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SOME IDEALS OF OPERATORS BETWEEN SPACES OF OPERATORS

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For $A \in L(X, Y)$, $B \in L(Z, T)$ we consider the operator $h : L(Y, Z) \to L(X, T)$, $h(U) = BUA$. We prove that in some hypotheses about A and B the operator h is in some ideal of operators. As a consequence we obtain that the ideals of Dunford-Pettis dual operator and weak*-norm sequentially continuous operators are projective tensor stable, $Nc^{dual} \otimes_\pi DP^{dual} \subset Nc^{dual}$, $GP^{dual} \otimes_\pi DP^{dual} \subset GP^{dual}$.

Let $U \in L(X, Y)$ be. U is called a Dunford-Pettis operator if: $x_n \to 0$ weak, implies $U(x_n) \to 0$ in norm. U is called an unconditionally converging operator if: for each weakly unconditionally Cauchy series $\sum_{n=1}^{\infty} x_n$, i.e. $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$, for each $x^* \in X^*$, it follows that: $U(x_n) \to 0$ in norm. A sequence $(x_n)_{n\in N} \subset X$ is called limited sequence if for every $x_n^* \to 0$ weak*, implies $x_n^*(x_n) \to 0$. An operator $U \in L(X, Y)$ is called a Gelfand-Phillips operator if for each $(x_n)_{n\in N} \subset X$ limited weakly null sequence it follows: $U(x_n) \to 0$ in norm. U has weak*-norm sequentially continuous dual if: $x_n^* \to 0$ weak*, implies $U^*(x_n^*) \to 0$ in norm. We denote DP, Nc, GP the ideal of the Dunford-Pettis, unconditionally converging, Gelfand-Phillips operators.

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Proposition 1. *Let $A \in DP^{dual}(X,Y)$ (resp. $A \in Nc^{dual}(X,Y)$), $B \in DP(Z,T)$ and $h : L(Y,Z) \to L(X,T)$, $h(U) = BUA$. Then h is a Dunford-Pettis operator (resp. an unconditionally converging operator).*

Proof. Suppose first that $A \in DP^{dual}(X,Y)$. Let $(U_n)_{n\in N} \subset L(Y,Z)$ such that: $U_n \to 0$ weak. For $n \in \mathbb{N}$ let $x_n \in X$, $\|x_n\| = 1$, such that:

$$\|h(U_n)\| - \frac{1}{n} < \|h(U_n)(x_n)\| = \|(BU_nA)(x_n)\|. \tag{1}$$

Let $z^* \in Z^*$, $y^{**} \in Y^{**}$ and $y^{**} \otimes z^*$: $L(Y,Z) \to R(C)$ be the functional, $(y^{**} \otimes z^*)(S) = y^{**}(S^*(z^*))$. Since $U_n \to 0$ weak, we obtain $\langle U_n, y^{**} \otimes z^*\rangle \to 0$, i.e. $\langle z^* \circ U_n, y^{**}\rangle \to 0$, or $y^{**} \in Y^{**}$ being arbitrary, that: $z^* \circ U_n \to 0$ weak. But $A \in DP^{dual}(X,Y)$ implies $A^*(z^* \circ U_n) \to 0$ in norm or, $z^* \circ U_n \circ A \to 0$ in norm of X^*, hence $(z^* \circ U_n \circ A)(x_n) \to 0$, i.e. $\langle (U_n \circ A)(x_n), z^*\rangle \to 0$ and since $z^* \in Z^*$ is arbitrary this shows that: $(U_n \circ A)(x_n) \to 0$ weak. Now $B \in DP(Z,T)$ and hence: $B((U_n \circ A)(x_n)) \to 0$ in norm of T, i.e. $(B \circ U_n \circ A)(x_n) \to 0$ in norm of T and the relation (1) implies $\|h(U_n)\| \to 0$, i.e. h is a Dunford-Pettis operator.

Suppose now $A \in Nc^{dual}(X,Y)$. Let be now $\sum_{n=1}^{\infty} U_n$ a weakly unconditionally Cauchy series. For $n \in \mathbb{N}$ let $x_n \in X$, $\|x_n\| = 1$, such that:

$$\|h(U_n)\| - \frac{1}{n} < \|h(U_n)(x_n)\| = \|(BU_nA)(x_n)\|. \tag{2}$$

Let $z^* \in Z^*$, $y^{**} \in Y^{**}$ and $y^{**} \otimes z^*$: $L(Y,Z) \to R(C)$ be the functional, $(y^{**} \otimes z^*)(S) = y^{**}(S^*(z^*))$. Since $\sum_{n=1}^{\infty} U_n$ is a weakly unconditionally Cauchy series we obtain: $\sum_{n=1}^{\infty} |\langle U_n, y^{**} \otimes z^*\rangle| < \infty$, i.e. $\sum_{n=1}^{\infty} |\langle z^* \circ U_n, y^{**}\rangle| < \infty$. But $y^{**} \in Y^{**}$ being arbitrary this means that: $\sum_{n=1}^{\infty} z^* \circ U_n$ is a weakly unconditionally Cauchy series. But $A \in Nc^{dual}(X,Y)$ implies $A^*(z^* \circ U_n) \to 0$ in norm or, $z^* \circ U_n \circ A \to 0$ in norm of X^*, from where: $(z^* \circ U_n \circ A)(x_n) \to 0$, i.e. $\langle (U_n \circ A)(x_n), z^*\rangle \to 0$ and since $z^* \in Z^*$ is arbitrary this shows that; $(U_n \circ A)(x_n) \to 0$ weak. Now $B \in DP(Z,T)$ and hence: $B((U_n \circ A)(x_n)) \to 0$ in norm of T, i.e. $(B \circ U_n \circ A)(x_n) \to 0$ in norm of T and the relation (2) implies $\|h(U_n)\| \to 0$, i.e. h is an unconditionally converging operator.

Let us observe that the same proof is still true if $h : K(Y,Z) \to K(X,T)$, $h(U) = BUA$, where $K(X,Y)$ is the space of all compact operators from X into the Y equipped with the operatorial norm.

Corollary 2. a) *The ideal of all Dunford-Pettis dual operators is projective tensor stable, i.e.* $DP^{dual} \otimes_\pi DP^{dual} \subset DP^{dual}$.
b) $Nc^{dual} \otimes_\pi DP^{dual} \subset Nc^{dual}$.

Proof. Let $U \in L(X, X_1)$, $V \in L(Y, Y_1)$ and $U \otimes_\pi V : X \otimes_\pi Y \to X_1 \otimes_\pi Y_1$ the projective tensor product. Then: $h = (U \otimes_\pi V)^* : L(X_1, Y_1^*) \to L(X, Y^*)$ has the action $h(\psi) = V^* \circ \psi \circ U$, i.e. is the operator h from Proposition 1 corresponding to $A = U$, $B = V^*$. The corollary follows from Proposition 1.

A naturally question is: the ideal of all dual unconditionally converging operators is projective tensor stable, i.e. $Nc^{dual} \otimes_\pi Nc^{dual} \subset Nc^{dual}$? The answer is no. Take the identity operator $i : l_2 \to l_2$ which has the dual unconditionally converging, since l_2 does not contain a copy of c_0 and use the well-known result of Pelczynski, but the dual of $i \otimes_\pi i : l_2 \otimes_\pi l_2 \to l_2 \otimes_\pi l_2$ is the identity operator on $L(l_2, l_2)$ which contains copy of c_0 and hence is not unconditionally convergent. (The application $\varphi : c_0 \to L(l_2, l_2)$, $x \to \varphi(x)$ where $\varphi(x) : l_2 \to l_2$, $\varphi(x)(y) = (x_n y_n)_{n\in\mathbb{N}}$, $x = (x_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}} \in l_2$ is an isometry). The point a) extend a result of [4].

Also in [3] is proved that if X has the Schur property, then $l_1^s(X)$, the space of all unconditionally convergent series $\sum_{n=1}^{\infty} x_n$, equipped with the norm $\varepsilon((x_n)_{n\in\mathbb{N}}) = \sup_{\|x^*\|\le 1} \sum_{n=1}^{\infty} |x^*(x_n)|$, has the Schur property. Since as is well-known $l_1^s(X) = K(c_0, X)$ and the identity operator correspond to $h : K(c_0, X) \to K(c_0, X)$, $h(U) = iUI$, $i : c_0 \to c_0$, $I : X \to X$ being the identity operator, then using the remark from the Proposition 1, we obtain that h is a Dunford-Pettis operator, i.e. $l_1^s(X) = K(c_0, X)$ has the Schur property.

