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BLOCKING SETS OF SMALL SIZE AND COLOURINGS IN FINITE AFFINE PLANES

SANDRO RAJOLA - MARIA SCAFATI TALLINI

Let $(S, \mathcal{L})$ be an either linear or semilinear space and $X \subset S$. Starting from X we define three types of colourings of the points of S. We characterize the Steiner systems $S(2, k, v)$ which have a colouring of the first type with $X = \{P\}$. By means of such colourings we construct blocking sets of small size in affine planes of order q. In particular, from the second and third type of colourings we get blocking sets B with $|B| \leq 2q - 2$.

1. Three different colourings in a semilinear space.

Let $\mathcal{S}$ be a semilinear space, that is a pair $(S, \mathcal{L})$ where S is a non-empty set of elements called points and $\mathcal{L}$ is a family of subsets of S called lines, such that $\mathcal{L}$ is a covering of S, every line has at least two points, through two distinct points there is at most one line. Let X be a subset of S, we define in $(S, \mathcal{L})$ three types of colourings of the points starting from X. We denote such colourings by X_1, X_2, X_3 colouring respectively.

X_1 colouring of $\mathcal{S}$:
The points of any external line to X have different colours.

X_2 colouring of $\mathcal{S}$:
The points of any tangent to X have different colours.

X_3 colouring of $\mathcal{S}$:

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The points of any secant s to X, ($|s \cap X| \geq 2$), have different colours.

2. The X_1 colourings of $\mathcal{S}$.

Let $\mathcal{L}_e(X) \neq \emptyset$ be the set of lines of $\mathcal{L}$ external to X. Set

$$k_e = \max_{s \in \mathcal{L}_e(X)} |s|.$$

Let C_1 be a X_1 colouring of $\mathcal{S}$ and let N_1 be the number of colours of C_1. Let $\mathcal{P}_1$ be the partition of S determined by C_1:

$$\mathcal{P}_1 = \{A_i\}_{i=1,\ldots,N_1}.$$

Lemma 1. *For every line s external to X and for every $i = 1, \ldots, N_1$ we have:*

$$|s \cap A_i| \leq 1.$$

Proof. The proof follows since the points of an external line to X have different colours.

Lemma 2. *The following inequality holds:*

$$N_1 \geq k_e.$$

Proof. Let s be a line of $\mathcal{S}$ external to X with $|s| = k_e$. Since the points of s have different colours, the proof follows.

If $N_1 = k_e$, we call C_1 *minimal*. Obviously, if C_1 is minimal, every external line to X of size k_e is tangent to every class A_i of $\mathcal{P}_1$.
A partial Steiner system is a semilinear space such that the lines have the same size [4]. Now let us consider the minimal X_1 colourings in a partial Steiner system $\mathcal{S} = (S, \mathcal{L})$. Let

$$|S| = v, \quad |s| = k,$$

for any $s \in \mathcal{L}$. Let X be a set of S having some external line and let C_1 be a minimal X_1 colouring of $\mathcal{S}$. We have:

$$\sum_{i=1}^{N_1} |A_i| = v.$$

Now let

$$m_1 = \min_{i=1,\dots,N_1} |A_i|.$$

It follows that

$$(1) \quad v = \sum_{i=1}^{N_1} |A_i| \geq N_1 m_1\,, \quad v = N_1 m_1 \Longleftrightarrow |A_i| = m_1\,, \quad i = 1, \dots, N_1.$$

By (1), since C_1 is minimal and $N_1 = k_e = k$, it follows that

$$(2) \qquad m_1 \leq v/k\,, \quad m_1 = v/k \Longleftrightarrow |A_i| = m_1\,, \quad i = 1, \dots, N_1\,.$$

Let $A_i^* \in \mathcal{P}_1$ with $|A_i^*| = m_1$. The following theorem holds:

Theorem 3. *The set $I = X \cup A_i^*$ of $\mathcal{S}$ is an intersection set (i.e. the set meets every line) such that*

$$|I| \leq |X| + v/k\,.$$

The equality holds, if and only if, $X \cap A_i^* = \emptyset$ and $|A_i| = m_1$, $i = 1, \dots, N_1$.

