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ON HIGHER SECANT VARIETIES OF RATIONAL NORMAL SCROLLS

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In this paper we study the higher secant varieties of rational normal scrolls, in particular we give them as determinantal varieties. From this we can obtain, in some cases, the sequence of secant defects, generalizing to a class of varieties and to every characteristic the counterexample given by Ådlandsvik to Zak's theorem of superadditivity.

Introduction.

Let $X \subset \mathbb{P}^N$ be a projective variety. The k^{th} secant variety X^{k+1} of X is the closure of the union of the k-planes spanned by $k+1$ points of X.

The interest for the properties of secant varieties arose first at the beginning of this century; we recall in particular the articles of Palatini [8], [9] and one of A. Terracini, [11]. In recent years the argument has been looked over again, especially by the work of F. Zak, see [12]. Some of the most interesting properties of higher secant varieties can be found in his study about the secant defects: the expected dimension of the k^{th} secant variety of a projective variety X, when not linear, is $(k+1)n+k$, where $n = \dim X$, so it is natural to define the k^{th} secant defect as the integer $\delta_k := s_{k-1} + n + 1 - s_k$, where $s_k = \dim X^{k+1}$. In [13], Zak proved that the sequence of secant defects is monotonic non decreasing; in the same paper, he also stated that this sequence is superadditive for every non-degenerate smooth projective variety $X \subset \mathbb{P}^N$

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in the interval $[0, k_0]$, where k_0 is the minimum integer such that the k_0^{th} secant variety X^{k_0+1} is the whole $\mathbb{P}^N$, i.e. if $k = k_1 + \ldots + k_r$, then $\delta_k \geq \delta_{k_1} + \ldots + \delta_{k_r}$. But the proof was wrong, as a counterexample given by B. Ådlandsvik in [15] shows; in particular he found that, in zero-characteristic, for a rational normal scroll of dimension two of the type $(1, a)$, with $a \geq 7$, the theorem fails.

The theorem of superadditivity was then proved in more restrictive hypothesis by B. Fantechi in [3], by assuming that the k_r^{th} secant variety is almost smooth, i.e. the tangent star to every point is contained in the join of the point and the variety itself. More recently, Holme and Roberts saw what the situation is about this problem, in [6].

In this paper we will study the higher secant varieties of rational normal scrolls. In particular, we prove Theorem 4.6 and Corollary 4.7 that, if $X_{a_1,\ldots,a_\ell}$ is a scroll of type $a_1, \ldots, a_\ell$, then $X^k_{a_1,\ldots,a_\ell}$ is a cone of vertex $L_{j_1} \ldots L_{j_h}$, and basis the variety $X^k_{a_1,\ldots,\widehat{a_{j_1}},\ldots,\widehat{a_{j_h}},\ldots,a_\ell}$ where L_j is the linear space associated to a_j, and $a_p < k$, $\forall p = j_1, \ldots, j_h$, $a_q \geq k$, $\forall q \neq j_1, \ldots, j_h$. From this we can obtain, in some cases, the sequence $\{\delta_k\}$, generalizing to a class of varieties and to every characteristic the example given by Ådlandsvik. We will prove this by considering the higher secant varieties of rational normal scrolls (and rational normal scrolls themselves, indeed) as linear determinantal varieties; in particular, our main result will be proved as a generalization of a new "geometric" proof of the fact that the higher secant varieties of a rational normal curve are given, set-theoretically, by the linear determinantal varieties of the catalecticant matrix associated to the curve (for other proofs see [2] and [10]).

In the first section we introduce the language of the present work and basic definitions. In the second one, we study linear determinantal varieties, in particular the varieties X for which X^k is given by the vanishing of the minors of order $k+1$ of the matrix defining X and we characterize a class of them. In the third section we apply these results to catalecticant matrices, whose linear determinantal varieties are rational normal curves. In the fourth section we prove the theorem about higher secant varieties of rational normal scrolls, and we give an example of calculation of the sequence $\{\delta_k\}$ for a rational normal scroll.

After sending this paper to the referee, prof. D. Eisenbud told me that a student of him, M. Johnson, in his Ph. D. thesis, obtained results similar to ours; recently he wrote an article, just appeared [7], reporting his main result on this argument.

I am thankful to prof. E. Mezzetti for suggesting me this argument and for her help and suggestions and to the referee also, for the careful reading and

useful remarks.

Notations. Let $\mathbb{K}$ be an algebraically closed field; if V is a $\mathbb{K}$-vector space of dimension $N+1$, we denote by $\mathbb{A}^{N+1} := \mathbb{V}(V)$ the affine space on V and by $\mathbb{P}^N = \mathbb{P}(V) = \mathrm{Proj}S(V)$ the projective space associated to V; $[v]$ denotes the point of $\mathbb{P}^N$ associated to $v \in V$.

By a *variety* we mean a reduced and irreducible algebraic $\mathbb{K}$-scheme.
For a variety Z of $\mathbb{P}^N$, we indicate with $\widehat{Z} \subset \mathbb{A}^{N+1}$ the affine cone of Z in $\mathbb{A}^{N+1}$.

1. Join of Varieties.

We recall the following definitions (see [14]).

Definition 1.1. Let X and Y be two varieties of $\mathbb{A}^{N+1}$; we call *sum of X and Y*, and we indicate with $X+Y$, the closure of the image of $X \times Y$ under the morphism determined by the addition of V.

Obviously we have $\dim(X+Y) \leq \dim X + \dim Y$.

The analogous of the sum of two affine varieties in the projective space is the join:

Definition 1.2. Let X and Y be two varieties of $\mathbb{P}^N$; the reduced scheme XY such that:

$$\widehat{XY} = \widehat{X} + \widehat{Y}$$

is called *join of X and Y*.

