BLOCKING SETS OF SMALL SIZE AND
COLOURINGS IN FINITE AFFINE PLANES

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Let \((S, \mathcal{L})\) be an either linear or semilinear space and \(X \subseteq S\). Starting from \(X\) we define three types of colourings of the points of \(S\). We characterize the Steiner systems \(S(2, k, v)\) which have a colouring of the first type with \(X = \{P\}\). By means of such colourings we construct blocking sets of small size in affine planes of order \(q\). In particular, from the second and third type of colourings we get blocking sets \(B\) with \(|B| \leq 2q - 2\).

1. Three different colourings in a semilinear space.

Let \(S\) be a semilinear space, that is a pair \((S, \mathcal{L})\) where \(S\) is a non-empty set of elements called points and \(\mathcal{L}\) is a family of subsets of \(S\) called lines, such that \(\mathcal{L}\) is a covering of \(S\), every line has at least two points, through two distinct points there is at most one line. Let \(X\) be a subset of \(S\), we define in \((S, \mathcal{L})\) three types of colourings of the points starting from \(X\). We denote such colourings by \(X_1, X_2, X_3\) colouring respectively.

\(X_1\) colouring of \(S\):
The points of any external line to \(X\) have different colours.

\(X_2\) colouring of \(S\):
The points of any tangent to \(X\) have different colours.

\(X_3\) colouring of \(S\):

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The points of any secant $s$ to $X$, $(|s \cap X| \geq 2)$, have different colours.

2. The $X_1$ colourings of $S$.

Let $\mathcal{L}_e(X) \neq \emptyset$ be the set of lines of $\mathcal{L}$ external to $X$. Set

$$k_e = \max_{s \in \mathcal{L}_e(X)} |s|.$$ 

Let $C_1$ be a $X_1$ colouring of $S$ and let $N_1$ be the number of colours of $C_1$. Let $\mathcal{P}_1$ be the partition of $S$ determined by $C_1$:

$$\mathcal{P}_1 = \{A_i\}_{i=1,\ldots,N_1}.$$ 

Lemma 1. For every line $s$ external to $X$ and for every $i = 1, \ldots, N_1$ we have:

$$|s \cap A_i| \leq 1.$$ 

Proof. The proof follows since the points of an external line to $X$ have different colours.

Lemma 2. The following inequality holds:

$$N_1 \geq k_e.$$ 

Proof. Let $s$ be a line of $S$ external to $X$ with $|s| = k_e$. Since the points of $s$ have different colours, the proof follows.

If $N_1 = k_e$, we call $C_1$ minimal. Obviously, if $C_1$ is minimal, every external line to $X$ of size $k_e$ is tangent to every class $A_i$ of $\mathcal{P}_1$.

A partial Steiner system is a semilinear space such that the lines have the same size [4]. Now let us consider the minimal $X_1$ colourings in a partial Steiner system $S = (S, \mathcal{L})$. Let

$$|S| = v, \quad |s| = k,$$

for any $s \in \mathcal{L}$. Let $X$ be a set of $S$ having some external line and let $C_1$ be a minimal $X_1$ colouring of $S$. We have:

$$\sum_{i=1}^{N_1} |A_i| = v.$$
Now let
\[ m_1 = \min_{i=1, \ldots, N_1} |A_i|. \]

It follows that
\[ v = \sum_{i=1}^{N_1} |A_i| \geq N_1 m_1, \quad v = N_1 m_1 \iff |A_i| = m_1, \quad i = 1, \ldots, N_1. \]

By (1), since \( C_1 \) is minimal and \( N_1 = k_e = k \), it follows that
\[ m_1 \leq v/k, \quad m_1 = v/k \iff |A_i| = m_1, \quad i = 1, \ldots, N_1. \]

Let \( A_i^* \in \mathcal{P}_1 \) with \( |A_i^*| = m_1 \). The following theorem holds:

**Theorem 3.** The set \( I = X \cup A_i^* \) of 8 is an intersection set (i.e. the set meets every line) such that
\[ |I| \leq |X| + v/k. \]

The equality holds, if and only if, \( X \cap A_i^* = \emptyset \) and \( |A_i| = m_1 \), \( i = 1, \ldots, N_1 \).

**Proof.** Let \( s \) be any line \( 8 \). If \( s \) meets \( X \), then \( s \cap (X \cup A_i^*) \neq \emptyset \). If \( s \) is external to \( X \), then \( s \) is tangent to \( A_i^* \), since \( C_1 \) is minimal and \( |s| = k \) for every \( s \in \mathcal{L} \).

