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A VERY AMPLENESS RESULT

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Let (M, L) be a polarized manifold. The aim of this paper is to establish a connection between the generators of the graded algebra $\bigoplus_{i\geq 1} H^0(M, iL)$ and the very ampleness of the line bundle rL. Some applications are given.

1. Introduction.

Let L be an ample line bundle on an algebraic manifold M, the problem of finding the least n such that nL is very ample is a basic one in the classification theory of polarized varieties. Many attempts were made in order to establish a general formula. A fundamental result due to Matsusaka [9] says that there is a constant c, depending only on the Hilbert polynomial of (M, L), such that cL is very ample. Moving in another direction Fujita conjectured ([5], § 2, Conjecture b) that $K_M + (n+2)L$ is very ample and $K_M + (n+1)L$ is spanned for every polarized manifold (M, L) with $n = \dim M$. In this paper we consider the graded algebra:

$$G(M, L) = \bigoplus_{i\geq 1} H^0(M, iL).$$

Now the generators of this algebra describe the whole structure of (M, L). They also allows us to determine the embedding of M via the linear system $|rL|$ for $r \gg 0$, so there must be a connection between these generators and the very ampleness of rL. The aim of this paper is pointing out this connection. The

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paper is organized as follows: in Section 2 we prove the main theorem asserting the very ampleness of rL, outside the base locus of L, for L a r-generated line bundle. This also implies the very ampleness of $2L$ for any ample 2-generated line bundle 2.7. In Section 3 some applications of the main Theorem 2.2 are given, concerning: a) sectionally hyperelliptic polarized varieties of type $(-)$, b) surfaces of general type, c) del Pezzo surfaces.

2. The main theorem.

All notation used in this paper are standard in algebraic geometry. Let L be a line bundle on a projective manifold M and consider the associated graded algebra $G(M, L)$. The following definition generalizes the notion of simply generated line bundle.

Definition 2.1. *Let $r \geq 1$ be an integer. A line bundle L on M is r-generated if $G(M, L)$ is generated by the sections of $H^0(M, L), \dots, H^0(M, rL)$. We also say that the pair (M, L) is r-generated.*

There is a connection between the r-generation of L and the very ampleness of the line bundle rL; this connection, extending a known fact holding for simply generated line bundles, is expressed by the following

Theorem 2.2. *Given a polarized manifold (M, L) with L effective and r-generated, then $\varphi_{|rL|}$ is an embedding of $M \setminus Bs\,|L|$.*

Proof. Set $N = M \setminus \mathrm{Bs}\,|L|$ and $\varphi = \varphi_{|rL|}$. First of all we observe that φ is well defined in N, since L is spanned on N and so are its multiples. By hypothesis there is an integer $k \geq 1$ such that kL is very ample. Take $p, q \in N$ and by contradiction suppose that no section of $H^0(iL)$ with $1 \leq i \leq r$ separates p and q. Then each section $s \in H^0(M, iL)$ that vanish at p must vanish at q too. Consider two sections $s_1, s_2 \in H^0(iL)$, and define γ, λ so that $s_1(p) = \gamma s_1(q)$ e $s_2(p) = \lambda s_2(q)$. If we take $s = s_1(p)s_2 - s_2(p)s_1$ we have that $s(p) = 0$ and necessarily $0 = s(q) = s_1(q)s_2(q)(\gamma - \lambda)$. This implies that $\gamma = \lambda$. We note that if one of the two sections vanishes at q it must vanish also at p, otherwise it will separate p and q, but in this situation we may equally take $\gamma = \lambda$.
The previous argument shows that there exist some constants γ_i associate to each $H^0(iL)$ for $1 \leq i \leq r$ such that for each section $s \in H^0(iL)$ we have $s(p) = \gamma_i s(q)$.
Now note that since $G(M, L)$ is an algebra, the constants γ_i have to satisfy the relations:

$$\gamma_i = \gamma^i \quad \text{where} \quad \gamma := \gamma_1.$$

For, since L is effective there exists $\sigma \in H^0(L)$, then $\sigma^i(p) = \gamma^i \sigma^i(q)$, but $\sigma^i \in H^0(iL)$ so we have also $\sigma^i(p) = \gamma_i \sigma^i(q)$ and this implies the previous equation.