One can prove that

Theorem 1.3. *Let X and Y be two (irreducible) varieties of $\mathbb{P}^N$; then we have:*

1) *XY is irreducible, therefore XY is a projective variety of $\mathbb{P}^N$;*
2) $\dim(XY) \leq \dim X + \dim Y + 1$.

Proof. See for example [3], (1.2.). □

A straightforward consequence of this theorem is the fact that the set var($\mathbb{P}^N$) of projective subvarieties of $\mathbb{P}^N$ becomes a commutative semiordered monoid by the operation given by the join. The order is given by the inclusion, and the empty variety is the unit of the monoid.

Definition 1.4. By *higher secant variety* of a given variety X of $\mathbb{P}^N$ we mean a power X^k of X in the monoid var($\mathbb{P}^N$).

It is easy to prove that this definition is equivalent to the classical one given at the beginning of the introduction and used, for example, by F. Zak (see [12], [13]).

An easy consequence of Theorem 1.3, 2) is that

$$\dim X^{k+1} \leq \dim X^k + \dim X + 1 \leq \cdots \leq (k+1)\dim X + k,$$

so we can give the following

Definition 1.5. The k^{th} *secant defect of* X is the integer δ_k defined by

$$\delta_k := \begin{cases} 0 & \text{if} \quad k = 0 \\ (s_{k-1} + s_0 + 1) - s_k & \text{if} \quad 1 \leq k+1 \leq k_0 \\ \dim X & \text{if} \quad k \geq k_0 \end{cases}$$

where s_k is the dimension of X^{k+1} (obviously we have $s_0 = \dim X$) and $k_0 = \min\{k \in \mathbb{N} \mid X^k = \mathbb{P}^N\}$.

2. Linear Determinantal Varieties.

Let us consider the projective space $M := \mathbb{P}^{nm-1}$ associated to the vector space of $n \times m$ matrices with entries in $\mathbb{K}$.

Definition 2.1. A *generic determinantal variety* $M_k \subset M$ is the locus of matrices of rank at most k.

It is well known that M_k is a subvariety of M, and its ideal is generated by the minors of order $k+1$ of

$$P = \begin{pmatrix} x_0 & \cdots & x_{m-1} \\ \vdots & & \\ x_{nm-m-1} & \cdots & x_{nm-1} \end{pmatrix}$$

where the x_i, $i = 0, \ldots, nm-1$, are coordinates on M (see for instance [1], pp. 67-75).

In particular M_1 is the locus of matrices of rank 1. Let us recall the following interpretation of M_1 as a Segre variety (cfr. [5], pp. 98-99). Let σ be the Segre map

$$\sigma : \mathbb{P}(\mathbb{K}^n) \times \mathbb{P}((\mathbb{K}^m)^*) \longrightarrow \mathbb{P}(\mathbb{K}^n \otimes (\mathbb{K}^m)^*)$$

defined by

$$\sigma(Y, Z) = \sigma([y_1, \ldots, y_n], [z_1, \ldots, z_m]) = [(y_1, \ldots, y_n) \otimes (z_1, \ldots, z_m)].$$

To give a $n \times m$ matrix of rank 1 means to give a linear map

$$A : \mathbb{K}^m \longrightarrow \mathbb{K}^n$$

of rank 1. To give such a matrix, up to a scalar multiplication, means to fix an image of dimension 1 and a kernel of dimension $m-1$, which is conceivable as an element of $(\mathbb{K}^m)^*$. Then we can write:

$$\mathrm{Im}\,(A) = [y_1, \ldots, y_n] \in \mathbb{P}(\mathbb{K}^n)$$

and

$$\mathrm{Ker}(A) = [z_1, \ldots, z_m] \in \mathbb{P}((\mathbb{K}^m)^*).$$

Under the canonical isomorphism

$$\psi : \mathrm{Hom}\,(\mathbb{K}^m, \mathbb{K}^n) \cong (\mathbb{K}^m)^* \otimes \mathbb{K}^n,$$

A corresponds to some (nonzero) multiple of

$$Y \otimes Z = [y_1, \ldots, y_n] \otimes [z_1, \ldots, z_m].$$

Remark 2.2. The matrices of rank $\leq k$ correspond, under ψ, to sums of k pure tensors.

This remark allows to get in a short way the well known characterization of higher secant varieties of generic determinantal varieties.

Theorem 2.3. $M_h^k = M_{hk}$.

Proof. First of all, we observe that it is sufficient to prove the formula for $h = 1$, since if $M_1^k = M_k$, then:

$$(M_h)^k = (M_1^h)^k = M_1^{hk} = M_{hk}\,.$$

So, let us fix $A_1, \ldots, A_k \in M_1$ and identify A_i with $Y_i \otimes Z_i$; then $P \in M_k$ if and only if it can be written as a sum of k pure tensors, by (2.2.), and then if and only if $P \in M_1^k$. $\square$

We will consider now linear sections of the generic determinantal variety, or, more precisely, sets of zeroes of minors of a $n \times m$ matrix of linear forms on a projective space $\mathbb{P}\ell$, obtained by fixing a linear rational map $i : \mathbb{P}^\ell \dashrightarrow M$. We give the following

Definition 2.4. Let $\Omega = (L_{ij})$ be a matrix of linear forms on $\mathbb{P}^\ell$; the *linear determinantal variety* $\Sigma_k(\Omega)$ is the pullback of the generic determinantal variety M_k under the rational map $i : \mathbb{P}^\ell \dashrightarrow M$ determined by the linear forms (Lij).