Proof. Let s be any line $\mathcal{S}$. If s meets X, then $s \cap (X \cup A_i^*) \neq \emptyset$. If s is external to X, then s is tangent to A_i^*, since C_1 is minimal and $|s| = k$ for every $s \in \mathcal{L}$. It follows that $s \cap (X \cup A_i^*) \neq \emptyset$.
We get

$$|I| = |X \cup A_i^*| \leq |X| + |A_i^*| = |X| + m_1 \leq |X| + v/k.$$

Moreover, if

$$|I| = |X| + v/k, \quad \text{then} \quad m_1 = v/k,$$
$$|X \cup A_i^*| = |X| + |A_i^*|,$$

and by (2)

$$|A_i| = m_1\,, \quad i = 1, \dots, N_1\,, \quad X \cap A_i^* = \emptyset.$$

Conversely, if the above conditions are satisfied, by (2) we get

$$m_1 = v/k\,, \quad X \cap A_i^* = \emptyset$$

and then

$$|I| = |X| + v/k.$$

The set I is called *intersection set related to X and C_1*. Obviously, for any A_i^* there is a unique set I. By Theorem 3 we get the following

Corollary 4. *Let $\mathcal{S}$ be a partial Steiner system. Let Y be any intersection set of $\mathcal{S}$. Let X be a subset of $\mathcal{S}$ such that $X \cap Y = \emptyset$, $\mathcal{S}$ has a minimal X_1 colouring C_1 and there is an intersection set $I = X \cup A_i^*$ related to X and C_1 such that*

$$I \cap Y = \emptyset.$$

Then I and Y are blocking sets (i.e. sets meeting every line and not containing a line) of $\mathcal{S}$, since I and Y are two disjoint intersection sets. Moreover, we get:

$$|I| \leq |X| + v/k.$$

Example 5. *Let π be a Fano plane. Let A, B, C, D, E, F, G be the points of π and $\{A, G, C\}$, $\{A, D, E\}$, $\{A, F, B\}$, $\{C, E, B\}$, $\{C, D, F\}$, $\{F, E, G\}$, $\{B, D, G\}$ the lines of π. Let $\mathcal{S}$ be the partial Steiner system whose points are those of π and whose lines are the lines of π not through A. Let $X = \{B\}$.*

A minimal X_1 colouring C_1 of $\mathcal{S}$ is the following:

$$\mathcal{P}_1 = \{\{G, C\}, \{D, E\}, \{B, F, A\}\}.$$

In $\mathcal{S}$ there are two intersection sets related to X and C_1:

$$X \cup \{G, C\}, \quad X \cup \{D, E\}.$$

We remark that they are both blocking sets. The blocking set $I = X \cup \{G, C\}$ is constructed from the Corollary by choosing $Y = \{D, E\}$ and $A_i^* = \{G, C\}$. The blocking set $I' = X \cup \{D, E\}$ is constructed similarly by choosing $Y = \{G, C\}$, $A_i^* = \{D, E\}$.

Example 6. *Let $(S, \mathcal{L}) = PG(2, 3)$ and let X be a 4-arc of $PG(2, 3)$, that is an irreducible conic of $PG(2, 3)$. Let e_1, e_2, e_3 be three external lines to X. Let E_1, E_2, E_3 be three non-collinear points such that:*

$$E_1 \in e_1 - e_1 \cap e_2 - e_1 \cap e_3,$$

$$E_2 \in e_2 - e_2 \cap e_1 - e_2 \cap e_3,$$

$$E_3 \in e_3 - e_3 \cap e_1 - e_3 \cap e_2.$$

Let

$$X = \{F, G, H, L\},$$

$$V = e_1 \cap e_3, Z = e_1 \cap e_2, T = e_2 \cap e_3,$$

$$M = e_1 - V - Z - E_1, P = e_3 - V - T - E_3, U = e_2 - Z - T - E_2.$$

A minimal X_1 colouring C_1 of $PG(2, 3)$ is the following:

$$\mathcal{P}_1 = \{\{V, U, F, L\}, \{T, M, G\}, \{E_1, E_2, E_3\}, \{Z, P, H\}\}.$$

In $PG(2, 3)$ there are three intersection sets related to X and C_1. As before they are blocking sets. They are

$$X \cup \{T, M, G\}, \quad X \cup \{E_1, E_2, E_3\}, \quad X \cup \{Z, P, H\}.$$

The blocking set $I = X \cup \{T, M, G\}$ is constructed by choosing Y as the complement of I. A similar construction holds for the other two blocking sets.