Now let us consider a section $s \in H^0(kL)$. By the r-generation hypothesis the sections of $H^0(M, kL)$ are linear combinations of products of sections in $H^0(iL)$ with $1 \leq i \leq r$. Then we can write $s = \sum \xi_j s_j$ where $\xi_j \in \mathbb{C}$ and $s_j = \prod_{i=1}^r \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}$ with $\sigma_{n_i} \in H^0(iL)$, where α_i is the number of sections of $H^0(iL)$ that appears in the product. Note that $\sum_{i=0}^r i\alpha_i = k$, because $s_j \in H^0(kL)$. We have

$$s(p) = \sum \xi_j s_j(p) = \sum \xi_j (\prod_{i=1}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(p)) = \sum \xi_j (\prod_{i=1}^{r} \prod_{n_i=1}^{\alpha_i} \gamma^i \sigma_{n_i}(q)) =$$

$$= \sum \xi_j (\prod_{i=1}^{r} (\gamma^{i\alpha_i} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(q))) = \sum \xi_j \gamma^{(\sum_{i=1}^{r} i\alpha_i)} (\prod_{i=1}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(q)) =$$

$$= \sum \xi_j \gamma^k (\prod_{i=1}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(q)) = \gamma^k \sum \xi_j s_j(q) = \gamma^k s(q).$$

We thus obtained that the sections of $H^0(kL)$ do not separate points, which is absurd. Then there exists a section $\sigma \in H^0(iL)$ which separates p and q, i.e. $\sigma(p) = 0$ and $\sigma(q) \neq 0$. Since L is spanned outside Bs $|L|$ we may take a section $\delta \in H^0(L)$ such that $\delta(q) \neq 0$. Now the section $\sigma \delta^{r-i} H^0(rL)$ separates p and q.

In order to show that the map φ is an immersion at each point $p \in N$ we have to show that for each vector $\tau \in T_p M$, there exists a section $\sigma \in H^0(iL)$ such that $d\sigma(\tau) \neq 0$ and $\sigma(p) = 0$. By contradiction suppose that such a section does not exist, then for each $1 \leq i \leq r$ there exists a constant η_i such that for each $s \in H^0(iL)$, $ds(\tau) = \eta_i s(p)$. To show this let $s_1, s_2 \in H^0(iL)$ and consider α, β such that $ds_1(\tau) = \alpha s_1(p)$ and $ds_2(\tau) = \beta s_2(p)$. Now consider the section $s = s_1(p)s_2 - s_2(p)s_1$; we have $s(p) = 0$ and $0 = ds(\tau) = s_1(p)s_2(p)(\beta - \alpha)$. This implies $\alpha = \beta$.

Define $\eta := \eta_1$. Take $\sigma^i \in H^0(iL)$ then $\eta_i \sigma^i(p) = d[\sigma^i](\tau) = i d\sigma(\tau)\sigma^{i-1}(p) = i\eta\sigma^i(p)$. Hence we have:

$$\eta_i = i\eta.$$

Now for a section $s \in H^0(kL)$ we have:

$$ds(\tau) = \sum \xi_j ds_j(\tau) = \sum \xi_j d[\prod_{i=1}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}](\tau) =$$

$$= \sum \xi_j (\sum_{h=1}^{r} d[\prod_{n_h=1}^{\alpha_h} \sigma_{n_h}](\tau) \prod_{i=1, i\neq h}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(p)) =$$

$$= \sum \xi_j (\sum_{h=1}^{r} (\sum_{m=0}^{\alpha_h} d\sigma_m(\tau) \prod_{n_h=1, n_h\neq m}^{\alpha_h} \sigma_{n_h}(p)) \prod_{i=1, i\neq h}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(p)) =$$