A matrix of linear forms Ω can be thought of as a linear application

$$\omega : V \longrightarrow \operatorname{Hom}(U, W)$$

where $U \cong \mathbb{K}^m$, $W \cong \mathbb{K}^n$ and $\mathbb{P}^\ell = \mathbb{P}(V)$, or as an element

$$\omega \in V^* \otimes U^* \otimes W,$$

because of the canonical isomorphism:

$$V^* \otimes U^* \otimes W \cong (V, \operatorname{Hom}(U, W)).$$

A further interpretation is given by the isomorphism:

$$V^* \otimes U^* \otimes W \cong \operatorname{Hom}(\operatorname{Hom}(U, W)^*, V^*)$$

under which ω corresponds to a surjective map:

$$\mu : \operatorname{Hom}(U, W)^* \cong \operatorname{Hom}(W, U) \cong U \otimes W^* \to V^*$$

with kernel:

$$V^\perp := \{\psi \in \operatorname{Hom}(W, U) \mid \langle \omega(\phi), \psi\rangle = 0 \ \forall \phi \in V\}$$

where, $\forall \phi \in \operatorname{Hom}(U, W)$ and $\forall \psi \in \operatorname{Hom}(W, U)$, we have:

$$\langle \phi, \psi \rangle := \psi(\phi)$$

(considering $\operatorname{Hom}(W, U)$ as the dual of $\operatorname{Hom}(U, W)$).

Let us recall that two matrices $\Omega, \Omega' \in \operatorname{Hom}(U, W)$, are said to be *conjugate* if there exist $A \in \operatorname{GL}(W)$ and $B \in \operatorname{GL}(U)$ such that:

$$\Omega' = A \circ \Omega \circ B.$$

We recall also that the left multiplication by a matrix $A \in \operatorname{GL}(W)$ (respectively the right multiplication by $B \in \operatorname{GL}(U)$) is called an invertible row (resp. column) operation.

We characterize now the class of matrices of linear forms for which the variety of secant k-planes of $(\Sigma_1(\Omega))^k$ is $\Sigma_k(\Omega)$:

Lemma 2.5. *Let Ω be a $n \times m$ matrix of linear forms on $\mathbb{P}^\ell$; then the following facts are equivalent:*

1) $(\Sigma_1(\Omega))^k = \Sigma_k(\Omega)$*;*
2) $(\Sigma_1(\Omega))^k \supset \Sigma_k(\Omega)$.

Proof. We observe that, as in the case of generic determinantal varieties, a matrix $\Lambda \in \Sigma_k(\Omega)$ can be thought of as an element of $U^* \otimes W$ which is the sum of k pure tensors; so from (2.2.) it follows that $(\Sigma_1(\Omega))^k \subset \Sigma_k(\Omega)$. □

To give a useful criterion for verifying if a matrix of linear forms Ω satisfies the condition $(\Sigma_1(\Omega))^k = \Sigma_k(\Omega)$ Theorem 2.8, we need the following

Lemma 2.6. *Let Ω be a $n \times m$ matrix of linear forms on $\mathbb{P}^\ell$. Assume that Ω satisfies the following condition:*

(*) *for every invertible matrix $n \times n$ A and for any choice of $n-k$ rows of $A \circ \Omega$, the linear space defined by the vanishing of the linear forms of these rows is contained in the reduced scheme X defined as follows: X is the union of the k joins of $(\Sigma_1(\Omega))^{k-1}$ and one of the k linear spaces defined by the vanishing of the linear forms contained in $n-1$ rows of $A \circ \Omega$ including the fixed $(n-k)$'s.*

Then:

$$(\Sigma_1(\Omega))^k = \Sigma_k(\Omega).$$

Proof. By (2.5.), to prove the theorem, it is sufficient to prove that if $\Lambda \in \Sigma_k(\Omega)$, it belongs to $(\Sigma_1(\Omega))^k$. To prove this, it is better thinking of Λ as an element of $\mathrm{Hom}\,(U, W)$, or as a matrix of rank at most k. Now, since

$$\mathrm{rank}\,(\Lambda) = \dim\,(\mathrm{Im}\,(\Lambda)),$$

Λ has rank at most k if there exists at least a subspace S of W with $\dim\,(S) = k$ that contains $\mathrm{Im}\,(\Lambda)$ i.e. if and only if there exists at least a projection:

$$\pi_S : W \longrightarrow \frac{W}{S}$$

(with $\dim\,(S) = k$) that, composed with Λ, gives the zero map. Therefore we can write:

$$\Sigma_k(\Omega) = \bigcup_{S \in \mathbb{G}(k-1, \mathbb{P}(W))} \{\Lambda \in i(\mathbb{P}^\ell) \mid \pi_S \circ \Lambda = 0\} \tag{2.7}$$

where $\mathbb{G}(k-1, \mathbb{P}(W))$ is the Grassmannian of $(k-1)$-planes of $\mathbb{P}(W)$ and $i : \mathbb{P}^\ell \hookrightarrow M$ is the map introduced in (2.4.). Fixed $S \in \mathbb{G}(k-1, \mathbb{P}(W))$, the projection π_S can be represented as a matrix of the type:

$$\pi_S = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \\ p_{n-k1} & \cdots & p_{n-kn} \end{pmatrix}$$

whose minors of order $n-k$ are the Plücker coordinates of S in $\mathbb{G}(k-1, \mathbb{P}(W))$. From this, if

$$\Lambda = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1m} \\ \vdots & & \\ \lambda_{n1} & \cdots & \lambda_{nm} \end{pmatrix}$$

we obtain the equations of the set $\{\Lambda \mid \pi_S \circ \Lambda = 0\}$:

$$0 = \pi_S \circ \Lambda = \begin{pmatrix} \sum_{i=1}^n p_{1i}\lambda_{i1} & \cdots & \sum_{i=1}^n p_{1i}\lambda_{im} \\ \vdots & & \\ \sum_{i=1}^n p_{n-ki}\lambda_{i1} & \cdots & \sum_{i=1}^n p_{n-ki}\lambda_{im} \end{pmatrix}.$$

