Under some additional assumptions on the weight giving the degeneracy, the previous condition is also necessary.

1. Introduction.

In recent years many Authors studied local regularity of solutions of the linear elliptic equation

$$(1.1) \qquad Lu := -\sum_{i,j=1}^{n}(a_{ij}u_{x_i})_{x_j} = -Vu$$

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assuming on the potential V more general hypotheses than the "classical" L^p ones (see [1], [2], [11], [4] and, in the quasilinear case, [10], [9] and [12]). All these papers do not deal with degenerate equations. For the degenerate equations the only result known is given by C. Gutierrez in his paper [8]. There he studies the degenerate elliptic equation (1.1), with the weight w giving the degeneracy belonging to A_2, assuming $\frac{V}{w}$ belonging to the class S (see Definition 2.8), that is the natural extension to the degenerate case of the Stummel-Kato class (see [1], [2] and [11]). In [8] Gutierrez, following very closely the technique developed in [2], proves a Harnack's inequality for the positive solutions of the degenerate equation (1.1). We notice that in [8] Gutierrez also claims that, under the same assumptions on $\frac{V}{w}$, the continuity of the solutions can be obtained. We give a proof of that in Theorem 3.1 of Section 3.

Our purpose in this note is to extend to the degenerate case some of the results of the papers [4] and [11], where hölder continuity of solutions of (1.1) is proved under the assumption that V belongs to the Morrey space $L^{1,\lambda}$, $\lambda > n-2$.

We consider the equation (1.1) where the coefficients $a_{ij}(x)$ are measurable functions such that

$$a_{ij}(x) = a_{ji}(x), \qquad i, j = 1, 2, \dots, n\,, \tag{1.2}$$

and

$$\begin{aligned} \exists\, \nu > 0 : \nu^{-1} w(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \\ \leq \nu w(x)|\xi|^2 \;\; a.e.\; x \in \mathbb{R}^n,\; \forall \xi \in \mathbb{R}^n, \end{aligned} \tag{1.3}$$

with the weight w belonging to the A_2 class.

The first problem one has to face is to understand what a "degenerate" Morrey space is. We introduce two such notions of degenerate Morrey space, $M_\sigma(w)$ and $L^{1,\varepsilon}(w)$ (see Definition 2.9 and Definition 2.10). Both give back, in the nondegenerate case, the "classical" Morrey space $L^{1,\lambda}$; in particular for $\varepsilon = \sigma > 0$ we have $L^{1,\varepsilon}(1) = M_\varepsilon(1) = L^{1,n-2+\varepsilon}$, as we prove in Remark 2.15.

In Theorem 3.2, assuming $\frac{V}{w} \in M_\sigma(w)$, we prove hölder-continuity of solutions of the degenerate elliptic equation (1.1), extending the results in [4] and [11]. The space $L^{1,\varepsilon}(w)$ in turn gives some interesting necessary conditions for hölder-continuity of solutions of (1.1), as we show in Section 4. We know that in general these two spaces are different even if they are the same in many non trivial situations (see Proposition 2.13 and Example 2.15).

2. Function spaces and preliminary results.

Let $p > 1$, a function $w : \mathbb{R}^n \to]0, +\infty[$, such that $w(x)$ and $[w(x)]^{-\frac{1}{p-1}}$ belong to $L^1_{loc}(\mathbb{R}^n)$, is said an A_p weight if and only if

$$\text{(2.1)} \quad \sup_{B_r} \left(\frac{1}{|B_r|} \int_{B_r} w(x)\, dx \right) \left(\frac{1}{|B_r|} \int_{B_r} [w(x)]^{-\frac{1}{p-1}}\, dx \right)^{p-1} = C_0 < +\infty,$$

where B_r (1) is a ball in $\mathbb{R}^n$; C_0 is said the A_p constant of w.

We now recall some results about A_p weights (see [3] and [7] for the proof).

Lemma 2.1. *Let $w(x)$ be an A_p weight, $p \in]1, +\infty[$, set $w(B_r) = \int_{B_r} w(x)dx$, then*

a) there exists a constant $C_d > 1$ such that

$$w(B(x, 2r)) \leq C_d w(B(x, r));$$

b) there exists a positive constant $K < 1$ such that

$$w(B(x, r)) \leq K w(B(x, 2r));$$

c) for any bounded subset Ω of $\mathbb{R}^n$ there exists a positive constant $C = C(w, \Omega)$ such that

$$|B_r|^p \leq C w(B_r)\,,$$

for any ball B_r contained in Ω.

Let Ω be an open bounded set in $\mathbb{R}^n$. Because of the local character of our results it is sufficient to assume $\Omega \equiv B(0, R)$.

Let $w(x)$ be an A_2 weight. We give the definitions of the spaces $L^p(\Omega, w)$, $H^{1,p}(\Omega, w)$, $H^{1,p}_{loc}(\Omega, w)$, $H^{1,p}_0(\Omega, w)$, $H^{-1,p}(\Omega, w)$, $p \in [1, +\infty[$ (see also [5]).