$$= \sum \xi_j (\sum_{h=1}^{r} (\sum_{m=0}^{\alpha_h} h\eta_1 \prod_{n_h=1}^{\alpha_h} \sigma_{n_h}(p)) \prod_{i=1, i\neq h}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(p)) =$$

$$= \sum \xi_j (\sum_{h=1}^{r} h\alpha_h \eta_1) \prod_{i=1}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(p) = \sum \xi_j k\eta_1 \prod_{i=1}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(p) = \eta_k s(p).$$

This implies that the sections of $H^0(kL)$ do not separate p from the vector τ, but this contradicts the very ampleness of kL. □

Theorem 2.2 immediately gives the following

Corollary 2.3. *Let (M, L) a polarized manifold with L spanned and r-generated, then tL is very ample.*

Note that Theorem 2.2 cannot be inverted. The following examples show very ample line bundles L which are not 1-generated.

Example 2.4. Let (S, L) be an abelian surface polarized by a very ample line bundle. Let $L^2 = 2d$, since S is abelian K_S is the trivial bundle, so we have $h^i(L) = h^i(K_S + L) = 0$ for $i = 1, 2$. It follows that $h^0(L) = \chi(L) = \chi\mathcal{O}_S + \frac{L^2 - LK_S}{2} = d$.
Let $S^2(H^0(L))$ denote the second symmetric power of $H^0(L)$. Then:

$$\dim S^2(H^0(L)) = \frac{d(d+1)}{2}.$$

Now if L is simply generated we have $S^2(H^0(L)) \cong H^0(2L)$, but $h^0(2L) = \chi(\mathcal{O}_S) + \frac{4L^2 - 2LK_S}{2} = 4d$. So if $d \leq 6$ we have $h^0(2L) = 4d > \frac{d(d+1)}{2} = S^2(H^0(L))$. It is well known that there exist abelian surfaces of degree $2d = 10$ in $\mathbb{P}^4$.

Example 2.5. Let $Q = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, we have $\text{Pic}(Q) \cong \mathbb{Z} \times \mathbb{Z}$. Let $L_1, L_2 \in \text{Pic}(Q)$ be two generators; then $2L_1 + 4L_2$ is very ample ([6], Ch. 2.18, p. 380). So there exists a smooth curve $C \in |2L_1 + 4L_2|$. Now consider the polarized curve (C, L), where $L = \mathcal{O}_{\mathbb{P}^3}(1)_C$. Since C is smooth we have: $2g(C) - 2 = (2L_1 + 4L_2 + K_Q)(2L_1 + 4L_2) = (2L_1 + 4L_2 - 2L_1 - 2L_2)(2L_1 + 4L_2) = 2L_2(2L_1 + 4L_2) = 4$. Hence $g(C) = 3$ and $\deg L = (L_1 + L_2)(2L_1 + 4L_2) = 6 > 2g(C) - 1$. Then $h^1(L) = 0$ and $h^0(L) = 4$. From this it follows that: $\dim S^2(H^0(L)) = h^0(2L) = 10$. Now consider the exact sequence:

$$0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_C \to 0.$$

Tensoring with $\mathcal{O}_{\mathbb{P}^3}(2)$ and taking cohomology, we obtain:

$$0 \to H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(2L) \to \cdots$$

Since C is contained in a quadric it follows that $H^0(\mathcal{I}_C(2)) \neq 0$, and recalling that $H^0(\mathcal{O}_{\mathbb{P}^3}(2)) = S^2(H^0(L))$ we immediately see that the map: $S^2(H^0(L)) \to H^0(2L)$ cannot be surjective.