If we fix $z_0, \dots, z_\ell$ coordinates on $\mathbb{P}^\ell$, the equations become:

$$\sum_{j=1}^n p_{ij}\lambda_{jk}[z_0, \dots, z_\ell] = 0 \qquad \forall i = 1, \dots, n-k, \qquad \forall k = 1, \dots, m.$$

Therefore, each of the elements in brackets of (2.7.) determines the linear space given by the linear forms of the rows

$$\Big(\sum_{j=1}^n p_{ij}\lambda_{j1}[z_0, \dots, z_\ell] \cdots \sum_{j=1}^n p_{ij}\lambda_{jm}[z_0, \dots, z_\ell]\Big) \qquad i = 1, \dots, n-k,$$

and our thesis easily follows from the hypothesis. $\square$

Theorema 2.8. *Let Ω be a $n \times m$ matrix of linear forms on $\mathbb{P}^\ell$. Assume that the following condition is fulfilled: for any choice of $n-k$ rows of Ω, the linear space defined by the vanishing of the linear forms of these rows is contained in the reduced scheme X defined as follows: X is the union of the k joins of $(\Sigma_1(\Omega))^{k-1}$ and one of the k linear spaces defined by the vanishing of the linear forms contained in $n-1$ rows of Ω including the fixed $(n-k)$'s. Then:*

$$(\Sigma_1(\Omega))^k = \Sigma_k(\Omega).$$

Proof. By the previous lemma, it is enough to verify the condition (*) of Lemma 2.6. In fact: let us consider $A \circ \Omega$, where $A \in \mathrm{GL}(W)$, i.e. an invertible row operation; interpret $\Sigma_1(\Omega)$ as a subvariety of the Segre variety $\mathbb{P}^m \times \mathbb{P}^n$. The operation just considered, geometrically means a projectivity of $\mathbb{P}^n$ (since A determines a projectivity $[A] \in \mathbb{P}\mathrm{GL}(n)$). Therefore, with the change of projective coordinates of $\mathbb{P}^n$ determined by $[A]^{-1}$, $A \circ \Omega$ becomes, in the new coordinates, again Ω. Then, since a projective change of coordinates maps linearly independent forms to linearly independent forms, the condition (*) is fulfilled. □

It is clear that, passing to the dual spaces and considering the matrix ${}^t\Omega$, we obtain an analogue proposition for the columns. Therefore in the following, it will not be restrictive to suppose $n \leq m$.

3. Rational Normal Curves.

In this section, using Theorem 2.8, we get, in an easy way, the well-known characterization of secant varieties of rational normal curves. Let us recall the following

Definition 3.1. A $n \times m$ matrix $A = (a_{ij})$ with entries in a ring R is called *catalecticant* (or *persymmetric*) if $a_{ij} = a_{hk}$ $\forall i, j, h, k$ such that $i + j = h + k$.

The following matrix

$$\mathrm{Cat}(m+1, n+1) = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_{m+1} \\ x_2 & x_3 & & & \\ x_3 & & & & \vdots \\ \vdots & & & & \\ x_{n+1} & & \cdots & & x_{m+n+1} \end{pmatrix}$$

with indeterminates entries is called the *generic catalecticant matrix*.

Let us note that $\mathrm{Cat}(m+1, n+1)$ can be interpreted as the matrix of the map:

$$\phi : \mathrm{Sym}^n V^* \otimes \mathrm{Sym}^m V^* \to \mathrm{Sym}^\ell V^* \qquad \ell = n + m$$

determined by the ordinary multiplication of polynomials of degrees m and n on a linear space V of dimension 2, or, (up to canonical isomorphisms) as a map:

$$\varphi : \mathrm{Sym}^\ell V \to \mathrm{Sym}^n V \otimes \mathrm{Sym}^m V.$$

We introduce now the rational normal curve $C^\ell := v_{\ell,1}(\mathbb{P}(V))$, image of the Veronese map:

$$v_{\ell,1} : \mathbb{P}(V) \longrightarrow \mathbb{P}(\mathrm{Sym}^\ell V).$$

The following proposition is well known (see [2], (4.2.)):

Proposition 3.2. $\Sigma_1(\mathrm{Cat}\,(m+1, n+1)) = C^\ell$.

We are finally able to find the equations of the higher secant varieties of an interesting class of linear determinantal varieties, and as a corollary we will find the higher secant varieties of the rational normal curve. First of all, we need the following

Lemma 3.3. *Given a projective variety $X \subset \mathbb{P}^N$ and a linear projection $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^M$, then we have $\overline{\pi(X^k)} = (\overline{\pi(X)})^k$.*

Proof. It is enough to prove the claim for open subsets, therefore our thesis follows from these equivalences:

$$P \in \pi(X^k) \iff P \in \pi(Q_1 \dots Q_k) = \pi(Q_1) \dots \pi(Q_k)\,,$$

where

$$Q_i \in X,\ \forall i = 1, \dots, k, \iff P \in \pi(X)^k\,. \qquad \square$$

From now on we will denote by P_i the i^{th} fundamental point of $\mathbb{P}^\ell$, whose coordinates x_j, vanish $\forall\, j = 1, \dots, \widehat{i}, \dots, \ell + 1$. Then, we prove the following

Theorem 3.4. $(\Sigma_1(\mathrm{Cat}\,(m+1, n+1))^h = \Sigma_h(\mathrm{Cat}\,(m+1, n+1))$ *(with $h \leq \min(m+1, n+1)$).*

Proof. Let us prove this theorem by induction on h. The case $h = 1$ is trivial, and for the case $h = 2$ it is sufficient to verify the hypotheses of Theorem 2.8: let us consider $n-2$ rows of $\mathrm{Cat}\,(m+1, n+1)$; it is easy to see that the only nontrivial case is if we consider the first $n-2$ rows (or, which is the same, for simmetry, the last $n-2$ ones), otherwise these rows give the empty set or a point of the curve.

