LE MATEMATICHE
Vol. LI (1996) – Fasc. I, pp. 27–41

ON STABILITY FOR PERTURBED DIFFERENTIAL EQUATIONS

ALEXANDER ANDREEV - GIUSEPPE ZAPPALÀ

This paper deals with a nonautonomous differential equation, precompact in the sense of G.R. Sell and Z. Artstein. We investigate the eventual asymptotic stability and total stability of this equation with infinitesimal perturbations using Liapunov function with semidefinite derivative.

Introduction.

Traditionally asymptotic stability has been studied either by Liapunov's direct method or by Poincare's geometric method. The first attempt, to unify the two procedures, was carried out by La Salle [8] by combining information, obtained from simple and natural Liapunov's functions, with information about geometric properties obtained from the invariance principle of limit set. In nonautonomous differential equations one obtains more results by using the limiting equations theory established by Sell [14], [15] and Artstein [3], [4], [5]; the main problem of locating the limit set, by using Liapunov functions, was studied in [1] and [2].

The purpose of this paper is to describe, essentially, the eventual asymptotic stability of the trivial solution by a differential equation under infinitesimal perturbation, as considered by Artstein.

Entrato in Redazione l'11 luglio 1995.

This research was supported by Foundation RFFI (Grant 96 - 01 - 01067) and by italian MURST.

The paper is organized as follows:

— in the Theorems 1.1 and 1.2 we analyze the properties of the limit set

— the Theorems 2.1, 2.2 and 2.3 highlight global attractivity and asymptotic stability

— from Theorem 2.4 we obtain uniform total stability

— three examples are given based on the explained theory.

Preliminaries, definitions, theorems on limit sets.

Let us consider the vectorial ordinary differential equation

$$\dot{y} = Y(t, y) + P(t, y) = Z(t, y), \tag{1.1}$$

where P plays the role of perturbation for the equation

$$\dot{y} = Y(t, y). \tag{1.2}$$

The vector functions $Y, P : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, ($\mathbb{R}^+ = [0, +\infty[$, $\mathbb{R}^n$ is a n-th dimension vector space with the norm $\|y\|^2 = (y_1^2 + \cdots + y_n^2))$ are such that for every point $(t_0, y_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ the solutions $y(t) = y(t, t_0, y_0)$ and $\tilde{y}(t) = \tilde{y}(t, t_0, y_0)$ exist and they are unique [6].

Suppose also that for every compact set $H \subset R^n$ there exist two locally L_1 functions $r_H(t)$ and $\eta_H(t) : R^+ \to R^+$ [3] such that (s.t.):

i) $\forall (t, y) \in R^+ \times H \quad \|Y(t, y)\| \leq r_H(t)$ and $\forall \varepsilon > 0 \exists \mu_H(\varepsilon) > 0$ s.t. if $E \subset [t, t+1] \subset R^+$ is a measurable set with measure $< \mu_H(\varepsilon)$ then

$$\int_E r_H(\tau)\, d\tau \leq \varepsilon .$$

2i) $\forall y', y" \in H, \forall t \in R^+$ we have

$$\|Y(t, y') - Y(t, y'')\| \leq \eta_H(t)\|y' - y''\|$$

(Lipschitz) and $\exists$ a constant $N_H > 0$ s.t.

$$\int_t^{t+1} \eta_H(\tau)\, d\tau \leq N_H \quad \forall t \in R^+.$$

As is shown in [3] these hypotheses guarantee the precompactness of the equation (1.2), in the restrict sense and the uniqueness of solutions for (1.2) and for $\dot{y} = \varphi(t, y)$ *limiting equation* [5] of (1.2).