$L^p(\Omega, w)$ is the space of measurable u in Ω, such that

$$\|u\|_{L^p(\Omega,w)} = \left(\int_\Omega |u(x)|^p w(x)\, dx \right)^{\frac{1}{p}} < +\infty.$$

(1) In this paper we will write $B(x, r)$ to denote the ball centered at x with radius r. Whenever x is not relevant we will write B_r.

$\mathrm{Lip}\,(\overline{\Omega})$ denotes the class of Lipschitz functions in $\overline{\Omega}$. $\mathrm{Lip}_0\,(\Omega)$ denotes the class of functions $\phi \in \mathrm{Lip}\,(\overline{\Omega})$ with compact support contained in Ω. If ϕ belongs to $\mathrm{Lip}\,(\overline{\Omega})$ we can define the norm

$$\|\phi\|_{H^{1,p}(\Omega,w)} := \|\phi\|_{L^p(\Omega,w)} + \sum_{i=1}^{n} \|\phi_{x_i}\|_{L^p(\Omega,w)}. \tag{2.2}$$

$H^{1,p}(\Omega, w)$ denotes the closure of $\mathrm{Lip}\,(\overline{\Omega})$ under the norm (2.2). We say that $u \in H^{1,p}_{loc}(\Omega, w)$ if $u \in H^{1,p}(\Omega', w)$ for every $\Omega' \subset\subset \Omega$.

$H^{1,p}_0(\Omega, w)$ denotes the closure of $\mathrm{Lip}_0\,(\Omega)$ under the norm (2.2).

$H^{-1,p'}(\Omega, w)$ is the dual space of $H^{1,p}_0(\Omega, w)$, where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. We have $T \in H^{-1,p'}(\Omega, w)$ if $\exists\, f_i \,:\, \dfrac{f_i}{w} \in L^{p'}(\Omega, w)$, $i = 1, 2, \dots, n$, with $T = \sum\limits_{i=1}^{n} (f_i)_{x_i}$.

Let $T \in H^{-1,2}(\Omega, w)$. We say that $u \in H^{1,2}_{loc}(\Omega, w)$ is a *local weak solution* of the equation

$$Lu = T$$

if

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} u_{x_i} \psi_{x_j}\, dx =< T, \psi > \quad \forall \psi \in C_0^\infty(\Omega). \tag{2.3}$$

We now recall some results which we will use in the following.

Theorem 2.2 (see [6], Theorem 2.3.12)**.** *Let u be a local solution of $Lu = 0$ in Ω. Then u is locally Hölder continuous in Ω. More precisely, there exist $M > 0$ and $0 < \alpha < 1$, depending only on the A_2 constant, such that if $x_0 \in \Omega$ and $B(x_0, r) \subset\subset \Omega$, then*

$$\sup_{|x-x_0|<\rho} |u(x) - u(x_0)| \le M \left(\frac{1}{w(B(x_0, r))} \int_{B(x_o,r)} u^2 w\, dx \right)^{\frac{1}{2}} \left(\frac{\rho}{r}\right)^{\alpha}, \tag{2.4}$$

for $\rho < r$.

Theorem 2.3 (see [6], Theorem 2.3.8)**.** *Let u be a positive local solution of $Lu = 0$ in Ω. Then there exists a constant $M > 1$ such that*

$$\max_B u \le M \min_B u\,, \tag{2.5}$$

for each ball $B \equiv B(x, r)$ such that $B(x, 2r) \subset \Omega$.

We now define a different class of solutions. Let μ be a bounded variation measure in Ω. We say that $u \in L^1(\Omega, w)$ is a *very weak solution vanishing on* $\partial\Omega$ of the equation

$$Lu = \mu \tag{2.6}$$

if

$$\int_\Omega u(x) L\psi(x)\, dx = \int_\Omega \psi(x)\, d\mu$$

for every $\psi \in H_0^{1,2}(\Omega, w) \cap C^0(\overline{\Omega})$ such that $L\psi \in C^0(\overline{\Omega})$. We observe that the class of test functions is not empty by Theorem 2.3.15 in [6].

Remark 2.4. For any bounded variation measure μ in Ω, there exists a unique very weak solution u of $Lu = \mu$ in Ω such that $u = 0$ on $\partial\Omega$ (see [5], Proposition 2.1). Moreover it is no difficult to show that if $u \in H_0^{1,2}(\Omega, w)$ is a weak solution of the equation $Lu = \mu$, i.e.

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} u_{x_i} \psi_{x_j}\, dx = \int_\Omega \psi(x)\, d\mu \quad \forall \psi \in C_0^\infty(\Omega)$$

then u is a very weak solution of the same equation.

Let $y \in \Omega$. Denote by $g_\Omega(x, y)$ the very weak solution vanishing on $\partial\Omega$ of the equation

$$Lu = \delta_y,$$

where δ_y is the Dirac mass at y. We call it the Green's function relative to the operator L in Ω.

We now recall some results, concerning the Green's function, proved in [5].