Let us consider then the first $n-2$ rows; these determine the straight line

$$x_{n+m} = \lambda, \qquad x_{n+m+1} = \mu,$$

and the case $h = 2$ easily follows from the observation that this line is the tangent to the rational normal curve at the point P_{m+n+1}.

Now, let us consider $n-h$ rows of $\mathrm{Cat}\,(m+1, n+1)$; it is easy to see, like in the proof of the case $h = 2$, that the only nontrivial case is if we consider the first $n - h$ rows, otherwise we have points of $\Sigma_g(\mathrm{Cat}(m+1, n+1))$, $g < h$ and our thesis follows by the inductive hypothesis.

Let us consider the first $n - h$ rows: these determine the h plane:

$$x_{n+m-h+2} = \lambda_1, \ldots, x_{n+m+1} = \lambda_h,$$

therefore, by the inductive hypothesis, it is sufficient to prove that $P_{n+m-h+2}$ belongs to $(\Sigma_1(\mathrm{Cat}\,(m+1, n+1)))^h$. To prove this, let us consider the linear projection

$$\pi([x_1, \ldots, x_{n-h+2}, x_{n-h+3}, \ldots, x_{n+m+1}]) = [x_1, \ldots, x_{n-h+2}]$$

which maps the curve C^{m+n} to the curve C^{n-h+1}, and our thesis follows from the Lemma 3.3 and the inductive hypothesis. □

Note 3.5. This theorem was proved by T.G. Room in [9] and by D. Eisenbud in [2]; the above new "geometric" proof will be suitable to be generalized to find higher secant varieties of rational normal scrolls.

As an obvious consequence we have the following

Corollary 3.6. $\Sigma_h(\mathrm{Cat}\,(m+1, n+1)) = \Sigma_h(\mathrm{Cat}\,(p+1, q+1))$ *with* $\ell = m + n = p + q$ *and* $h \leq \min\{m+1, n+1, p+1, q+1\}$.

Note 3.7. This corollary is proved in more general hypoteses in [4] (pag. 9, Lemma 2.3). It is used in [2] just to prove (3.4.).

4. Rational Normal Scrolls.

Let $a_1, \ldots, a_k$ be integers such that $a_i \geq 0$, $\forall i = 1, \ldots, k$, and $a_j > 0$ for at least one index j. Let us take k linear supplementary subspaces

$$L_i \subset \mathbb{P}^N, \qquad i = 1, \ldots, k$$

with $\dim(L_i) = a_i$. For $a_i \neq 0$, we consider the rational normal curve $C^i \subset L_i$ image of the morphism:

$$\phi_i := v_{a_i,1} : \mathbb{P}^1 \longrightarrow L_i.$$

If $a_i = 0$, we put $C^i = L_i$ and ϕ_i the constant map.

Definition 4.1. A *rational normal scroll of type* $a_1, \dots, a_k$, is the variety:

$$X_{a_1\dots a_k} := \bigcup_{P\in\mathbb{P}^1} \phi_1(P)\dots\phi_k(P).$$

We show now how rational normal scrolls can be seen as linear determinantal varieties. We fix homogeneous coordinates $x_0^{(i)}, \dots, x_{a_i}^{(i)}$ on L_i, $i = 1, \dots, k$, so that $x_0^{(1)}, \dots, x_{a_1}^{(1)}, \dots, x_0^{(i)}, \dots, x_{a_i}^{(i)}, \dots, x_0^{(k)}, \dots, x_{a_k}^{(k)}$ are coordinates on $\mathbb{P}^N$. We may assume that $a_i = 0\ \forall i = 1, \dots, h-1$ and $a_j \neq 0$ otherwise. Let us consider the matrix

$$M_{a_1\dots a_k} := \begin{pmatrix} x_0^{(h)} & \cdots & x_m^{(h)} \\ x_1^{(h)} & \cdots & x_{m+1}^{(h)} \\ \vdots & & \\ x_{a_h-m}^{(h)} & \cdots & x_{a_h}^{(h)} \\ x_0^{(h+1)} & \cdots & x_m^{(h+1)} \\ \vdots & & \\ x_0^{(k)} & \cdots & x_m^{(k)} \\ x_1^{(k)} & \cdots & x_{m+1}^{(k)} \\ \vdots & & \\ x_{a_k-m}^{(k)} & \cdots & x_{a_k}^{(k)} \end{pmatrix},$$

where m is an integer such that

$$1 \leq m < \begin{cases} \min\{a_h, \dots, a_k\} & \text{if} \quad 1 \neq a_h = \dots = a_k, \\ \min\{a_h, \dots, a_k\} + 1 & \text{otherwise.} \end{cases}$$

It is given by the "concatenation" of $k-h+1$ catalecticant matrices $M_h, \dots, M_k$.

The proof of the following classical theorem is standard.

Theorem 4.2. $X_{a_1\dots a_k} = \Sigma_1(M_{a_1\dots a_k})$.