Theorem 2.2 can be regarded as a generalization of the following well known fact

Proposition 2.6. *Let (M, L) be a polarized manifold and assume that L is simply generated (i.e. 1-generated), then L is very ample.*

We have only to show that in this case $\mathrm{BS}|L| = \emptyset$. By contradiction suppose that there exists $p \in \mathrm{Bs}\,|L|$ and let tL be very ample. Since L is 1-generated, each section of tL is a sum of products of sections belonging to $H^0(L)$, then tL would not be spanned at p.
Theorem 2.2 also allows us to prove the following

Theorem 2.7. *Let (M, L) be a polarized manifold with effective and 2-generated line bundle; then $2L$ is very ample.*

Proof. From Theorem 2.2 we have only to show that:

- $2L$ is spanned.
- $2L$ separates p, q with at least one in $\mathrm{Bs}\,|L|$.
- $2L$ separates $p \in \mathrm{Bs}\,|L|$ from each $\tau \in T_pM$.

Let tL, $t \geq 2$, be a very ample line bundle. Each section $s \in H^0(tL)$ is of the form

$$s = \sum \left(\prod_{i=0}^{a} \alpha_i \prod_{j=0}^{b} \beta_j\right), \tag{1}$$

with $\alpha_i \in H^0(2L)$ and $\beta_j \in H^0(L)$ so that $2a + b = t$. Now $2L$ is spanned outside $\mathrm{Bs}\,|L|$, since $\mathrm{Bs}\,|2L| \subset \mathrm{Bs}\,|L|$. Let $p \in \mathrm{Bs}\,|L|$, if every $\alpha \in H^0(2L)$ vanishes at p then each product in 1 must vanish, so tL is not spanned.
If $p \in \mathrm{Bs}\,|L|$ and $q \notin \mathrm{Bs}\,|L|$ take $\beta \in H^0(L)$ such that $\beta(q) \neq 0$, $\beta^2 \in H^0(2L)$ separates p and q. Let $p, q \in \mathrm{Bs}\,|L|$. As in the proof of Theorem 2.2 the same shows that there exists a section in $H^0(L)$ or in $H^0(2L)$ separating p from q, then $2L$ separates p and q.
If $p \in \mathrm{Bs}\,|L|$ and $\tau \in T_pM$, then as before we know that exists a section of $H^0(L)$ or of $H^0(2L)$ separating p and τ. But in this case we may have a $\beta \in H^0(L)$ with $d\beta_p(\tau) \neq 0$. This doesn't tell anything on $H^0(2L)$ because $\beta^2(p) = 0$ but also $d\beta_p^2(\tau) = 0$. Now we prove that, if $2L$ does not separate p and τ, such a β does not exists. We have two cases:
t is even. Let η be as in Theorem 2.2 such that $d\alpha_p(\tau) = \eta\alpha(p)$, for each $\alpha \in H^0(2L)$. Let $t = 2r$ and consider $s \in H^0 = (tL)$:

$$ds_p(\tau) = \sum(d(\prod_{i=1}^{a}\alpha_i \prod_{2j=0}^{b}\beta_j)_p(\tau)) = \sum(d(\prod_{i=1}^{r}\alpha_i)_p(\tau)) =$$
$$= \sum\sum_{i=1}^{r}(\prod_{j\neq i}\alpha_j(p))d\alpha_{ip}(\tau) = \sum\sum_{i=1}^{r}\eta(\prod_{j=1}^{r}\alpha_j(p)) =$$
$$= \sum r\eta(\prod_{j=1}^{r}\alpha_j(p)) = r\eta s(p).$$

So tL does not separate p and τ.
t is odd. Since $\beta_j(p) = 0$ for each j, we have:

$$s(p) = \sum(\prod_{i=0}^{a}\alpha_i(p)\prod_{2j+1=0}^{b}\beta_j(p)) = 0.$$

Then tL would not span in p. □

3. Some applications.

In this section we discuss some applications of Theorem 2.2.
First we consider a polarized manifold (M, L) with $\mathrm{Bs}|L| = \{p_1, \ldots, p_n\}$ (a finite set) and we obtain some information on the generator of $G(M, L)$. To apply the main technique: "the Apollonius method" see [3], recall that

given (M, L) be a polarized manifold, a sequence of irreducible and reduced subvarieties D_i of M such that $D_i \in |L_{D_{i-1}}|$ and $D_1 \in |L|$ is called a *ladder* of (M, L) and the D_i are said the *rungs* of the ladder. In the second applications we study some connection between the generator of the canonical ring $G(S, K_S)$ of a surface of general type and the very ampleness of tK_S. Finally we study the very ampleness of $-tK_S$ for a del Pezzo surface.