Theorem 2.5 (see [5], Proposition 2.4)**.** *Let $B_r \subseteq \Omega$ be a ball and $g_{B_r}(x, y)$ be the Green's function of L in B_r, then $g_{Br}(., y) \in H^{1,2}(B_r \setminus B(y, \varepsilon), w)$ for any $\varepsilon > 0$.*

Theorem 2.6 (see [5], Lemma 2.7)**.** *Let $B_r \subseteq \Omega$ be a ball and $g_{B_r}(x, y)$ be the Green's function of L in B_r, then*

$$u(x) = \int_{B_r} g_{B_r}(x, y)\, d\mu(y)$$

is the very weak solution vanishing on ∂B_r of (2.6) *in B_r.*

Denoting by $g_{B(0,4R)}$ the Green's function in $B(0, 4R)$ we have the following result

Theorem 2.7. *Let $x, y \in B(x_0, r) \subseteq \Omega$. Then there exist two positive constants C_1 and C_2, independent of x and y, such that*

$$C_1 < \frac{g_{B(x_0,4r)}(x, y)}{\int_{|x-y|}^{4r} \frac{s^2}{w(B(x,s))} \frac{ds}{s}} < C_2. \tag{2.7}$$

Moreover

$$g_{\Omega''} \leq g_{\Omega'} \leq g_{B(0,4R)} \tag{2.8}$$

for any $\Omega'' \subseteq \Omega' \subseteq B(0, 4R)$.

Proof. For (2.7) see Theorem 3.3 in [5]. (2.8) follows by maximum principle. □

Remark 2.8. There is $C_1 > 0$, independent from x, ρ_1 and ρ_2, with $0 < \rho_1 < \rho_2$, such that

$$\int_{\rho_1}^{\rho_2} \frac{s^2}{w(B(x, s))} \frac{ds}{s} \geq C_1 \frac{{\rho_1}^2}{w(B(x, \rho_1))}. \tag{2.9}$$

Moreover if $K < \frac{1}{4}$, where K is the constant in Lemma 2.1b, there is $C_2 > 0$, independent from x, ρ_1 and ρ_2, such that

$$\int_{\rho_1}^{\rho_2} \frac{s^2}{w(B(x, s))} \frac{ds}{s} \leq C_2 \frac{{\rho_1}^2}{w(B(x, \rho_1))}. \tag{2.10}$$

Indeed if N be a positive nonnegative integer number such that

$$2^N \rho_1 \leq \rho_2 < 2^{N+1} \rho_1,$$

then we have

$$\int_{\rho_1}^{\rho_2} \frac{s^2}{w(B(x, s))} \frac{ds}{s} \leq \int_{\rho_1}^{2\rho_1} \frac{s^2}{w(B(x, s))} \frac{ds}{s},$$

if $N = 0$, and

$$\sum_{k=1}^{N} \int_{2^{k-1}\rho_1}^{2^k \rho_1} \frac{s^2}{w(B(x, s))} \frac{ds}{s} \leq \int_{\rho_1}^{\rho_2} \frac{s^2}{w(B(x, s))} \frac{ds}{s} \leq \sum_{k=1}^{N+1} \int_{2^{k-1}\rho_1}^{2^k \rho_1} \frac{s^2}{w(B(x, s))} \frac{ds}{s},$$

if $N > 1$.

By a) and b) of Lemma 2.1 we have the conclusion.

We now define some function spaces.

Definition 2.9 (see [8]). *Let $\eta(r)$ be a nondecreasing function defined in* $]0, +\infty[$ *such that* $\lim_{r\to 0}\eta(r) = 0$. *We set*

$$S(\Omega, w) = \{V \in L^1(\Omega, w) :$$
$$\sup_{x\in\Omega}\int_{\{y\in\Omega\,:\,|x-y|<r\}} |V(y)| \int_{|x-y|}^{4R} \frac{s^2}{w(B(x,s))}\frac{ds}{s} w(y)\,dy \le \eta(r)\}.$$

Definition 2.10 (see [11]). *Let $\sigma > 0$, $C > 0$ and $0 < r < 2R$. We set*

$$M_\sigma(\Omega, w) = \{V \in L^1(\Omega, w) :$$
$$\sup_{x\in\Omega}\int_{\{y\in\Omega\,:\,|x-y|<r\}} |V(y)| \int_{|x-y|}^{4R} \frac{s^2}{w(B(x,s))}\frac{ds}{s} w(y)\,dy \le Cr^\sigma\}.$$

Definition 2.11. *Let $\varepsilon \in \mathbb{R}$. We set*

$$L^{1,\varepsilon}(\Omega, w) = \{V \in L^1(\Omega, w) :$$
$$\|V\|_{1,\varepsilon} = \sup_{\substack{x\in\Omega\\ 0<r<2R}} \frac{r^{2-\varepsilon}}{w(B(x,r))}\int_{\{y\in\Omega\,:\,|x-y|<r\}} |V(y)|w(y)\,dy < +\infty\}.$$

Remark 2.12. We note that in the nondegenerate case, i.e. $w = 1$, $S(\Omega, w)$ is the Kato-Stummel class (see [1] and [2]), $M_\sigma(\Omega, w)$ and $L^{1,\varepsilon}(\Omega, w)$ coincide with the classical Morrey space $L^{1,\lambda}$ for some opportune λ (see [4] and [12]); in particular for $\sigma = \varepsilon > 0$ we obtain $L^{1,\lambda}$ with $\lambda = n - 2 + \varepsilon$ (see Remark 2.15).