It follows that \( s \cap (X \cup A_i^*) \neq \emptyset \).

We get
\[ |I| = |X \cup A_i^*| \leq |X| + |A_i^*| = |X| + m_1 \leq |X| + v/k. \]

Moreover, if
\[ |I| = |X| + v/k, \quad \text{then} \quad m_1 = v/k, \quad |X \cup A_i^*| = |X| + |A_i^*|, \]

and by (2)
\[ |A_i| = m_1, \quad i = 1, \ldots, N_1, \quad X \cap A_i^* = \emptyset. \]

Conversely, if the above conditions are satisfied, by (2) we get
\[ m_1 = v/k, \quad X \cap A_i^* = \emptyset \]

and then
\[ |I| = |X| + v/k. \]

The set \( I \) is called *intersection set related to \( X \) and \( C_1 \).* Obviously, for any \( A_i^* \) there is a unique set \( I \). By Theorem 3 we get the following
**Corollary 4.** Let $S$ be a partial Steiner system. Let $Y$ be any intersection set of $S$. Let $X$ be a subset of $S$ such that $X \cap Y = \emptyset$, $S$ has a minimal $X$-colouring $C_1$ and there is an intersection set $I = X \cup A^*_1$ related to $X$ and $C_1$ such that

$$I \cap Y = \emptyset.$$ 

Then $I$ and $Y$ are blocking sets (i.e. sets meeting every line and not containing a line) of $S$, since $I$ and $Y$ are two disjoint intersection sets. Moreover, we get:

$$|I| \leq |X| + v/k.$$ 

**Example 5.** Let $\pi$ be a Fano plane. Let $A, B, C, D, E, F, G$ be the points of $\pi$ and $\{A, G, C\}, \{A, D, E\}, \{A, F, B\}, \{C, E, B\}, \{C, D, F\}, \{F, E, G\}, \{B, D, G\}$ the lines of $\pi$. Let $S$ be the partial Steiner system whose points are those of $\pi$ and whose lines are the lines of $\pi$ not through $A$. Let $X = \{B\}$.

A minimal $X_1$ colouring $C_1$ of $S$ is the following:

$$\mathcal{P}_1 = \{\{G, C\}, \{D, E\}, \{B, F, A\}\}.$$ 

In $S$ there are two intersection sets related to $X$ and $C_1$:

$$X \cup \{G, C\}, \quad X \cup \{D, E\}.$$ 

We remark that they are both blocking sets. The blocking set $I = X \cup \{G, C\}$ is constructed from the Corollary by choosing $Y = \{D, E\}$ and $A^*_1 = \{G, C\}$. The blocking set $I' = X \cup \{D, E\}$ is constructed similarly by choosing $Y = \{G, C\}, A^*_1 = \{D, E\}$.

**Example 6.** Let $(S, \mathcal{L}) = PG(2, 3)$ and let $X$ be a 4-arc of $PG(2, 3)$, that is an irreducible conic of $PG(2, 3)$. Let $e_1, e_2, e_3$ be three external lines to $X$. Let $E_1, E_2, E_3$ be three non-collinear points such that:

$$E_1 \in e_1 - e_1 \cap e_2 - e_1 \cap e_3,$$

$$E_2 \in e_2 - e_2 \cap e_1 - e_2 \cap e_3,$$

$$E_3 \in e_3 - e_3 \cap e_1 - e_3 \cap e_2.$$ 

Let

$$X = \{F, G, H, L\},$$

$$V = e_1 \cap e_3, Z = e_1 \cap e_2, T = e_2 \cap e_3,$$

$$M = e_1 - V - Z - E_1, P = e_3 - V - T - E_3, U = e_2 - Z - T - E_2.$$
A minimal $X_1$ colouring $C_1$ of $PG(2, 3)$ is the following:

$$\mathcal{P}_1 = \{\{V, U, F, L\}, \{T, M, G\}, \{E_1, E_2, E_3\}, \{Z, P, H\}\}.$$ 

In $PG(2, 3)$ there are three intersection sets related to $X$ and $C_1$. As before they are blocking sets. They are

$$X \cup \{T, M, G\}, \quad X \cup \{E_1, E_2, E_3\}, \quad X \cup \{Z, P, H\}.$$ 

The blocking set $I = X \cup \{T, M, G\}$ is constructed by choosing $Y$ as the complement of $I$. A similar construction holds for the other two blocking sets.