Example 7. *Let π_q be a finite projective plane of order q. Let O be a point of π_q and $s_1, \dots, s_{q+1}$ the lines through O. Let X be a set of π_q such that:*

$$X \subset s_{q+1}, \quad O \in X, \quad 1 \leq |X| \leq q.$$

A minimal X_1 colouring C_1 of π_q is the following:

$$\mathcal{P}_1 = \{s_1 \cup X, s_2 - \{O\}, \dots, s_q - \{O\}, s_{q+1} - X\}.$$

If $|X| = 1$, that is $X = \{O\}$, this colouring is called *radial colouring* with centre O.
We remark that there are semilinear spaces $\mathcal{S}$ where not any subset X gives rise to a minimal X_1 colouring, as the following example shows.

Example 8. *Let $\mathcal{S}$ be the following linear space:*

$$S = \{A, B, C, D, E, F, G, H, L, M\},$$
$$\mathcal{L} = \{\{A, B, C, D\}, \{A, E, F, G\}, \{D, G, H, L\}, \{C, F, M, L\}, \{B, E, M, H\},$$
$$\{A, M\}, \{A, L\}, \{B, F\}, \{B, G\}, \{B, L\},$$
$$\{A, H\}, \{C, E\}, \{C, G\}, \{C, H\}, \{D, E\},$$
$$\{D, F\}, \{D, M\}, \{E, L\}, \{F, H\}, \{G, M\}\}.$$

If $X = \{A\}$, then $\mathcal{S}$ does not have any minimal X_1 colouring. If C_1 is a minimal X_1 colouring, we get $|\mathcal{P}_1| = 4$ and the points B, E, M, H have different colours, since they belong to the same external line to X. Let b, e, m, h be the colours of B, E, M, H respectively. The point C has none of the colours e, m, h, since the 2-line CE, the 4-line $CFML$ and the 2-line CH are external to $X = \{A\}$. Therefore C has necessarily the colour b. Similarly, D has the colour b and F, G have the colour e. Then L has the colour h, since the line $CFML$ is external to X. We get a contradiction, since the line $DGHL$, external to X, contains the points L, H of the same colour h. The contradiction proves that $\mathcal{S}$ has not a minimal $\{A\}_1$ colouring.

The Corollary can be used to construct a blocking set B of a non-desarguesian finite affine plane α_q, with $|B| \leq t$, where t is a fixed integer. In order to do this, we choose in α_q a blocking set Y and a set X such that

$$X \cap Y = \emptyset, \quad |X| + v/k = |X| + q \leq t,$$

that is such that $|X| \leq t - q$. Then we construct a minimal X_1 colouring of α_q and we consider $I = X \cup A_i^* = B$. If $B \cap Y = \emptyset$, then B is a blocking set of α_q such that $|B| \leq t$. We prove the following theorem:

Theorem 9. *Let P be a point of a Steiner system $S(2, k, v)$. If S has a minimal $\{P\}_1$ colouring C_1, then S is a projective plane and C_1 is a radial colouring with centre P.*

Proof. Let C_1 be a minimal X_1 colouring of S. If r denotes the number of lines through any point of S, let $s_1, \dots, s_r$ be the lines through P. If s_{j1}, s_{j2} are two lines through P and P_1, P_2 are two points such that $P_1 \in s_{j1} - \{P\}$, $P_2 \in s_{j2} - \{P\}$, the line P_1, P_2 is external to $X = \{P\}$. The P_1 and P_2 have different colours. It follows that

$$N_1 \geq r = (v-1)/(k-1).$$

Since

$$r \geq k,$$

we get

$$N_1 \geq (v-1)/(k-1) \geq k.$$

Since C_1 is minimal, we get

$$N_1 = k = (v-1)/(k-1).$$

It follows that

$$v = k^2 - k + 1,$$

that is S is an $S(2, k, k^2 - k + 1)$, i.e. a projective plane of order $q = k - 1$ and then $r = k$. Since $N_1 = k = r$ and two points $x, y \in S$ with $x \neq y$, $x \neq P$,

$xP \neq yP$ have different colours, it follows that C_1 is radial with centre P. So the theorem is proved.