Remark 2.13. If $2 < \varepsilon$ then $L^{1,\varepsilon}(\Omega, w) = \{0\}$, indeed

$$\frac{1}{w(B(x,r))}\int_{\{y\in\Omega\,:\,|x-y|<r\}} |V(y)|w(y)\,dy \le \|V\|_{1,\varepsilon} r^{\varepsilon-2}.$$

If $\varepsilon < 2 - 2n$ then $L^{1,\varepsilon}(\Omega, w) = L^1(\Omega, w)$, indeed by Lemma 2.1c

$$\frac{r^{2-\varepsilon}}{w(B(x,r))}\int_{\{y\in\Omega\,:\,|x-y|<r\}} |V(y)|w(y)\,dy \le$$
$$\le Cr^{2-2n-\varepsilon}\int_{\{y\in\Omega:|x-y|<r\}} |V(y)|w(y)\,dy.$$

We wish now to compare the spaces introduced above.

Proposition 2.14. *We have*

i) $M_\sigma(\Omega, w) \subseteq S(\Omega, w)$*;*

ii) $M_\sigma(\Omega, w) \subseteq L^{1,\sigma}(\Omega, w),\ \sigma > 0$*;*

iii) $L^{1,\varepsilon}(\Omega, w) \subseteq M_\varepsilon(\Omega, w),\ \varepsilon > 0,\ if\ K < \frac{1}{4}$*, where* K *is the constant in Lemma 2.1b.*

Proof. $i)$ is trivial; we prove $ii)$ and $iii)$. Let $V \in M_\sigma(\Omega, w)$, $0 < r < 2R$ using (2.9), we have

$$\int_{\{y\in\Omega:\,|x-y|<r\}} |V(y)| w(y)\, dy \leq$$

$$\leq \left(\int_r^{4R} \frac{s^2}{w(B(x,s))}\frac{ds}{s}\right)^{-1} \int_{\{y\in\Omega:\,|x-y|<r\}} |V(y)| \int_{|x-y|}^{4R} \frac{s^2}{w(B(x,s))}\frac{ds}{s} w(y)\, dy \leq$$

$$\leq C \left(\int_r^{4R} \frac{s^2}{w(B(x,s))}\frac{ds}{s}\right)^{-1} r^\sigma \leq C w(B(x,r)) r^{\sigma-2},$$

and that proves $ii)$.

Let $V \in L^{1,\varepsilon}(\Omega, w)$, $0 < r < 2R$, we have

$$\int_{\{y\in\Omega:\,|x-y|<r\}} |V(y)| \int_{|x-y|}^{4R} \frac{s^2}{w(B(x,s))}\frac{ds}{s} w(y)\, dy =$$

$$= \sum_{k=0}^{+\infty} \int_{\{y\in\Omega:\,\frac{r}{2^{k+1}} \leq |x-y| < \frac{r}{2^k}\}} |V(y)| \int_{|x-y|}^{4R} \frac{s^2}{w(B(x,s))}\frac{ds}{s} w(y)\, dy \leq$$

$$\leq \sum_{k=0}^{+\infty} \int_{\frac{r}{2^{k+1}}}^{4R} \frac{s^2}{w(B(x,s))}\frac{ds}{s} \int_{\{y\in\Omega:\,|x-y|<\frac{r}{2^k}\}} |V(y)| w(y)\, dy \leq$$

$$\leq \sum_{k=0}^{+\infty} \int_{\frac{r}{2^{k+1}}}^{4R} \frac{s^2}{w(B(x,s))}\frac{ds}{s} w(B(x, \frac{r}{2^k}))(\frac{r}{2^k})^{\varepsilon-2} \|V\|_{1,\varepsilon}.$$

By (2.10) we have

$$\sum_{k=0}^{+\infty} \int_{\frac{r}{2^{k+1}}}^{4R} \frac{s^2}{w(B(x,s))}\frac{ds}{s} w(B(x, \frac{r}{2^k}))(\frac{r}{2^k})^{\varepsilon-2} \leq C r^\varepsilon,$$

and the conclusion is obtained. □

Remark 2.15. It is trivial that, if $n > 2$, $M_\varepsilon(\Omega, 1) = L^{1,\varepsilon}(\Omega, 1) = L^{1,n-2+\varepsilon}(\Omega)$, showing that both $M_\varepsilon(\Omega, w)$ and $L^{1,\varepsilon}(\Omega, w)$ are generalizations of $L^{1,n-2+\varepsilon}(\Omega)$.

The following example shows that for some weight $w \in A_2$ there exists $\varepsilon > 0$ such that $M_\varepsilon(\Omega, w) \notin L^{1,\varepsilon}(\Omega, w)$.

Example 2.16. Let $\Omega = B(0, 1)$, $0 < \sigma < \varepsilon < 2$, $w(y) = |y|^{\sigma-n}$ and $V(y) = \chi_\Omega(y)$, where $\chi_\Omega(y)$ is the characteristic function of Ω. We claim that $V(y) \notin M_\varepsilon(\Omega, w)$ while $V(y) \in L^{1,\varepsilon}(\Omega, w)$.