We will now obtain the equations of higher secant varieties of rational normal scrolls. We need the following three lemmas:

Lemma 4.3. $X^n_{a_1\dots a_k} = \Sigma_n(M_{a_1\dots a_k})$, *with* $n \leq m$.

Proof. We can follow the proof of Theorem 3.4 on the columns instead of the rows. □

Lemma 4.4. *Let* $X_{a_1\dots a_i\dots a_k}$ *be a rational normal scroll; let us fix* i *and let* $P^0, \dots, P^{a_i}$ *be* $a_i + 1$ *points such that* $L_i = P^0 \dots P^{a_i}$*; let* $Z := X_{a_1\dots a_{i-1}0\dots 0a_{i+1}\dots a_k}$ *be the rational normal scroll generated by the linear spaces* $L_1, \dots, L_{i-1}, P^0, \dots, P^{a_i}, L_{i+1}, \dots, L_k$*. Then:*

$$Z = X_{a_1\dots\widehat{a_i}\dots a_k} L = X_{a_1\dots a_i\dots a_k} L.$$

Proof. The inclusions:

$$Z \subset X_{a_1\dots\widehat{a_i}\dots a_k} L \subset X_{a_1\dots a_i\dots a_k} L$$

are obvious. To prove the others, consider a point $Q \in X_{a_1\dots a_i\dots a_k} L$. Then we have $Q \in xl$, where $x \in X_{a_1\dots a_i\dots a_k}$, $l \in L$ so there exists a point $P \in \mathbb{P}^1$ such that

$$x \in \phi_1(P) \dots \phi_i(P) \dots \phi_k(P),$$

therefore

$$x \in \phi_1(P) \dots \widehat{\phi_i(P)} \dots \phi_k(P) L$$

and finally

$$x \in \phi_1(P) \dots \widehat{\phi_i(P)} \dots \phi_k(P) P^0 \dots P^h$$

i.e. $Q \in Z$. □

Lemma 4.5. *In the hypotesis of the previous lemma, if* $n > a_i$*, we have:*

$$X^n_{a_1\dots a_i\dots a_k} = X^n_{a_1\dots a_{i-1}0\dots 0a_{i+1}\dots a_k}.$$

Proof. From $X_{a_1\dots a_i\dots a_k} \subset X_{a_1\dots a_{i-1}0\dots 0a_{i+1}\dots a_k}$, we obtain

$$X^n_{a_1\dots a_i\dots a_k} \subset X^n_{a_1\dots a_{i-1}0\dots 0a_{i+1}\dots a_k}.$$

Viceversa. From the previous lemma (and from the fact that $L^n = L$, since L is linear) it suffices to prove that

$$X^n_{a_1\dots\widehat{a_i}\dots a_k} L \subset X^n_{a_1\dots a_i\dots a_k}.$$

We consider a point $x \in X^n_{a_1\dots\widehat{a_i}\dots a_k}L$: by definition,

$$x \in y_1 \dots y_n l$$

where $y_1, \dots, y_n \in X_{a_1\dots\widehat{a_i}\dots a_k}$, $l \in L$, and, in particular there exist $P_1, \dots, P_n \in \mathbb{P}^1$ for which we have:

$$y_\ell \in \phi_1(P_\ell) \dots \widehat{\phi_i(P_\ell)} \dots \phi_k(P_\ell) \quad \ell = 1, \dots, n.$$

We have also that $y_1, \dots, y_n \in X_{a_1\dots a_i\dots a_k}$, therefore there exist $l_1, \dots, l_n \in L$ such that

$$\phi_i(P_1) = l_1, \dots, \phi_i(P_n) = l_n\ ,$$

moreover, since $n > a_i$, we have

$$l_1 \dots l_n = L$$

and

$$y_\ell \in \phi_1(P_\ell) \dots l_\ell \dots \phi_k(P_\ell) \quad \ell = 1, \dots, n\,;$$

therefore, since

$$\begin{aligned} &\big(\phi_1(P_1) \dots \widehat{\phi_i(P_1)}) \dots \phi_k(P_1) \dots (\phi_1(P_n) \dots \widehat{\phi_i(P_n)} \dots \phi_k(P_n)\big)L = \\ &= \big(\phi_1(P_1) \dots l_1 \dots \phi_k(P_1)) \dots (\phi_1(P_n) \dots l_n \dots \phi_k(P_n)\big) \subset X^n_{a_1\dots a_i\dots a_k} \end{aligned}$$

we conclude that $x \in X^n_{a_1\dots a_i\dots a_k}$. $\square$

It is clear that a permutation of the integers $a_1, \dots, a_k$ induces a projective transformation of rational normal scrolls. So it is not restrictive to suppose $0 \leq a_1 \leq \dots \leq a_k$.

Theorem 4.6. *Let $X_{a_1\dots a_k}$ be a rational normal scroll, with $0 \leq a_1 \leq \dots \leq a_k$; then $X^n_{a_1\dots a_k} = \Sigma_n(M_{a_1\dots a_k,n})$, where $M_{a_1\dots a_k,n}$ is the following matrix*

$$M_{a_1\ldots a_k,n} := \begin{pmatrix} x_0^{(j)} & \cdots & x_m^{(j)} \\ x_1^{(j)} & \cdots & x_{m+1}^{(j)} \\ \vdots & & \\ x_{a_j-m}^{(j)} & \cdots & x_{a_j}^{(j)} \\ x_0^{(j+1)} & \cdots & x_m^{(j+1)} \\ \vdots & & \\ x_0^{(k)} & \cdots & x_m^{(k)} \\ x_1^{(k)} & \cdots & x_{m+1}^{(k)} \\ \vdots & & \\ x_{a_k-m}^{(k)} & \cdots & x_{a_k}^{(k)} \end{pmatrix}$$

where j is the minimum integer such that $a_j \geq n$ and $1 \leq m \leq a_j$.