**Example 7.** Let $\pi_q$ be a finite projective plane of order $q$. Let $O$ be a point of $\pi_q$ and $s_1, \ldots, s_{q+1}$ the lines through $O$. Let $X$ be a set of $\pi_q$ such that:

$$X \subset s_{q+1}, \quad O \in X, \quad 1 \leq |X| \leq q.$$ 

A minimal $X_1$ colouring $C_1$ of $\pi_q$ is the following:

$$\mathcal{P}_1 = \{s_1 \cup X, s_2 - \{O\}, \ldots, s_q - \{O\}, s_{q+1} - X\}.$$ 

If $|X| = 1$, that is $X = \{O\}$, this colouring is called *radial colouring* with centre $O$.

We remark that there are semilinear spaces $S$ where not any subset $X$ gives rise to a minimal $X_1$ colouring, as the following example shows.

**Example 8.** Let $S$ be the following linear space:

$$S = \{A, B, C, D, E, F, G, H, L, M\},$$


If $X = \{A\}$, then $S$ does not have any minimal $X_1$ colouring. If $C_1$ is a minimal $X_1$ colouring, we get $|\mathcal{P}_1| = 4$ and the points $B, E, M, H$ have different colours, since they belong to the same external line to $X$. Let $b, e, m, h$ be the colours of $B, E, M, H$ respectively. The point $C$ has none of the colours $e, m, h$, since the 2-line $CE$, the 4-line $CFML$ and the 2-line $CH$ are external.
to $X = \{A\}$. Therefore $C$ has necessarily the colour $b$. Similarly, $D$ has the colour $b$ and $F, G$ have the colour $e$. Then $L$ has the colour $h$, since the line $CFML$ is external to $X$. We get a contradiction, since the line $DGHL$, external to $X$, contains the points $L, H$ of the same colour $h$. The contradiction proves that $\S$ has not a minimal $\{A\}_1$ colouring.

The Corollary can be used to construct a blocking set $B$ of a non-desarguesian finite affine plane $\alpha_q$, with $|B| \leq t$, where $t$ is a fixed integer. In order to do this, we choose in $\alpha_q$ a blocking set $Y$ and a set $X$ such that

$$X \cap Y = \emptyset, \quad |X| + v/k = |X| + q \leq t,$$

that is such that $|X| \leq t - q$. Then we construct a minimal $X_1$ colouring of $\alpha_q$ and we consider $I = X \cup A^* = B$. If $B \cap Y = \emptyset$, then $B$ is a blocking set of $\alpha_q$ such that $|B| \leq t$. We prove the following theorem:

**Theorem 9.** Let $P$ be a point of a Steiner system $S(2, k, v)$. If $S$ has a minimal $\{P\}_1$ colouring $C_1$, then $S$ is a projective plane and $C_1$ is a radial colouring with centre $P$.

**Proof.** Let $C_1$ be a minimal $X_1$ colouring of $S$. If $r$ denotes the number of lines through any point of $S$, let $s_1, \ldots, s_v$ be the lines through $P$. If $s_{j1}, s_{j2}$ are two lines through $P$ and $P_1, P_2$ are two points such that $P_1 \in s_{j1} - \{P\}, P_2 \in s_{j2} - \{P\}$, the line $P_1, P_2$ is external to $X = \{P\}$. The $P_1$ and $P_2$ have different colours. It follows that

$$N_1 \geq r = (v - 1)/(k - 1).$$

Since

$$r \geq k,$$

we get

$$N_1 \geq (v - 1)/(k - 1) \geq k.$$

Since $C_1$ is minimal, we get

$$N_1 = k = (v - 1)/(k - 1).$$

It follows that

$$v = k^2 - k + 1,$$

that is $S$ is an $S(2, k, k^2 - k + 1)$, i.e. a projective plane of order $q = k - 1$ and then $r = k$. Since $N_1 = k = r$ and two points $x, y \in S$ with $x \neq y, x \neq P$,
$xP \neq yP$ have different colours, it follows that $C_1$ is radial with centre $P$. So
the theorem is proved.

We remark that in Theorem 9 the hypothesis that $S$ is a Steiner system is
essential. The following example proves that there are linear spaces, which are
not Steiner systems, and therefore not projective planes, having a minimal $\{P\}_1$
colouring.