We have

$$r^{-\varepsilon} \int_{|y|<r} V(y) \int_{|y|}^{4} \frac{s^2}{w(B(0, s))} \frac{ds}{s} w(y)\, dy =$$

$$= \frac{\sigma}{(2-\sigma)\omega_n} r^{-\varepsilon} \int_{|y|<r} [4^{2-\sigma} - |y|^{2-\sigma}]|y|^{\sigma-n}\, dy =$$

$$= \frac{\sigma}{2-\sigma} r^{-\varepsilon} \left(\frac{4^{2-\sigma} r^\sigma}{\sigma} - \frac{r^2}{2} \right).$$

Since the last term is unbounded in $]0, 1[$, we conclude that $V(y) \notin M_\varepsilon(\Omega, w)$.

Consider the function

$$\psi(r, x) = \frac{r^{2-\varepsilon}}{w(B(x, r))} \int_{|x-y|<r} |V(y)|w(y)\, dy =$$

$$= r^{2-\varepsilon} \frac{\int_{|x-y|<r} \chi_\Omega(y)|y|^{\sigma-n}\, dy}{\int_{|x-y|<r} |y|^{\sigma-n}\, dy}$$

with $x \in \Omega$ and $r \in]0, 2[$. It is trivial that there exists $M > 0$ such that $\psi(r, x) \le M$ for every $(x, r) \in \Omega \times]0, 2[$, that is $V(y) \in L^{1,\varepsilon}(\Omega, w)$.

3. Continuity and Hölder-continuity of local solutions.

We begin this section proving the continuity of solutions of the elliptic equation

$$Lu + Vu = 0 \tag{3.1}$$

such that (1.2) and (1.3) hold and $\frac{V}{w} \in S(\Omega, w)$.

We will say that $u \in H^{1,2}_{loc}(\Omega, w)$ is a *local weak solution* of the equation (3.1) if

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} u_{x_i} \psi_{x_j} \, dx + \int_{\Omega} V u \psi \, dx = 0 \quad \forall \psi \in C_0^{\infty}(\Omega). \tag{3.2}$$

(3.2) is meaningful by Lemma 3.3 in [8].

The continuity result is only stated in [8]. We give a proof much in the line of the analogous theorem in [2], because a slight modification of the proof will give our main result: Theorem 3.2.

Theorem 3.1. *Let u be a local weak solution of (3.1) in Ω. If $\frac{V}{w} \in S(\Omega, w)$, then there exist positive numbers α, r_0 and C, independent of u, such that for any ball $B(x_0, r)$, with $B(x_0, 16r) \subseteq \Omega$, $0 < r \leq \dfrac{r_0}{8}$ and any $x \in B(x_0, r)$ we have*

$$|u(x) - u(x_0)| \leq C \left(\sup_{B(x_0,4r)} |u| \right) \Big[|x - x_0|^{\frac{\alpha}{2}} r^{-\frac{\alpha}{2}} \eta(2r) + \tag{3.3}$$
$$+ |x - x_0|^{\alpha} r^{-\alpha} + \eta(r^{\frac{1}{2}} |x - x_0|^{\frac{1}{2}} + |x - x_0|) \Big].$$

Proof. Let u be a local weak solution of (3.1), i.e. $u \in H^{1,2}_{loc}(\Omega, w)$ such that (3.2) holds.

From Theorem 3.8 in [8] u is locally bounded in Ω. More precisely there exist two positive constant r_0 and C, independent of u such that if $r \leq \dfrac{r_0}{8}$ then

$$\sup_{B(x_0,4r)} |u| \leq C \left(\frac{1}{w(B(x_0, 8r))} \int_{B(x_0,8r)} |u(x)|^2 w(x) \, dx \right)^{\frac{1}{2}}.$$

Let $\phi \in C_0^{\infty}(\Omega)$ be such that $0 \leq \phi \leq 1$ in Ω, $\phi = 1$ in $B(x_0, \frac{3}{2}r)$, $\phi = 0$ outside $B(x_0, 2r)$, $|\nabla \phi| \leq \frac{4}{r}$, where $0 < r \leq \dfrac{r_0}{8}$.

By (3.2) we have

$$\sum_{i,j=1}^{n} \int_{B(x_0,2r)} a_{ij} (u\phi)_{x_i} \psi_{x_j} \, dx = - \int_{B(x_0,2r)} V u \psi \phi \, dx +$$
$$+ \sum_{i,j=1}^{n} \int_{B(x_0,2r)} a_{ij} u \phi_{x_i} \psi_{x_j} \, dx - \sum_{i,j=1}^{n} \int_{B(x_0,2r)} a_{ij} u_{x_i} \phi_{x_j} \psi \, dx,$$

i.e. $u\phi$ is a weak solution (hence a very weak solution) of

$$Lv = -Vu\phi - \sum_{i,j=1}^{n} a_{ij}u_{x_i}\phi_{x_j} - \sum_{i,j=1}^{n}(a_{ij}u\phi_{x_i})_{x_j}$$

in $B(x_0, 2r)$.