a) Hyperelliptic polarized manifold of type $(-)$.

In this subsection we apply the main theorem to provide an evidence for the first conjecture of Fujita in a special case. First of all we recall Fujita's conjecture, see [5] (§ 2, Conjecture b).

Conjecture 3.1. *For every polarized manifold* (M, L) *with* $\dim M = n$, $K_M + (n+2)L$ *is very ample.*

We recall that the delta-genus $\Delta(M, L)$ of a polarized manifold (M, L) is defined as $\Delta(M, L) = L^n + n - h^0(M, L)$. Here we deal with polarized manifolds with $\Delta(M, L) = 1$. In this case for $L^n \geq 3$ it is known [3] that L is very ample. It is also easy to prove that for $L^n = 2$ the line bundle $(g+1)L$ is very ample [7].
The case $L^n = 1$ is very difficult to classify and there are only partial results, see [4]. Here we consider only polarized manifolds sectionally hyperelliptic of type $(-)$. In this case Fujita [4] proved the following facts:

(2) $$H^q(M, tL) = 0 \text{ for } q \geq 1, t \in \mathbb{Z},$$

(3) $$K_M = (2g - n - 1)L \text{ where } g = g(M, L) \text{ and } n = \dim M,$$

(4) $$\text{Pic } M \cong \mathbb{Z}, \text{ generated by } L,$$

(5) $$(M, L) \text{ has a ladder.}$$

Properties 2 and 3 are true also for the rungs (D, L_D) of this ladder. In this situation we may apply Theorem 2.2 to prove the very ampleness of $K_M + (n+2)L = (2g+1)L$.
We start with a lemma on the r-generation inspired by [3], 2.3.

Lemma 3.2. *Let* (M, L) *be a polarized manifold with* $\Delta(M, L) = d(M, L) = 1$*, sectionally hyperelliptic of type* $(-)$*; if the line bundle* L_C *is* r*-generated, then* L *is* r*-generated.*

Proof. Consider a ladder of $(M, L) : (M_1, L_1) = (M, L), \ldots, (M_n, L_n) = (C, L_C)$ and proceed by induction on n.

For $n = 1$ the assumption is true. Now suppose that it is true for $k + 1$ and consider the exact sequence:

$$0 \to \mathcal{O}_{M_k}(-L_k) \to \mathcal{O}_{M_k} \to \mathcal{O}_{M_{k+1}} \to 0,$$

tensor with $\mathcal{O}_{M_k}(tL_k)$ and consider the exact cohomology sequence, recalling 2 we get:

$$(6) \qquad 0 \to H^0(M_k, (t-1)L_k) \to H^0(M_k, tL_k) \to H^0(M_{k+1}, tL_{k+1}) \to 0.$$

So for each t the restriction map ψ:

$$H^0(M_k, tL_k) \to H^0(M_{k+1}, tL_{k+1})$$

is surjective. Let us take a set of generators $\gamma_1, \dots, \gamma_i$ of $G(M_{k+1}, L_{k+1})$ and their inverse images $\eta_1, \dots, \eta_i$ in $G(M_k, L_k)$, if $\sigma \in H^0(M_k, L_k)$ is the defining section of M_{k+1} then $G(M_k, L_k)$ is generated by the η_j and σ. To show this, observe that by 6 $H^0(M_k, L_k)$ is generated by σ and by the η_j that belong to $H^0(M_{k+1}, L_{k+1})$. Proceeding by induction on t it is simple to prove the assertion. Finally observe that $\eta_1, \dots, \eta_i$ and σ belong to $H^0(M_k, tL_k)$ with $t \leq r$. This completes the proof. □