Proposition 3. *The ideal of all dual weak*-norm sequentially continuous operators is projective tensor stable.*

Proof. Let $A \in L(X, Y)$, $B \in L(Z, T)$ be two operators with dual weak*-norm sequentially continuous, $A \otimes_\pi B : X \otimes_\pi Z \to Y \otimes_\pi T$ the projective tensor product and $h = (U \otimes_\pi V)^* : L(Y, T^*) \to L(X, Z^*)$ the dual of $U \otimes_\pi V$, which has the action $h(\psi) = B^* \circ \psi \circ A$. Let $\psi_n \to 0$ weak*. For $n \in \mathbb{N}$, let $x_n \in X$, $z_n \in Z$, $\|x_n\| = \|z_n\| = 1$, such that:

$$\|h(\psi_n)\| - \frac{1}{n} < \|[h(\psi_n)(x_n)](z_n)\| = \|[(B^* \circ \psi_n \circ A)(x_n)](z_n)\|. \tag{3}$$

Then for each $y \in Y$ and $t \in T$, $\langle t, \psi_n(y)\rangle = \langle y \otimes t, \psi_n\rangle \to 0$, i.e. $\psi_n(y) \to 0$ weak*. Since $B^* : T^* \to Z^*$ is weak*-norm sequentially continuous we have: $B^*(\psi_n(y)) \to 0$ in norm of Z^*, and thus $[(B^* \circ \psi_n)(y)](z_n) \to 0$. Denoting for $z \in Z$ by $\hat{z} \in Z^{**}$ the canonical mapping associated to z into the bidual the above relation shows that: $(\widehat{z_n} \circ B^* \circ \psi_n)(y) \to 0$, i.e. $\widehat{z_n} \circ B^* \circ \psi_n \to 0$ weak*. Now since $A^* : Y^* \to X^*$ is weak*-norm sequentially continuous, we have: $A^*(\widehat{z_n} \circ B^* \circ \psi_n) \to 0$ in norm of X^* and hence: $[A^*(\widehat{z_n} \circ B^* \circ \psi_n)](x_n) \to 0$ in norm of X, $[\widehat{z_n} \circ B^* \circ \psi_n](Ax_n) = [(B^* \circ \psi_n \circ A)(x_n)](z_n) \to 0$, in norm of X. The relation (3) implies that: $\|h(\psi_n)\| \to 0$, i.e. h is weak*-norm sequentially continuous and the proposition is proved.

Proposition 4. a) *Let $A \in GP^{dual}(X, Y)$, $B \in GP(Z, T)$ and $h : K(Y, Z) \to K(X, T)$, $h(U) = BUA$. Then h is a Gelfand-Phillips operator.*

b) *Let $A \in GP^{dual}(X, Y)$, $B \in DP(Z, T)$ and $h : L(Y, Z) \to L(X, T)$, $h(U) = BUA$. Then h is a Gelfand-Phillips operator.*

Proof. a) Let $(U_n)_{n\in\mathbb{N}} \subset K(Y, Z)$ such that: $U_n \to 0$ weak and $(U_n)_{n\in\mathbb{N}}$ is a limited sequence. For $n \in \mathbb{N}$, let $x_n \in X$, $\|x_n\| = 1$, such that:

$$\|h(U_n)\| - \frac{1}{n} < \|h(U_n)(x_n)\| = \|(BU_nA)(x_n)\|. \tag{4}$$

Let $z^* \in Z^*$, $y^{**} \in Y^{**}$ and $y^{**} \otimes z^* : K(Y, Z) \to R(C)$ be the functional, $(y^{**}\otimes z^*)(S) = y^{**}(S^*(z^*))$. Since $U_n \to 0$ weak we obtain $\langle U_n, y^{**}\otimes z^*\rangle \to 0$, i.e. $\langle z^* \circ U_n, y^{**}\rangle \to 0$, or $y^{**} \in Y^{**}$ being arbitrary that: $z^* \circ U_n \to 0$ weak. Also $(z^* \circ U_n)_{n\in\mathbb{N}} \subset Y^*$ is a limited sequence. Indeed, if $y_n^{**} \to 0$ weak*, then $(y_n^{**} \otimes z^*)_{n\in\mathbb{N}} \subset (K(Y, Z))^*$ defined as above is clearly weak* converging to 0, and since $(U_n)_{n\in\mathbb{N}}$ is a limited sequence $\langle U, y_n^{**} \otimes z^*\rangle \to 0$, $y_n^{**}(U^*(z^*)) \to 0$. But $A \in GP^{dual}(X, Y)$ implies $A^*(z^* \circ U_n) \to 0$ in norm or, $z^* \circ U_n \circ A \to 0$ in norm of X^*, hence $(z^* \circ U_n \circ A)(x_n) \to 0$, i.e. $\langle (U_n \circ A)(x_n), z^*\rangle \to 0$ and since $z^* \in Z^*$ is arbitrary this shows that: $(U_n \circ A)(x_n) \to 0$ weak. Also $((U_n \circ A)(x_n))_{n\in\mathbb{N}} \subset Z$ is a limited sequence. Indeed, for $z_n^* \to 0$ weak*, let $\alpha_n : K(Y, Z) \to R(C)$ be the functional $\alpha_n(S) = z_n^*((S \circ A)(x_n)) = [S^*(z_n^*)](Ax_n)$. Since $S \in K(Y, Z)$, S^* is compact and hence weak*-norm sequentially continuous, i.e. $S^*(z_n^*) \to 0$ in norm and in particular, $\alpha_n(S) = [S^*(z_n^*)](Ax_n) \to 0$. Thus $\alpha_n \to 0$ weak*, hence $(U_n)_{n\in\mathbb{N}}$ being a limited sequence $\alpha_n(U_n) \to 0$, $z_n^*((U_n \circ A)(x_n)) \to 0$, i.e. $((U_n \circ A)(x_n))_{n\in\mathbb{N}} \subset Z$ is a limited sequence. Now $B \in GP(Z, T)$ and hence: $B((U_n \circ A)(x_n)) \to 0$ in norm of T, i.e. $(B \circ U_n \circ A)(x_n) \to 0$ in norm of T and the relation (4) implies: $\|h(U_n)\| \to 0$, i.e. h is a Gelfand-Phillips operator.