**Example 10.**

$$S = \{X, Y, Z, T, U, V\},$$
$$\mathcal{L} = \{\{X, Y, T\}, \{V, X, Z\}, \{V, T\}, \{Y, Z\},$$
$$\{V, Y\}, \{T, Z\}, \{V, U\}, \{Y, U\}, \{X, U\}\}.$$

*Since the lines of $S$ have not the same size, then $S$ is not a Steiner system.*

This linear space $S$ has the following minimal $\{X\}_1$ colouring.

$$\mathcal{P}_1 = \{\{V, X, Z\}, \{Y, T\}, \{U\}\}.$$

3. **The $X_2$ colourings of $S$.**

Assume that the subset $X$ has some tangent line and let $\mathcal{L}_t(X)$ be the set
of lines of $\mathcal{L}$ tangent to $X$. Set:

$$k_t = \max_{x \in \mathcal{L}_t} |x|.$$

Let $C_2$ be a $X_2$ colouring of $S = (S, \mathcal{L})$ and $N_2$ the number of colours of $C_2$.
Let $\mathcal{P}_2$ be the partition of $S$ determined by $C_2$, that is

$$\mathcal{P}_2 = \{A_i\}_{i=1, \ldots, N_2}.$$

**Lemma 11.** *For any line s tangent to X and for any $i = 1, \ldots, N_2$ we get:*

$$|s \cap A_i| \leq 1.$$

The proof is like in Lemma 1.

**Lemma 12.** *We get*

$$N_2 \geq k_t.$$
The proof is like in Lemma 2.

If \( N_2 = k \), we call \( C_2 \) minimal. Obviously, if \( C_2 \) is minimal, then every line of size \( k \), tangent to \( X \) is tangent to every class \( A'_i, i = 1, \ldots, N_2 \), of \( \mathcal{P}_2 \). Now we consider the minimal \( X_2 \) colourings in a partial Steiner system. Let \( \mathcal{S} = (S, \mathcal{L}) \) be a partial Steiner system and let

\[ v = |S|, \quad k = |s|, \]

for any \( s \in \mathcal{L} \).

Let \( X \) be a set of \( \mathcal{S} \) having some tangent line and let \( C_2 \) be a minimal \( X_2 \) colouring of \( \mathcal{S} \). With the same notations of the previous section, we get:

\[ \sum_{i=1}^{N_2} |A'_i| = v. \]

Now let

\[ m_2 = \min_{i=1, \ldots, N_2} |A'_i|. \]

It follows that

(3) \[ v = \sum_{i=1}^{N_2} |A'_i| \geq N_2 m_2, \quad v = N_2 m_2 \iff |A'_i| = m_2, \quad i = 1, \ldots, N_2. \]

By the above equalities it follows (since \( C_2 \) is minimal and then \( N_2 = k \)):

(4) \[ m_2 \leq v/N_2 = v/k, \quad m_2 = v/k \iff |A'_i| = m_2, \quad i = 1, \ldots, N_2. \]

Now we give two examples of minimal \( X_2 \) colourings.

**Example 13.**

\[ S = \{A, B, C, D, E, F, G, H, L\}, \]

\[ \mathcal{L} = \{\{A, B, C\}, \{D, E, F\}, \{G, H, L\}, \{A, D, G\}, \{B, E, H\}, \{C, F, L\}, \{A, E, L\}, \{G, E, C\}\}. \]

Let \( X = \{A, B\} \). A minimal \( X_2 \) colouring of \( S \) is the following:

\[ \mathcal{P}_2 = \{\{A, B, C\}, \{D, E, F\}, \{G, H, L\}\}. \]

We remark that there are semilinear spaces \( \mathcal{S} \) where not any subset \( X \) gives rise to a minimal \( X_2 \) colouring, as the following example shows.
Example 14.

\[ S = (S, \mathcal{L}) = PG(2, 2), \]

\[ S = \{A, B, C, D, E, F, G\}, \]

\[ \mathcal{L} = \{(A, B, C), (C, D, E), (A, F, E), (A, G, D), (E, G, B), (C, G, F), (B, D, F)\}. \]

The set \( X = \{A, B, C\} \) has some tangent line, but does not give rise to any minimal \( X_2 \) colouring of \( PG(2, 2) \). For, let \( C_2 \) be a minimal \( X_2 \) colouring of \( PG(2, 2) \). The line \( \{A, F, E\} \) is tangent to \( X \) and let \( a, f, e \) the colours of \( A, F, E \) respectively. The point \( G \) must have one of the colours \( a, f, e \), since \( X_2 \) is minimal and the lines of \( PG(2, 2) \) have three points. The point \( G \) cannot have the colours \( a, f, e \), since the lines \( \{A, G, D\}, \{C, G, F\}, \{B, G, E\} \) are tangent to \( X \), a contradiction which proves that a minimal \( X_2 \) colouring of \( PG(2, 2) \) does not exist.