Considering the ladder of (M, L), by adjunction from 3 we have:

$$K_{M_i} = (2g - i - 1)L_i.$$

Then $K_C = (2g-2)L_C = (2g-2)p$, with $p = \mathrm{Bs}\,|L|$. Hence $g(M, L) = g = g(C)$ the genus of the curve C. Then since C is an hyperelliptic curve we have that L_C is exactly $2g + 1$-generated. Then by the lemma (M, L) is $2g + 1$-generated.
The following theorem extends what proved in [8] for $g = 1$.

Theorem 3.3. *Let* (M, L) *be a polarized manifold with* $\Delta(M, L) = L^n = 1$, *sectionally hyperelliptic of type* $(-)$, *then* $(2g + 1)L$ *is very ample.*

Proof. Since L is $(2g+1)$-generated, by 2.2 we know that $\varphi_{|(2g+1)L|}$ is an embedding of $M \setminus \{p\}$.
To see that $(2g + 1)L$ is spanned on p we proceed by induction. Clearly $(2g + 1)L_C$ is very ample. Consider a smooth $D \in |L|$ (we know that such a D exists by the ladder property), then consider the exact sequence:

$$0 \to H^0(2gL) \to H^0((2g+1)L) \to H^0((2g+1)L_D) \to 0.$$

By induction we have that $(2g + 1)L_D$ is spanned on p then there exists $\sigma \in H^0((2g + 1)L_D)$ with $\sigma(p) \neq 0$. By the suriectivity of the restriction map we have a $\gamma \in H^0((2g + 1)L)$ such that $\gamma(p) \neq 0$.
To see that $(2g + 1)L$ defines an embedding in p, consider a vector $\tau \in T_pM$. We say that exists a section $\eta \in H^0(M, L)$ such that $d\eta(\tau) \neq 0$. To construct η we may take an hypersurface $D \in |L|$ such that $\tau \notin T_pD$ then the section that define D clearly have a differential that does not vanish on τ. Suppose by contradiction that for each $D \in |L| \tau \in T_pD$ then the intersection of n such D_i cannot be transverse, so $L^n > 1$, absurd. Section η^{2g+1} defines an embedding in p.
Now consider a point $q \in M$ different from p, we have to find a section in $H^0((2g + 1)L)$ that separates p and q, i.e. that vanishes on p but not on q. To see this observe that every section of $H^0(L)$ vanish on p, since $p = \mathrm{Bs}\,|L|$, but there is a section $\sigma \in H^0(L)$ that does not vanish on q, otherwise $q \in \mathrm{Bs}\,|L|$. Now consider $\sigma^{2g+1} \in H^0((2g + 1)L)$, this section separates p from q. ☐

Theorem 3.3 is as best as possible as proved in the following:

Proposition 3.4. *Given a polarized manifold (M, L) with $\Delta(M,L)=d(M, L)=1$, sectionally hyperelliptic of type $(-)$, the least integer t such that tL is very ample is $2g + 1$, i.e. $K_M + rL$ is not very ample for $r \leq n + 1$.*