Proof. Let us observe that the matrix $M_{a_1\ldots a_k,n}$ may be seen as a catalecticant matrix to which some rows have been taken away. So, if $a_k - m + 1 \geq 2$, our claim follows from (3.7.) and Lemma 4.5.

If $a_k - m + 1 = 1$, we have $a_j = \cdots = a_k = n + 1$, therefore $\Sigma_1(M_{a_1\ldots a_k,n})$ is a cone of vertex $L := L_0 \ldots L_{j-1}$ and basis the Segre variety $\Sigma_{m,k-j} := \Sigma_1(M_{a_1\ldots a_k,n}) \cap (L_j \ldots L_k)$. This cone contains $X_{a_1\ldots a_k}$, so $X^n_{a_1\ldots a_k} \subset \Sigma_n(M_{a_1\ldots a_k,n})$.

To prove the other inclusion, let us consider a point $T \in \Sigma_n(M_{a_1\ldots a_k,n})$; therefore there exist points $S_1, \ldots, S_n$ such that $S_i \in LP_i^{(j)} \ldots P_i^{(k)}$, $i = 1, \ldots, n$, where $P_i^{(\ell)} \in L_i$, $\forall i = j, \ldots, k$, and $P_i^{(j)} \ldots P_i^{(k)} \cong \mathbb{P}^{k-j} \times Q_i \subset \Sigma_{m,k-j}$ is an element of the family of $(k-j)$-planes of $\Sigma_{m,k-j}$. Then, we have

$$T \in L(P_1^{(j)} \ldots P_1^{(k)}) \ldots (P_n^{(j)} \ldots P_n^{(k)}) = L(P_1^{(j)} \ldots P_n^{(j)}) \ldots (P_1^{(k)} \ldots P_n^{(k)}) .$$

From the facts that, $\forall i = 1, \ldots, k$, the rational normal curve C^i generates the L_i, and the spaces $P_i^{(j)} \ldots P_i^{(k)}$, as i varies, are each contained in a space of the same family of $(k-j)$-planes of $\Sigma_{m,k-j}$, it follows that there exist n points $[s_1, t_1], \ldots, [s_n, t_n] \in \mathbb{P}^1$ such that

$$(P_1^{(\ell)} \ldots P_n^{(\ell)}) = \phi_\ell([s_1, t_1]) \ldots \phi_\ell([s_n, t_n]), \qquad \ell = j, \ldots, k.$$

Therefore $T \in X^n_{a_1\ldots a_k}$. $\square$

The geometrical meaning of Theorem 4.6 is the following:

Corollary 4.7. *The* $(n-1)^{\text{th}}$ *secant variety of the rational normal scroll* $X_{a_1\dots a_k}$ *is a cone of vertex the span generated by the linear spaces* L_i *such that* $\dim L_i < n$ * *and basis the* $(n-1)^{\text{th}}$ *secant variety of the rational normal scroll determined by the intersection of* $X_{a_1\dots a_k}$ *with the span of the linear spaces such that* $\dim L_i \geq n$.

From Theorem 4.6 we can compute the dimension of some higher secant varieties and the sequence of secant defects.

Lemma 4.8. *Let* $X_{a_1\dots a_k}$ *be a rational normal scroll, with* $0 \leq a_1 \leq \dots \leq a_k$, n *an integer and* j *the minimum integer such that* $a_j \geq n$*; then*

$$\dim X^n_{a_1\dots a_k} = \sum_{i=1}^{j-1}(a_i+1) + \dim\big((\sum_n (M_{a_1\dots a_k,n})) \cap (L_j \dots L_k)\big),$$

where the matrix $M_{a_1\dots a_k,n}$ *is defined in the previous theorem.*

Proof. It follows from the fact that $X^n_{a_1\dots a_k}$ is a cone of vertex $L_1 \dots L_{j-1}$ and basis $\sum_n (M_{a_1\dots a_k,n}) \cap (L_j \dots L_k)$, the $(n-1)^{\text{th}}$ secant variety of the rational normal scroll $X_{a_j\dots a_k} \subset (L_j \dots L_k)$ of type $a_j, \dots, a_k$. □

Lemma 4.9. *Let* $X_{a_1\dots a_k}$ *be a rational normal scroll, with* $0 \leq a_1 \leq \dots \leq a_k$ *and* $n \leq a_1$*, then*

$$\dim X^n_{a_1\dots a_k} = \min\{N, nk+n-1\}.$$

Proof. If $\dim X^n_{a_1\dots a_k} = N$, the lemma is trivial; therefore from now on we will suppose $\dim X^n_{a_1\dots a_k} < N$.

Let us define the following variety:

$$S := \overline{\{(P_1, \dots, P_n; Q) \mid P_i \in \mathbb{P}^1, Q \in \phi_1(P_1)\dots\phi_k(P_1)\dots\phi_1(P_n)\dots\phi_k(P_n)\}} \subset$$
$$\subset (\mathbb{P}^1)^n \times \mathbb{P}^N$$

and the projections $\mathbb{P}i_1 : S \dashrightarrow \mathbb{P}^1 \times \dots \times \mathbb{P}^1$ and $\pi_2 : S \dashrightarrow \mathbb{P}^N$, whose image is $X^n_{a_1\dots a_k}$. The generic fibre of the map π_1 at the point $(P_1, \dots, P_n)$ is the variety

$$\pi_1^{-1}(P_1, \dots, P_n) = (P_1, \dots, P_n; \phi_1(P_1)\dots\phi_k(P_1)\dots\phi_1(P_n)\dots\phi_k(P_n))$$

* i.e. these linear spaces are "filled up" by the $(n-1)^{\text{th}}$ secants of the rational normal curves.

of dimension $nk-1$. These facts follow from the observation that n distinct points of a rational normal curve of degree less or equal to $n-1$ generate a linear space of dimension $n-1$.