Definition 1.1. The perturbation $P(t, y)$ is said to be *integrally converging to zero as* $t \to +\infty$ if, for every sequence of continuous function $\{u_r(t) : [a, b] \to R^n\}$ uniformly converging to $\varphi : [a, b] \to R^n$ and for every sequence $t_r \to +\infty$, we have [5]

$$\lim_{r \to +\infty} \int_a^b P(t_r + \tau, u_r(\tau))\, d\tau = 0. \tag{1.3}$$

Remark. The previous definition is more general than the convergence to zero introduced in [10], [16]. If we suppose that for each H there exists a function $\sigma_H : R^+ \to R^+$ so that $\forall t \in R^+ \quad \forall y \in H$ we have [16]

$$\|P(t, y)\| \leq \sigma_H(t) \quad \text{and} \quad \lim_{s \to +\infty} \int_s^{s+1} \sigma_H(\tau)\, d\tau = 0$$

then (1.3) holds.

Definition 1.2. The perturbation $P(t, y)$ is said to be *infinitesimal on* R^+: if $\exists \sigma : R^+ \to R^+$ so that $\forall t \in R^+, \forall y \in R^n$ we have $\|P(t, y)\| \leq \sigma(t)$ and

$$\int_0^{+\infty} \sigma(\tau)\, d\tau < +\infty. \tag{1.4}$$

Remark. A perturbation *infinitesimal on* R^+ is integrally converging to zero as $t \to +\infty$.

Remark. If $P(t, y)$ is integrally converging to zero as $t \to +\infty$ then

$$\lim_{r \to +\infty} \int_a^b (Y + P)(t_r + \tau, u_r(\tau))\, d\tau = \lim_{r \to +\infty} \int_a^b Y(t_r + \tau, u_r(\tau))\, d\tau \tag{1.5}$$

it follows that (1.1) and (1.2) share the *same family of limiting equations* [5].

In virtue of this connection we obtain fundamental results on the *asymptotic behaviour* of the solution of (1.1) by using auxiliary function [11] denoted by V, W, h.

Definition 1.3. The function $h \in C^0(R^+ \to R^+)$ is said to be "a function of class K in the sense of Hahn" [11] if it is strictly increasing and $h(0) = 0$. A function $h \in K$ is said to be a function of class $\overline{K}$ if $\lim_{r \to +\infty} h(r) = +\infty$.

We will denote [11], for every Liapunov function $V \in C^1(R^+ \times R^n \to R)$:

$$\dot{V}_{(1.2)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial y} \cdot Y, \quad \dot{V}_{(1.1)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial y} \cdot Z \tag{1.6}$$

where $a \cdot b = a^T b$ is the scalar product.

We will assume [1] that the scalar function $W(t, y) \geq 0$, $W(t, 0) = 0$, is bounded on $(t, y) \in R^+ \times H$ and satisfies the Lipschitz conditions in t and y on each compact $[t_0, t_0 + \upsilon] \times H$ $(t_0 \geq 0, \upsilon > 0)$.

Then the set of functions $\omega(t, y)$ limiting to $W(t, y)$ will be nonempty, and the convergence of $W_n(t, y) = W(t_n + t, y)$ to $\omega(t, y)$ as $t \to +\infty$ will be uniform in each compact mentioned.

Definition 1.4. The pair (φ, ω) is said to be [1] a "limit pair" of (Y, W) when φ and ω derive from Y, W by using the identical sequence $\{t_r\}$.

Definition 1.5. Let (φ, ω) be a limit pair correspondent to (Y, W), we assume

$$M^+ = \{z(t) \in R^n; \dot{z}(t) = \varphi[t, z(t)], z(t) \in [\omega(t, y) = 0] \quad \forall t \in R^+\}$$
$$M_*^+ = \left\{\cup M^+ \quad \forall(\varphi, \omega)\right\}.$$

We shall prove our main results on the limit sets in this section.

Theorem 1.1. *Assume the following hypotheses:*

1) $\exists V \in C^1(R^+ \times R^n \to R^+)$ *bounded, on every* H, *such that*

$$\dot{V}_{(1.2)}(t, y) \leq -W(t, y) \leq 0;$$

2) *the perturbation* $P(t, y)$ *verify* (1.3) *and* $\frac{\partial V}{\partial y} \cdot P \leq 0$;
3) *the solution* $y(t)$ *belongs to* H.