Proof. By contradiction suppose that $2gL$ is very ample and consider a regular ladder of (M, L). Since the ladder is regular, the very ampleness of $2gL$ implies the very ampleness of the line bundles L_{M_i}.
In particular we have that $2gp$ is very ample, where $p = \mathrm{Bs}\,|L|$ and $2gp$ is a divisor on C, where C is the one-dimensional element of the ladder. Note that C is an hyperelliptic curve and that $g(M, L) = g(C)$.
Now we may compute canonical bundle by adjunction. By 6 we have $K_M = (2g - n - 1)L$, then we obtain: $K_D = (K_M + L)_D = (2g - n)L_D$, and by induction: $K_C = (2g - 2)L_C$. But $L_C = \mathrm{Bs}\,|L| = p$ so we have $K_C = (g - 1)2p$. Since C is hyperelliptic we know that $h^0(2p) = 2$. Now consider the canonical map $\varphi_{|K_C|}$ on C and let $\langle \sigma, \tau \rangle$ is a basis of $H^0(2p)$, a basis of $H^0(K_C)$ is: $\langle \sigma^{g-1}, \sigma^{g-2}\tau, \dots, \tau^{g-1} \rangle$.
Now observe that $\langle \sigma^g, \sigma^{g-1}\tau \dots, \tau^g \rangle$ is a basis of $H^0(K_C + 2p)$. In fact by Riemann-Roch we have $h^0(K_C + 2p) = g + 1$. This theorem implies that also $\varphi_{|K_C+2p|}$ is a double covering of $\mathbb{P}^1$ followed by the Veronese embedding, but this implies that $(2g + 1)L_C = K_C + 2p$ is not very ample. ☐

b) Surfaces of general type.

Let S be a minimal surface of general type (i.e. of Kodaira dimension 2) and let K_S be its canonical bundle. It is known [1] that K is ample outside the set of

(-2)-curves, i.e. $KC = 0$ if and only if C is a (-2)-curve.
In literature there are results [1] on the C-isomorphism property of the maps $\varphi_n = \varphi_{|nL|}$, there are also results [2] on the generation of the canonical ring:

$$\bigoplus_{t \geq 1} H^0(S, tK_S).$$

As Theorem 2.2 shows there must be a connection between these results. In the following table the results on the generation of the canonical ring, for surfaces with $q = H^1(S, \mathcal{O}_S) = 0$, are compared with the very ampleness of rK_S. For each pair of values of K_S^2 and $P_g(S) = h^0(S, K_S)$ are compared the C-isomorphism property of $\varphi_{|rK_S|}$ and the r-generation of K_S.

K_S^2	$\varphi r K_S$ is a C-isomorphism	P_g	K_S is r-generated
1	5	0	≤ 6
		1	≤ 4
		2	5
2	4	0	≤ 6
		1	≤ 4
		2	3($\star$)
		3	4
3	$3K_S$ is spanned and φ_3 is a birational morphism	0	≤ 6
		1	≤ 4
		2	3($\star$)
		3	2($\bullet$)

Where ($\star$) means that the result is true if $|K_S|$ contains an irreducible element and ($\bullet$) means that the result is true if K_S is base point free and $\varphi_{|K_S|}(S)$ is not a rational surface of degree $r-1$ in $\mathbb{P}^r (r = h^0(K_S))$.
We observe that r-generation of K_S is not so far from the very ampleness of rK_S and by Theorem 2.2 we have also the following propositions:

Proposition 3.5. *Let S be a surface of general type with $P_g = K^2 = 2$ and $q = 0$. If the generic element of $|K_S|$ is irreducible, then $\varphi_{|3K_S|}$ is a C-isomorphism outside $Bs\,|K_S|$.*

Proof. By Theorem 3.7 of [2], K_S is 3-generated so the proof follows from Theorem 2.2. □

Similarly one can prove

Proposition 3.6. *Let S a surface of general type with $q = 0$, $P_g \geq 3$, $K_S^2 \geq 3$ and K_S spanned, then we have two cases:*

- $\varphi_{|K_S|}(S)$ *is a rational surface of degree $r - 1$ in $\mathbb{P}^r (r = h^0(K_S))$,*
- $2K_S$ *is a C-isomorphism.*

c) Del Pezzo surfaces.

In the following table is resumed the correspondence between very ampleness and r-generation for a Del Pezzo Surface. Here $L = -K_S$

degree	very ampleness	r-generation
$L^2 = 3$	L	1-generated
$L^2 = 2$	2L	2-generated
$L^2 = 1$	3L	3-generated

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