We remark that in Theorem 9 the hypothesis that S is a Steiner system is essential. The following example proves that there are linear spaces, which are not Steiner systems, and therefore not projective planes, having a minimal $\{P\}_1$ colouring.

Example 10.

$$S = \{X, Y, Z, T, U, V\},$$

$$\mathcal{L} = \{\{X, Y, T\}, \{V, X, Z\}, \{V, T\}, \{Y, Z\},$$

$$\{V, Y\}, \{T, Z\}, \{V, U\}, \{Y, U\}, \{X, U\}\}.$$

Since the lines of S have not the same size, then S is not a Steiner system.

This linear space $\mathcal{S}$ has the following minimal $\{X\}_1$ colouring.

$$\mathcal{P}_1 = \{\{V, X, Z\}, \{Y, T\}, \{U\}\}.$$

3. The X_2 colourings of $\mathcal{S}$.

Assume that the subset X has some tangent line and let $\mathcal{L}_t(X)$ be the set of lines of $\mathcal{L}$ tangent to X. Set:

$$k_t = \max_{s \in \mathcal{L}_t} |s|.$$

Let C_2 be a X_2 colouring of $\mathcal{S} = (S, \mathcal{L})$ and N_2 the number of colours of C_2. Let $\mathcal{P}_2$ be the partition of S determined by C_2, that is

$$\mathcal{P}_2 = \{A'_i\}_{i=1,\ldots,N_2}.$$

Lemma 11. *For any line s tangent to X and for any $i = 1, \ldots, N_2$ we get:*

$$|s \cap A'_i| \leq 1.$$

The proof is like in Lemma 1.

Lemma 12. *We get*

$$N_2 \geq k_t.$$

The proof is like in Lemma 2.

If $N_2 = k_t$, we call C_2 *minimal*. Obviously, if C_2 is minimal, then every line of size k_t tangent to X is tangent to every class A_i', $i = 1, \dots, N_2$, of $\mathcal{P}_2$.
Now we consider the minimal X_2 colourings in a partial Steiner system.
Let $\mathcal{S} = (S, \mathcal{L})$ be a partial Steiner system and let

$$v = |S|, \quad k = |s|,$$

for any $s \in \mathcal{L}$.
Let X be a set of $\mathcal{S}$ having some tangent line and let C_2 be a minimal X_2 colouring of $\mathcal{S}$. With the same notations of the previous section, we get:

$$\sum_{i=1}^{N_2} |A_i'| = v.$$

Now let

$$m_2 = \min_{i=1,\dots,N_2} |A_i'|.$$

It follows that

$$\text{(3)} \qquad v = \sum_{i=1}^{N_2} |A_i'| \geq N_2 m_2, \ v = N_2 m_2 \Longleftrightarrow |A_i'| = m_2, \ i = 1, \dots, N_2 .$$

By the above equalities it follows (since C_2 is minimal and then $N_2 = k$):

$$\text{(4)} \qquad m_2 \leq v/N_2 = v/k, \quad m_2 = v/k \Longleftrightarrow |A_i'| = m_2, \ i = 1, \dots, N_2.$$

Now we give two examples of minimal X_2 colourings.

Example 13.

$$S = \{A, B, C, D, E, F, G, H, L\},$$
$$\mathcal{L} = \{\{A, B, C\}, \{D, E, F\}, \{G, H, L\}, \{A, D, G\},$$
$$\{B, E, H\}, \{C, F, L\}, \{A, E, L\}, \{G, E, C\}\}.$$

Let $X = \{A, B\}$. A minimal X_2 colouring of S is the following:

$$\mathcal{P}_2 = \{\{A, B, C\}, \{D, E, F\}, \{G, H, L\}\}.$$

We remark that there are semilinear spaces $\mathcal{S}$ where not any subset X gives rise to a minimal X_2 colouring, as the following example shows.

