LE MATEMATICHE
Vol. LI (1996) – Fasc. I, pp. 23–25

AMBIVALENT GROUPS HAVING A FAITHFUL MONOMIAL IRREDUCIBLE CHARACTER

ION ARMEANU

In this note we shall study the structure of the finite groups having a nonlinear faithful monomial irreducible character of minimal degree.

The notation and terminology are standard (see for example [2] and [3]). All groups will be finite.

Definition. *i) A rational group is a group all whose irreducible characters are rational valued.*

ii) An ambivalent group is a group all whose irreducible characters are real valued.

Theorem 1. *Let G be an ambivalent group having a nonlinear faithful monomial irreducible character of minimal degree. Then:*

i) $G = G'E_2$, *with* G' *abelian and* E_2 *an elementary abelian* 2*-group.*
ii) $G' = O(G)P$, *where* $O(G)$ *is the maximal normal odd order subgroup of* G *and* $P \in Syl_2(G')$.
iii) E_2 *inverts all elements of* G'.

Entrato in Redazione l'11 luglio 1995.

Proof. We shall prove first that G' is abelian. Suppose the contrary, thus $G'' \neq 1$. For an ambivalent group G/G' is an elementary abelian 2-group (see [1]). Let χ a faithful monomial nonlinear irreducible character of minimal degree. Let $H \leq G$ and $\mu \in Irr(H)$ a linear character such that $\mu^G = \chi$. Let λ be any irreducible constituent of $(1_H)^G$. It is clear that

$$\chi(1) = \mu^G(1) = (1_H)^G(1) > \lambda(1),$$

hence $\lambda(1) < \chi(1)$ and λ must be linear by the minimality of the degree of χ. Then $\ker(\lambda) > G'$ (see [2], p. 25). Thus

$$G' \leq \cap \ker(\lambda) = \ker(1_H)^G = \cap_{g \in G} g^{-1} H g \leq H.$$

Since G/G' is an elementary abelian 2-group and G' is characteristic in G, it follows that H is normal in G and the inertia group $I_G(\lambda) = H$. By Clifford's theorem (see [2]) $\chi_H = \sum_{j=1}^{k} \mu_j$, where $\mu_j \in \mathrm{Irr}(H)$ are the distinct conjugates of μ in G. Since μ_j are linear we have $\ker(\mu_j) \geq H' \geq G''$ and hence $\ker \chi \geq \ker \chi_H = \cap_j \ker(\mu_j) \geq G'' \neq 1$ which contradicts the faithfulness of χ.

Let now $O(G)$ be the maximal odd order normal subgroup of G. Since G is ambivalent, $O(G) \leq G'$ (see [1]) and hence $O(G)$ is abelian and $G' = O(G) \times P$ with $P \in Syl_2(G')$. Let $S \in Syl_2(G)$ such that $P \leq S$. Then $S \simeq G/O(G)$ so that G is 2-nilpotent. Since G is ambivalent and G' is abelian, it follows that $G/G' \simeq E_2$ where E_2 is an elementary abelian 2-group which inverts all elements of G'.

Corollary 2. *Let G be a rational group having a nonlinear faithful monomial irreducible character of minimal degree. Then $G \simeq (E_3 \times P)E_2$ where E_3 is an elementary abelian* 3*-group, $P \in Syl_2(G')$ has* $\exp(P) \leq 4$ *and E_2 inverts all elements of $E_3 \times P$.*

Proof. Since G is a rational group, it is easy to see that

$$N_G(< x >)/C_G(x) \simeq Aut(< x >)$$

for every $x \in G$. By the form of $Aut(< x >)$ (see [3]) the statement follows.

REFERENCES

[1] I. Armeanu, *About ambivalent groups,* to appear.

[2] I.M. Isaacs, *Character Theory of Finite Groups,* Academic Press, 1976.

[3] H. Kurzweil, *Endliche Gruppen,* Springer-Verlag, 1977.

Mathematics Department,
Physics Faculty,
Bucharest University,
P.O. Box MG-11,
Bucharest-Magurele (ROMANIA)