4. The \( X_3 \) colourings of \( S \).

Assume that the subset \( X \) of \( S \) has some secant line \( z \) and let \( \mathcal{L}_s(X) \) be the set of lines of \( \mathcal{L} \) secant to \( X \). Set

\[ k_s = \max_{z \in \mathcal{L}_s(X)} |z|. \]

Let \( C_3 \) be a \( X_3 \) colouring of \( S \) and let \( N_3 \) be the number of colours of \( C_3 \). Let \( \mathcal{P}_3 \) be the partition determined by \( C_3 \):

\[ \mathcal{P}_3 = \{A_i'\}_{i=1,..,N_3}. \]

Lemma 15. For any line \( z \) secant to \( X \) and for any \( i = 1, \ldots, N_3 \), we get

\[ |z \cap A_i' | \leq 1. \]

The proof is like in Lemma 1.

Lemma 16. We get

\[ N_3 \geq k_s. \]
The proof is like in Lemma 2.

If \( N_3 = k_3 \), we call \( C_3 \) minimal. Obviously, if \( C_3 \) is minimal, then every line of size \( k_3 \), secant to \( X \) is tangent to every class \( A''_i \), \( i = 1, \ldots, N_3 \) of \( \mathcal{P}_3 \).

Now we consider the minimal \( X_3 \) colourings in a partial Steiner system. Let \( S = (S, \mathcal{L}) \) be a partial Steiner system and let

\[
v = |S|, \quad k = |z|,
\]

for any \( z \in \mathcal{L} \).

Let \( X \) be a set of \( S \) having some secant line and let \( C_3 \) be a minimal \( X_3 \) colouring of \( S \). With the same notations of the previous Section we get:

\[
\sum_{i=1}^{N_3} |A''_i| = v.
\]

Now let

\[
m_3 = \min_{i=1,\ldots,N_3} |A''_i|.
\]

It follows that

\[
v = \sum_{i=1}^{N_3} |A''_i| \geq N_3 m_3, \quad v = N_3 m_3 \iff |A''_i| = m_3, \quad i = 1, \ldots, N_3.
\]

By the above equation it follows (since \( C_3 \) is minimal and then \( N_3 = k \)):

\[
m_3 \leq v/N_3 = v/k, \quad m_3 = v/k \iff |A''_i| = m_3, \quad i = 1, \ldots, N_3.
\]

**Example 17.**

\[
S = (S, \mathcal{L}) = PG(2, 2),
\]

\[
S = \{A, B, C, D, E, F, G\},
\]

\[
\mathcal{L} = \{\{A, B, C\}, \{C, D, E\}, \{A, F, E\}, \{E, G, B\}, \{F, G, C\}, \{A, G, D\}, \{F, D, B\}\}.
\]

Let \( X = \{A, C, E\} \). A minimal \( X_3 \) colouring of \( PG(2, 2) \) is the following:

\[
\mathcal{P}_3 = \{\{A, D\}, \{C, F\}, \{E, G, B\}\}.
\]

**Example 18.**

\[
S = (S, \mathcal{L}) = PG(2, 2).
\]

\( S \) and \( \mathcal{L} \) are like in Example 17.
Let \( X = \{A, E\} \). A minimal \( X_3 \) colouring of \( PG(2, 2) \) is the following:

\[
\mathcal{P}_3 = \{\{A, B, C\}, \{F, G\}, \{E, D\}\}.
\]

We remark that there are semilinear spaces \( \mathcal{L} \) where not any subset \( X \) gives rise to a minimal \( X_3 \) colouring, as the following example shows.

**Example 19.**

\( S = (S, \mathcal{L}) = PG(2, 2) \).

\( S \) and \( \mathcal{L} \) are like Example 17.