Hence by Theorem 2.5 we have

$$u(x)\phi(x) = -\int_{B(x_0,2r)} g_0(x, y)V(y)u(y)\phi(y)\,dy -$$
$$-\sum_{i,j=1}^{n}\int_{B(x_0,2r)} g_0(x, y)a_{ij}(y)u_{y_i}(y)\phi_{y_j}(y)\,dy -$$
$$-\sum_{i,j=1}^{n}\int_{B(x_0,2r)} g_0(x, y)[a_{ij}(y)u(y)\phi_{y_i}(y)]_{y_j}\,dy,$$

where $g_0(x, y)$ is the Green's function of L in $B(x_0, 2r)$.

Then for every $x \in B(x_0, r)$ we have

$$u(x) - u(x_0) = -\int_{B(x_0,2r)} V(y)u(y)\phi(y)[g_0(x, y) - g_0(x_0, y)]\,dy -$$
$$-\sum_{i,j=1}^{n}\int_{B(x_0,2r)}[g_0(x, y) - g_0(x_0, y)]a_{ij}(y)u_{y_i}(y)\phi_{y_j}(y)\,dy +$$
$$+\sum_{i,j=1}^{n}\int_{B(x_0,2r)}\{[g_0(x, y)]_{y_j} - [g_0(x_0, y)]_{y_j}\}a_{ij}(y)u(y)\phi_{y_i}(y)\,dy =$$
$$= -I - II + III.$$

We begin by estimating II. From (2.4), (2.5), (2.7) and (2.8) we have that if $y \in B(x_0, 2r)\backslash B(x_0, \frac{3}{2}r)$

$$|g_0(x, y) - g_0(x_0, y)| \le C|x - x_0|^{\alpha}r^{-\alpha}\int_{|x_0-y|}^{4r}\frac{s^2}{w(B(x_0, s))}\frac{ds}{s}$$

and then by Lemma 3.2 in [8] and our Lemma 2.1 - a) we obtain

$$(3.4)\quad |II| \le Cr^{-1}|x - x_0|^{\alpha}r^{-\alpha}\int_{r}^{4r}\frac{s^2}{w(B(x_0, s))}\frac{ds}{s}\int_{B(x_0,2r)}|\nabla u|w(y)\,dy \le$$

$$\leq C|x-x_0|^{\alpha} r^{-\alpha+1}\left(\frac{1}{w(B(x_0,r))}\int_{B(x_0,2r)}|\nabla u|^2 w(y)\,dy\right)^{\frac{1}{2}}\leq$$

$$\leq C|x-x_0|^{\alpha} r^{-\alpha}\sup_{B(x_0,4r)}|u|.$$

We now estimate III. From the Caccioppoli inequality we have

$$|III|\leq \frac{C}{r}\left(\int_{B(x_0,2r)}|u|^2 w(y)\,dy\right)^{\frac{1}{2}}\cdot$$

$$\cdot\left(\int_{B(x_0,2r)\setminus B(x_0,\frac{3}{2}r)}\sum_{i=1}^{n}|[g_0(x,y)]_{y_i}-[g_0(x_0,y)]_{y_i}|^2 w(y)\,dy\right)^{\frac{1}{2}}\leq$$

$$\leq \frac{C}{r^2}\left(\int_{B(x_0,2r)}|u|^2 w(y)\,dy\right)^{\frac{1}{2}}\cdot$$

$$\cdot\left(\int_{B(x_0,3r)\setminus B(x_0,\frac{5}{4}r)}|[g_0(x,y)]-[g_0(x_0,y)]|^2 w(y)\,dy\right)^{\frac{1}{2}}.$$

Proceeding as before we have

$$|III|\leq \frac{C}{r^2}\frac{|x-x_0|^{\alpha}}{r^{\alpha}}\int_r^{4r}\frac{s^2}{w(B(x_0,s))}\frac{ds}{s}\cdot \tag{3.5}$$

$$\cdot\left(\int_{B(x_0,2r)}|u|^2 w(y)\,dy\right)^{\frac{1}{2}} w^{\frac{1}{2}}(B(x_0,r))\leq C\frac{|x-x_0|^{\alpha}}{r^{\alpha}}\sup_{B(x_0,4r)}|u|.$$

Finally to bound I we note that

$$|I|\leq \int_{\{y\in B(x_0,2r)\,:\,|x_0-y|>N|x-x_0|\}}|V(y)||u(y)||\phi(y)||g_0(x,y)-g_0(x_0,y)|\,dy+$$

$$+\int_{\{y\in B(x_0,2r)\,:\,|x_0-y|\leq N|x-x_0|\}}|V(y)||u(y)||\phi(y)||g_0(x,y)-g_0(x_0,y)|\,dy$$

where $N>1$ will be determined later.