b) Let $(U_n)_{n\in\mathbb{N}} \subset L(Y, Z)$ such that: $U_n \to 0$ weak and $(U_n)_{n\in\mathbb{N}}$ is a limited sequence. For $n \in \mathbb{N}$, let $x_n \in X$, $\|x_n\| = 1$, such that:

$$\|h(U_n)\| - \frac{1}{n} < \|h(U_n)(x_n)\| = \|(BU_nA)(x_n)\|. \tag{5}$$

Let $z^* \in Z^*$, $y^{**} \in Y^{**}$ and $y^{**} \otimes z^*$: $L(Y, Z) \to R(C)$ be the functional, $(y^{**} \otimes z^*)(S) = y^{**}(S^*(z^*))$. Since $U_n \to 0$ weak, we obtain $\langle U_n, y^{**} \otimes z^* \rangle \to 0$, i.e. $\langle z^* \circ U_n, y^{**} \rangle \to 0$ or $y^{**} \in Y^{**}$ being arbitrary that: $z^* \circ U_n \to 0$ weak. Also $(z^* \circ U_n)_{n \in \mathbb{N}} \subset Y^*$ is a limited sequence. Indeed, if $y_n^{**} \to 0$ weak*, then $(y_n^{**} \otimes z^*)_{n \in \mathbb{N}} \subset (K(Y, Z))^*$ defined as above is clearly weak* converging to 0, and since $(U_n)_{n \in \mathbb{N}}$ is a limited sequence $\langle U, y_n^{**} \otimes z^* \rangle \to 0$, $y_n^{**}(U^*(z^*)) \to 0$. But $A \in GP^{dual}(X, Y)$ implies $A^*(z^* \circ U_n) \to 0$ in norm or, $z^* \circ U_n \circ A \to 0$ in norm of X^*, hence $(z^* \circ U_n \circ A)(x_n) \to 0$, i.e. $\langle (U_n \circ A)(x_n), z^* \rangle \to 0$ and since $z^* \in Z^*$ is arbitrary this shows that: $(U_n \circ A)(x_n) \to 0$ weak. Now $B \in DP(Z, T)$ and hence: $B((U_n \circ A)(x_n)) \to 0$ in norm of T, i.e. $(B \circ U_n \circ A)(x_n) \to 0$ in norm of T and the relation (5) implies: $\|h(U_n)\| \to 0$, i.e. h is a Gelfand-Phillips operator.

The point a) is an extension of Corollary 2.3 from [1] and the point b) is an extension of Theorem 2 from [2].

Corollary 5. $GP^{dual} \otimes_\pi DP^{dual} \subset GP^{dual}$.

Proof. Let $U \in L(X, X_1)$, $V \in L(Y, Y_1)$ and $U \otimes_\pi V : X \otimes_\pi Y \to X_1 \otimes_\pi Y_1$ the projective tensor product. Then: $h = (U \otimes_\pi V)^* : L(X_1, Y_1^*) \to L(X, Y^*)$ has the action $h(\psi) = V^* \circ \psi \circ U$, i.e. is the operator h from Proposition 4 corresponding to $A = U$, $B = V^*$. The corollary follows from Proposition 4.

The same example as above, i.e. the identity operator on $L(l_2, l_2)$, which is the dual of $i \otimes_\pi i : l_2 \otimes_\pi l_2 \to l_2 \otimes_\pi l_2$ ($i : l_2 \to l_2$ is a Gelfand-Phillips operator, l_2 is separable) shows that the dual of Gelfand-Phillips operators is not projective tensor stable, since $L(l_2, l_2)$ contains a copy of l_∞ and l_∞ is not a Gelfand-Phillips space.

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