Example 14.

$$\mathcal{S} = (S, \mathcal{L}) = PG(2, 2),$$
$$S = \{A, B, C, D, E, F, G\},$$
$$\mathcal{L} = \{\{A, B, C\}, \{C, D, E\}, \{A, F, E\},$$
$$\{A, G, D\}, \{E, G, B\}, \{C, G, F\}, \{B, D, F\}\}.$$

The set $X = \{A, B, C\}$ has some tangent line, but does not give rise to any minimal X_2 colouring of $PG(2, 2)$. For, let C_2 be a minimal X_2 colouring of $PG(2, 2)$. The line $\{A, F, E\}$ is tangent to X and let a, f, e the colours of A, F, E respectively. The point G must have one of the colours a, f, e, since X_2 is minimal and the lines of $PG(2, 2)$ have three points. The point G cannot have the colours a, f, e, since the lines $\{A, G, D\}$, $\{C, G, F\}$, $\{B, G, E\}$ are tangent to X, a contradiction which proves that a minimal X_2 colouring of $PG(2, 2)$ does not exist.

4. The X_3 colourings of $\mathcal{S}$.

Assume that the subset X of $\mathcal{S}$ has some secant line z and let $\mathcal{L}_s(X)$ be the set of lines of $\mathcal{L}$ secant to X. Set

$$k_s = \max_{z \in \mathcal{L}_s(X)} |z|.$$

Let C_3 be a X_3 colouring of $\mathcal{S}$ and let N_3 be the number of colours of C_3. Let $\mathcal{P}_3$ be the partition determined by C_3:

$$\mathcal{P}_3 = \{A_i''\}_{i=1,\dots,N_3}.$$

Lemma 15. *For any line z secant to X and for any $i = 1, \dots, N_3$, we get*

$$|z \cap A_i''| \leq 1.$$

The proof is like in Lemma 1.

Lemma 16. *We get*

$$N_3 \geq k_s.$$

The proof is like in Lemma 2.

If $N_3 = k_s$, we call C_3 minimal. Obviously, if C_3 is minimal, then every line of size k_s secant to X is tangent to every class A''_i, $i = 1, \dots, N_3$ of $\mathcal{P}_3$. Now we consider the minimal X_3 colourings in a partial Steiner system. Let $\mathcal{S} = (S, \mathcal{L})$ be a partial Steiner system and let

$$v = |S|\,, \quad k = |z|,$$

for any $z \in \mathcal{L}$.
Let X be a set of $\mathcal{S}$ having some secant line and let C_3 be a minimal X_3 colouring of $\mathcal{S}$. With the same notations of the previous Section we get:

$$\sum_{i=1}^{N_3} |A''_i| = v.$$

Now let

$$m_3 = \min_{i=1,\dots,N_3} |A''_i|.$$

It follows that

$$(5) \qquad v = \sum_{i=1}^{N_3} |A''_i| \geq N_3 m_3,\ v = N_3 m_3 \iff |A''_i| = m_3,\ i = 1, \dots, N_3.$$

By the above equation it follows (since C_3 is minimal and then $N_3 = k$):

$$(6) \qquad m_3 \leq v/N_3 = v/k\,, \quad m_3 = v/k \iff |A''_i| = m_3\,,\ i = 1, \dots, N_3.$$

Example 17.

$$\mathcal{S} = (S, \mathcal{L}) = PG(2, 2),$$
$$S = \{A, B, C, D, E, F, G\},$$
$$\mathcal{L} = \{\{A, B, C\}, \{C, D, E\}, \{A, F, E\}, \{E, G, B\},$$
$$\{F, G, C\}, \{A, G, D\}, \{F, D, B\}\}.$$

Let $X = \{A, C, E\}$. A minimal X_3 colouring of $PG(2, 2)$ is the following:

$$\mathcal{P}_3 = \{\{A, D\}, \{C, F\}, \{E, G, B\}\}.$$

Example 18.

$$\mathcal{S} = (S, \mathcal{L}) = PG(2, 2).$$

S and $\mathcal{L}$ are like in Example 17.

Let $X = \{A, E\}$. A minimal X_3 colouring of $PG(2, 2)$ is the following:

$$\mathcal{P}_3 = \{\{A, B, C\}, \{F, G\}, \{E, D\}\}.$$

We remark that there are semilinear spaces $\mathcal{L}$ where not any subset X gives rise to a minimal X_3 colouring, as the following example shows.