By (2.4), (2.5), (2.7) and (2.8) we have

$$|I|\leq CN^{-\alpha}\int_{\{y\in B(x_0,2r)\,:\,|x_0-y|>N|x-x_0|\}}|V(y)||u(y)||\phi(y)|\cdot$$

$$\cdot \int_{|x_0-y|}^{4r} \frac{s^2}{w(B(x_0,s))}\frac{ds}{s}\, dy +$$

$$+ C \int_{\{y\in B(x_0,2r):\, |x_0-y|\le N|x-x_0|\}} |V(y)||u(y)||\phi(y)|\Big(\int_{|x-y|}^{4r} \frac{s^2}{w(B(x,s))}\frac{ds}{s} +$$

$$+ \int_{|x_0-y|}^{4r} \frac{s^2}{w(B(x_0,s))}\frac{ds}{s}\Big)\, dy \le$$

$$\le CN^{-\alpha} \sup_{B(x_0,2r)} |u| \int_{B(x_0,2r)} |V(y)| \int_{|x_0-y|}^{4R} \frac{s^2}{w(B(x_0,s))}\frac{ds}{s}\, dy +$$

$$+ C \sup_{B(x_0,2r)} |u| \Big(\int_{\{y\in B(x_0,2r):\, |x-y|\le (N+1)|x-x_0|\}} |V(y)| \int_{|x-y|}^{4R} \frac{s^2}{w(B(x,s))}\frac{ds}{s}\, dy +$$

$$+ \int_{\{y\in B(x_0,2r):\, |x_0-y|\le N|x-x_0|\}} |V(y)| \int_{|x_0-y|}^{4R} \frac{s^2}{w(B(x_0,s))}\frac{ds}{s}\, dy\Big).$$

Choosing $N = \left(\dfrac{r}{|x-x_0|}\right)^{\frac{1}{2}}$ we obtain

$$|I| \le C \sup_{B(x_0,2r)} |u| \Big[\left(\frac{|x-x_0|}{r}\right)^{\frac{\alpha}{2}} \eta(2r) + \tag{3.6}$$

$$+\eta\left(r^{\frac{1}{2}}|x-x_0|^{\frac{1}{2}} + |x-x_0|\right) + \eta\left(r^{\frac{1}{2}}|x-x_0|^{\frac{1}{2}}\right)\Big].$$

(3.4), (3.5) and (3.6) give the desired conclusion. $\square$

By the inclusion $M_\sigma(\Omega, w) \subseteq S(\Omega, w)$ and Theorem 3.1 we obtain the following Hölder-continuity result for the local solutions of $Lu + Vu = 0$ that extends to the degenerate case the analogous result contained in [4] and [11].

Theorem 3.2. *In the same hypotheses of Theorem 3.1, assuming* $\frac{V}{w} \in M_\sigma(\Omega, w)$, *$u$ is locally Hölder-continuous in Ω.*

4. Necessary condition for hölder-continuity of solutions.

Let $V \in L^1(\Omega)$, $V \le 0$, we wish to point out that, in order that positive local weak solutions of the equation

$$Lu + Vu = 0 \tag{4.1}$$

be Hölder-continuous, it is necessary that $\frac{V}{w} \in L^{1,\varepsilon}(\Omega, w)$ with $\varepsilon > 0$.

We begin to prove the following lemma

Lemma 4.1. *Let* $V \in L^1(\Omega)$ *and let* $u \in C^{0,\alpha}(\Omega)$, $\alpha \in]0,1[$, *be a local weak solution of equation* (4.1). *Then, for any ball* $B(x_0,r)$ *such that* $B(x_0,2r) \subset\subset \Omega$, *we have*

$$\int_{B(x_0,r)} |\nabla u|^2 w\, dx \le C\left(r^\alpha \int_{B(x_0,2r)} |V|\,|u|\, dx + w(B(x_0,r)) r^{2\alpha-2}\right).$$

Proof. Let $\eta(x) \in C_0^\infty(B(x_0,2r))$ be such that $\eta(x) = 1$ in $B(x_0,r)$, $0 \le \eta(x) \le 1$, $|\nabla\eta| \le \dfrac{2}{r}$. Considering the test function

$$\phi(x) = \eta^2(x)[u(x) - u_{2r}],$$

where $u_{2r} = \dfrac{1}{|B(x_0,2r)|}\displaystyle\int_{B(x_0,2r)} u\, dx$, we have

$$\sum_{i,j=1}^n \int_{B(x_0,2r)} a_{ij} u_{x_i}[2\eta\eta_{x_j}(u-u_{2r}) + \eta^2 u_{x_j}]\, dx + \int_{B(x_0,2r)} V u \eta^2 (u-u_{2r})\, dx = 0.$$

Using the ellipticity hypothesis we obtain

$$\nu^{-1}\int_{B(x_0,2r)} |\nabla u|^2 \eta^2 w\, dx \le$$

$$\le 2\sum_{i,j=1}^n \int_{B(x_0,2r)} |a_{ij} u_{x_i} \eta\eta_{x_j}[u-u_{2r}]|\, dx + \int_{B(x_0,2r)} |V u \eta^2 [u-u_{2r}]|\, dx \le$$

$$\le 2\nu \int_{B(x_0,2r)} |\nabla u|\,|\nabla\eta|\,\eta\,|u-u_{2r}| w\, dx + \int_{B(x_0,2r)} |V|\,|u|\,\eta^2 |u-u_{2r}|\, dx \le$$

$$\le \varepsilon\nu \int_{B(x_0,2r)} |\nabla u|^2\eta^2 w\, dx + \frac{1}{\varepsilon}\nu \int_{B(x_0,2r)} |\nabla\eta|^2 |u-u_{2r}|^2 w\, dx +$$

$$+ \int_{B(x_0,2r)} |V|\,|u|\,\eta^2|u-u_{2r}|\, dx,$$

for any $\varepsilon > 0$.