Besides, for the same reason, the generic fibre of π_2 is a point, i.e. $\dim S = \dim X^n_{a_1\dots a_k}$ and from this we obtain our claim. □

Lemma 4.10. *Let $X_{a_1\dots a_k}$ be a rational normal scroll, with $0 \le a_1 \le \dots \le a_k$, n an integer and j the minimum integer such that $a_j \ge n$; then*

$$\dim X^n_{a_1\dots a_k} = \min\left\{N, \sum_{i=1}^{j-1}(a_i+1) + nk - nj + 2n - 1\right\}. \tag{4.11}$$

Proof. It is an easy dimensional count from the previous two lemmas. □

This theorem gives a class of counterexamples to Zak's theorem of superadditivity; they are highly unbalanced scrolls. For example we can obtain Ådlandsvik's counterexample; but we note that our theorem holds in every characteristic, while Ådlandsvik restricts himself to a field of zero-characteristic, because he uses the strong Terracini lemma (see [14], (1.11., (2))).

Example. Let $X_{a_1\dots a_k}$ be a rational normal scroll, with $0 < a_1 \le \dots \le a_k$ (i.e. a smooth scroll), and $3 \le \left[\frac{a_k+1}{2}\right] - a_{k-1}$. By (4.10.), and the fact that $\dim (C^{a_k})^n = 2n-1$, if $(C^{a_k})^n \ne L_k$ (see, for example, [14], (1.5)), we obtain

$$\dim X^n_{a_1\dots a_k} = \sum_{i=1}^{k-1}(a_i+1) + nk - nk + 2n - 1 = \sum_{i=1}^{a_k-1}(a_i+1) + 2n - 1$$

for $a_{k-1} < n \le \left[\frac{a_k+1}{2}\right]$. Then, if $a_{k-1} < n \le \left[\frac{a_k+1}{2}\right]$, we get:

$$\begin{aligned}\delta_n = \sum_{i=1}^{k-1}(a_i+1) + 2n + \dim X_{a_1\dots a_k} - \sum_{i=1}^{k-1}(a_i+1) - 2n - 1 =\\ = \dim X_{a_1\dots a_k} - 1 = k-1\end{aligned}$$

and the sequence is not superadditive.

Note 4.13. If we assume that our scroll $X_{a_1\dots a_k}$ is smooth (i.e. $a_1 > 0$), we have that:

$$\delta_1 = 2k+1 - \dim X^2_{a_1\dots a_k} = 2k+1 - \left(\sum_{i=1}^{j-1}(a_i+1) - 1 + 2k - 2j + 4\right) = 0,$$

so this is not in contraddiction with the Zak's claim stating that the theorem of superadditivity should hold for smooth varieties with $\delta_1 > 0$.

Note 4.14. It is easy to see that these examples do not satisfy the almost smoothness required in [3] to restore Zak's statement.

REFERENCES

[1] E. Arbarello - M. Cornalba - P.A. Griffiths - J. Harris, *Geometry of Algebraic Curves, Volume I,* Springer-Verlag, 1985.

[2] D. Eisenbud, *Linear sections of determinantal varieties,* Amer. J. Math., 110 (1988), pp. 541-575.

[3] B. Fantechi, *On the superadditivity of secant defects,* Bull. Soc. Math. France, 118 (1990), pp. 85-100.

[4] L. Gruson - C. Peskine, *Courbes de l'espace projectif., variétés de sécants, in enumerative geometry and classical algebraic geometry,* ed. P. Le Barz and Y. Hervier. Progress in Math., 24 (1982).

[5] J. Harris, *Algebraic Geometry, a First Course,* Springer-Verlag, 1992.

[6] A. Holme - J. Roberts, *Zak's theorem on superadditivity,* Ark. Mat., 32 (1994), pp. 99-120.

[7] M. Johnson, *The possible dimensions of the higher secant varieties,* Amer. J. Math., 118 (1996), pp. 355-361.

[8] F. Palatini, *Sulle varietà algebriche per le quali sono di dimensione minore dell'ordinario, senza riempire lo spazio ambiente, una o alcuna delle varietà formata da spazii seganti,* Atti Accad. Torino, 44 (1909), pp. 362-375.

[9] F. Palatini, *Sulle superficie algebriche i cui $S_h(h+1)$-seganti non riempiono lo spazio ambiente,* Atti Accad. Torino, 41 (1906), pp. 634-640.

[10] T.G. Room, *The Geometry of Determinantal Loci,* Cambridge at the University Press, 1938.

[11] A. Terracini, *Sulle V_k per cui la varietà degli $S_h(h+1)$-seganti ha dimensione minore dell'ordinario,* Rend. Circ. Mat. Palermo, 31 (1911), pp. 392-396.

[12] F.L. Zak, *Tangents and Secants of Algebraic Varieties,* American Mathematical Society, 1993.

[13] F.L. Zak, *Linear systems of hyperplane sections on varieties of small codimension,* Funktsional. Anal. i Prilozhen., 19 (1985), no. 3, 1-10; English transl. in Functional Anal. Appl., 19 (1985), pp. 165-173.

[14] B. Ådlandsvik, *Joins and higher secant varieties,* Math. Scand., 61 (1987), pp. 213-222.

[15] B. Ådlandsvik, *Higher Secant Varieties,* Thesis, Univ. Bergen, Bergen, Norway, 1987.

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