Example 19.

$$\mathcal{S} = (S, \mathcal{L}) = PG(2, 2).$$

S and $\mathcal{L}$ are like Example 17.

Let $X = \{A, B, G, F\}$. Let C_3 be a minimal X_3 colouring of $\mathcal{S}$. The line $\{A, B, C\}$ is secant to X and the points A, B, C have different colours a, b, c, respectively. The colours a, b, c are all the colours of C_3, since C_3 is minimal. The point G cannot have the colours a, b, c, since the lines $\{A, G, D\}$, $\{E, G, B\}$, $\{F, G, C\}$ are secant to X. A contradiction which proves that a minimal X_3 colouring of $PG(2, 2)$ does not exist.

5. The minimal skew irregular X_j colourings of a partial Steiner system and the construction of blocking sets.

Let $S = (S, \mathcal{L})$ be a partial Steiner system and let X be a subset of S. Let C_j be a minimal X_j colouring of $\mathcal{S}$, $j = 1, 2, 3$, and let m_j be the minimal size of the classes of $\mathcal{P}_j$. We say that C_j is *skews* if there is a class $A^* \in \mathcal{P}_j$ with $|A^*| = m_j$, such that $X \cap A^* = \emptyset$. We say that C_j is *irregular* if there are two classes of $\mathcal{P}_j$ having different sizes. The minimal X_2 colouring of Example 13 is skew and not irregular. The minimal X_1 colouring of Examples 5, 6, 7 are skew and irregular. The minimal X_3 colouring of Example 17 is irregular and not skew. The minimal X_3 colouring of Example 18 is irregular and skew.

The minimal X_j, $j = 2, 3$ colourings can be applied to find blocking sets in finite affine planes as follows.

Let α_q be a finite affine plane of order q and let X be an intersection set of α_q having some tangent line. Assume that α_q has a minimal X_2 skew irregular colouring C_2 and a minimal X_3 skew irregular colouring C_3. Let A_2^* be a class of $\mathcal{P}_2$ such that $|A_2^*| = m_2$ and $A_2^* \cap X = \emptyset$ and let A_3^* be a class of $\mathcal{P}_3$ such that $|A_3^*| = m_3$ and $A_3^* \cap X = \emptyset$. We remark that the set $A^* = A_2^* \cup A_3^*$ is a blocking set of α_q and then X is a blocking set too. In fact, A^* does not contain lines, since $A^* \cap X = \emptyset$ and X is an intersection set. Moreover, any line of α_q is either tangent to X (and then it is tangent also to A_2^*, since C_2 is minimal), or it is secant to X (and then it is tangent to A_3^*, since C_3 is minimal). Therefore every line meets $A_2^* \cup A_3^*$. Moreover, X is an intersection set not containing lines (that is a blocking set). For, if X contains a line, such a line is external to A^* which is a blocking set. This proves the remark. We get

$$|A^*| \leq 2q - 2.$$

In fact, by (4) and (6) it follows that $m_2 \leq q$, $m_3 \leq q$ and therefore $|A_2^*| \leq q$, $|A_3^*| \leq q$. The case $|A_2^*| = q$ is impossible, otherwise the other classes of $\mathcal{P}_2$ (which have at least q points) have all the same size q. A contradiction, since C_2 is irregular. Similarly we prove that $|A_3^*| = q$ is impossible too. Therefore we get:

$$|A_2^*| \leq q - 1, \quad |A_3^*| \leq q - 1.$$

This implies

$$|A^*| \leq |A_2^*| + |A_3^*| \leq 2q - 2.$$

Thus the following Theorem is proved:

Theorem 20. *Let X be an intersection set of a non-desarguesian finite affine plane α_q. If α_q has both a minimal skew irregular X_2 colouring C_2 and a minimal skew irregular X_3 colouring C_3, then α_q contains a blocking set B, with $|B| \leq 2q - 2$. Such a blocking set B coincides with $A_2^* \cup A_3^*$, where A_2^* is a class of C_2 of minimal size and such that $A_2^* \cap X = \emptyset$ and A_3^* is a class of C_3 of minimal size such that $A_3^* \cap X = \emptyset$.*

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Dipartimento di Matematica,
Università di Roma "La Sapienza",
Piazzale A. Moro 2,
00185 Roma (ITALY)