Fixing $\varepsilon = \dfrac{1}{4\nu^2}$ we have

$$\int_{B(x_0,2r)} |\nabla u|^2 \eta^2 w\, dx \le$$

$$\le C\left\{ \int_{B(x_0,2r)} |\nabla\eta|^2 |u-u_{2r}|^2 w\, dx + \int_{B(x_0,2r)} |V|\,|u|\,\eta^2 |u-u_{2r}|\, dx\right\}$$

and the thesis follows because u was assumed to be in $C^{0,\alpha}(\Omega)$. □

We prove now the desired result

Theorem 4.2. *Let* $V \le 0$*,* $V \in L^1(\Omega)$ *and let* $u \in C^{0,\alpha}(\Omega)$ *be, with* $\alpha \in]0,1[$*, a local weak solution of equation* (4.1)*, such that* $l < u$*, where* l *is a positive constant. Then* $\frac{V}{w} \in L^{1,\alpha}(\Omega, w)$*.*

Proof. Let $B(x_0, r)$ be such that $B(x_0, 4r) \subset\subset \Omega$ and $\eta(x) \in C_0^\infty(\Omega)$ be such that $\eta(x) = 1$ in $B(x_0, r)$, $0 \le \eta(x) \le 1$, $|\nabla \eta| \le \dfrac{2}{r}$.

We have, using the inequality

$$0 \le 2ab \le \varepsilon r^{-\alpha} a^2 + \varepsilon^{-1} r^\alpha b^2, \quad \forall \varepsilon > 0\,,$$

$$\int_{B(x_0,r)} |V| u \, dx \le \sum_{i,j=1}^n \int_{B(x_0,2r)} a_{ij} u_{x_i} \eta_{x_j} \, dx \le$$

$$\le C(\nu, n) \left\{ \varepsilon r^{-\alpha} \int_{B(x_0,2r)} |\nabla u|^2 w \, dx + \varepsilon^{-1} r^\alpha \int_{B(x_0,2r)} |\nabla \eta|^2 w \, dx \right\}.$$

By Lemma 4.1 and Lemma 2.1 - a) we obtain

$$\int_{B(x_0,r)} |V| u \, dx \le C(\nu, n, C_d) \left\{ \varepsilon \int_{B(x_0,4r)} |V| u \, dx + \right.$$

$$\left. + \left(\varepsilon + \frac{1}{\varepsilon} \right) r^{\alpha-2} w(B(x_0, r)) \right\}.$$

Putting

$$\omega(r) = \int_{B(x_0,r)} |V| u \, dx, \ \sigma = \varepsilon C(\nu, n, C_d) \ \text{ and } \ H(\sigma) = \sigma + \frac{C^2(\nu, n, C_d)}{\sigma}$$

we have

$$\omega(r) \le \sigma \omega(4r) + H(\sigma) r^{\alpha-2} w(B(x_0, r)), \tag{4.2}$$

for every $\sigma > 0$ and $r \in]0, \rho[$, with $\rho = \dfrac{R}{4}$.

Let $r \in \left[\dfrac{\rho}{4}, \rho\right[$ be and

$$M = \sup_{[\frac{\rho}{4}, \rho[} \frac{\omega(r)}{r^{\alpha-2} w(B(x_0, r))}$$

then

$$\omega(r) \le M r^{\alpha-2} w(B(x_0, r)). \tag{4.3}$$

If $r \in \left[\frac{\rho}{4^2}, \frac{\rho}{4}\right[$, by (4.2) and (4.3) and Lemma 2.1 - a) we have

$$\begin{aligned} \omega(r) &\le \sigma\omega(4r) + H(\sigma) r^{\alpha-2} w(B(x_0, r)) \le \\ &\le \sigma M (4r)^{\alpha-2} w(B(x_0, 4r)) + H(\sigma) r^{\alpha-2} w(B(x_0, r)) \le \\ &\le M\sigma \left[2^{\alpha-2} C_d\right]^2 r^{\alpha-2} w(B(x_0, r)) + H(\sigma) r^{\alpha-2} w(B(x_0, r)). \end{aligned}$$

Fixing $\sigma = \dfrac{1}{2\left[2^{\alpha-2} C_d\right]^2}$, we obtain

$$\omega(r) \le \frac{M}{2} r^{\alpha-2} w(B(x_0, r)) + H\left(\frac{1}{2\left[2^{\alpha-2} C_d\right]^2}\right) r^{\alpha-2} w(B(x_0, r)). \tag{4.4}$$

Iterating this procedure, if $r \in \left[\frac{\rho}{4^{i+1}}, \frac{\rho}{4^i}\right]$, we have

$$\omega(r) \le \left\{\frac{M}{2^i} + H\left(\frac{1}{2\left[2^{\alpha-2} C_d\right]^2}\right) \sum_{k=0}^{i-1} \frac{1}{2^k}\right\} r^{\alpha-2} w(B(x_0, r)),$$

that is

$$\omega(r) \le C(\nu, n, \alpha, C_d) r^{\alpha-2} w(B(x_0, r))$$

and, being $0 < l < u$, the conclusion is obtained. $\square$

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