Let \( X = \{A, B, G, F\} \). Let \( C_3 \) be a minimal \( X_3 \) colouring of \( S \). The line \( \{A, B, C\} \) is secant to \( X \) and the points \( A, B, C \) have different colours \( a, b, c \), respectively. The colours \( a, b, c \) are all the colours of \( C_3 \), since \( C_3 \) is minimal. The point \( G \) cannot have the colours \( a, b, c \), since the lines \( \{A, G, D\}, \{E, G, B\}, \{F, G, C\} \) are secant to \( X \). A contradiction which proves that a minimal \( X_3 \) colouring of \( PG(2, 2) \) does not exist.

5. **The minimal skew irregular \( X_j \) colourings of a partial Steiner system and the construction of blocking sets.**

Let \( S = (S, \mathcal{L}) \) be a partial Steiner system and let \( X \) be a subset of \( S \). Let \( C_j \) be a minimal \( X_j \) colouring of \( S \), \( j = 1, 2, 3 \), and let \( m_j \) be the minimal size of the classes of \( \mathcal{P}_j \). We say that \( C_j \) is *skews* if there is a class \( A^* \in \mathcal{P}_j \) with \( |A^*| = m_j \) such that \( X \cap A^* = \emptyset \). We say that \( C_j \) is *irregular* if there are two classes of \( \mathcal{P}_j \) having different sizes. The minimal \( X_2 \) colouring of Example 13 is skew and not irregular. The minimal \( X_1 \) colouring of Examples 5, 6, 7 are skew and irregular. The minimal \( X_3 \) colouring of Example 17 is irregular and not skew. The minimal \( X_3 \) colouring of Example 18 is irregular and skew.

The minimal \( X_j \), \( j = 2, 3 \) colourings can be applied to find blocking sets in finite affine planes as follows.

Let \( \alpha_q \) be a finite affine plane of order \( q \) and let \( X \) be an intersection set of \( \alpha_q \) having some tangent line. Assume that \( \alpha_q \) has a minimal \( X_2 \) skew irregular colouring \( C_2 \) and a minimal \( X_3 \) skew irregular colouring \( C_3 \). Let \( A^*_2 \) be a class of \( \mathcal{P}_2 \) such that \( |A^*_2| = m_2 \) and \( A^*_2 \cap X = \emptyset \) and let \( A^*_3 \) be a class of \( \mathcal{P}_3 \) such that \( |A^*_3| = m_3 \) and \( A^*_3 \cap X = \emptyset \). We remark that the set \( A^* = A^*_2 \cup A^*_3 \) is a blocking set of \( \alpha_q \) and then \( X \) is a blocking set too. In fact, \( A^* \) does not contain lines, since \( A^* \cap X = \emptyset \) and \( X \) is an intersection set. Moreover, any line of \( \alpha_q \) is either tangent to \( X \) (and then it is tangent also to \( A^*_2 \), since \( C_2 \) is minimal), or it is secant to \( X \) (and then it is tangent to \( A^*_3 \), since \( C_3 \) is minimal). Therefore
every line meets $A^*_2 \cup A^*_3$. Moreover, $X$ is an intersection set not containing lines (that is a blocking set). For, if $X$ contains a line, such a line is external to $A^*$ which is a blocking set. This proves the remark. We get

$$|A^*| \leq 2q - 2.$$ 

In fact, by (4) and (6) it follows that $m_2 \leq q$, $m_3 \leq q$ and therefore $|A^*_2| \leq q$, $|A^*_3| \leq q$. The case $|A^*_2| = q$ is impossible, otherwise the other classes of $P_2$ (which have at least $q$ points) have all the same size $q$. A contradiction, since $C_2$ is irregular. Similarly we prove that $|A^*_3| = q$ is impossible too. Therefore we get:

$$|A^*_2| \leq q - 1, \quad |A^*_3| \leq q - 1.$$ 

This implies

$$|A^*| \leq |A^*_2| + |A^*_3| \leq 2q - 2.$$ 

Thus the following Theorem is proved:

**Theorem 20.** Let $X$ be an intersection set of a non-desarguesian finite affine plane $\alpha_q$. If $\alpha_q$ has both a minimal skew irregular $X_2$ colouring $C_2$ and a minimal skew irregular $X_3$ colouring $C_3$, then $\alpha_q$ contains a blocking set $B$, with $|B| \leq 2q - 2$. Such a blocking set $B$ coincides with $A^*_2 \cup A^*_3$, where $A^*_2$ is a class of $C_2$ of minimal size and such that $A^*_2 \cap X = \emptyset$ and $A^*_3$ is a class of $C_3$ of minimal size such that $A^*_3 \cap X = \emptyset$. 
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