Then the limit set $\Omega^+(y(t)) \subset M_*^+$ *i.e.* $y(t) \to M_*^+$.

Proof. By Malkin's formula [11] we obtain

$$\dot{V}_{(1.1)}(t, y) = \dot{V}_{(1.2)}(t, y) + \frac{\partial V(t, y)}{\partial y} \cdot P(t, y) \leq -W(t, y) \leq 0. \tag{1.7}$$

From (1.5) it follows that (Y, W) and $(Y + P, W)$ have the same family of limit pair. On the basis of Theorem 2.1 of [1] we have the result required.

Theorem 1.2. *Let* $P(t, y) = P_1(t, y) + P_2(t, y)$. *Then the results of the previous theorem are confirmed if*

$$\frac{\partial V}{\partial y} \cdot P_1 \leq 0, \quad \|P_2(t, y)\| \leq \sigma(t),$$

$\sigma(t)$ *verifies* (1.4) *and* $\exists S > 0$ *such that*

$$\|\frac{\partial V}{\partial y}\| \leq S, \quad \forall (t, y) \in R^+ \times R^n .$$

Proof. If we choose the auxiliary function

$$V_1(t, y) = V(t, y) + S \int_t^{+\infty} \sigma(\tau)\, d\tau$$

we obtain

$$\dot{V}_{1(1.1)}(t, y) = \dot{V}_{(1.2)}(t, y) + \frac{\partial V}{\partial y} \cdot P_1 + \frac{\partial V}{\partial y} \cdot P_2 - S\sigma(t) \leq -W(t, y) \leq 0$$

and so we can conclude the proof.

1. Theorems on the global attractivity and the eventual stability (of $y = 0$).

Assume that $Y(t, 0) = 0$ for every $t \in R^+$, so that, for the differential equation (1.2), the origin is an equilibrium or critical point. Then we obtain sufficient conditions for the eventual stability of this solution [7].

For the right maximal interval where $y(t, t_0, y_0)$ is defined we write $J^+(t_0, y_0)$.

Definition 2.1. The solution $y = 0$ of (1.2) is said to be *eventually stable* with respect to (1.1) if $(\forall \varepsilon > 0)$ $(\exists \tau = \tau(\varepsilon) > 0)$ $(\forall t_0 > \tau)$ $(\exists \delta = \delta(t_0, \varepsilon) > 0)$ $(\forall y_0 : \|y_0\| \leq \delta)$ one has $(\|y(t, t_0, y_0)\| < \varepsilon \; \forall t \geq t_0)$.

Definition 2.2. The solution $y = 0$ of (1.2) is said to be *eventually uniformly stable* with respect to (1.1) if $(\forall \varepsilon > 0)$ $(\exists \tau = \tau(\varepsilon) > 0)$ $(\forall t_0 > \tau)$ $(\exists \delta = \delta(\varepsilon) > 0)$ such that $(\forall y_0 : \|y_0\| \leq \delta)$ one has $(\|y(t, t_0, y_0)\| < \varepsilon$ $\forall t \geq t_0)$.

Definition 2.3. The solution $y = 0$ of (1.2) is said to be *globally attractive* with respect to (1.1) if $\forall (t_0, y_0) \in R^+ \times R^n \lim_{t \to +\infty} y(t, t_0, y_0) = 0$.

Definition 2.4. The solution $y = 0$ of (1.2) is said to be *globally equiattractive* with respect to (1.1) if $(\forall t_0 \in R^+)$ $(\forall y_0 \in R^n)$ $(J^+(t_0, y_0) = [t_0, +\infty[)$ and $(\forall \varepsilon > 0)$ $(\exists T = T(t_0, \varepsilon) > 0)$ s.t. $(\|y(t, t_0, y_0)\| < \varepsilon\ \forall t \geq t_0 + T)$.

Definition 2.5. The solution $y = 0$ of (1.2) is said to be *globally uniformly attractive* with respect to (1.1) if $(\forall t_0 \in R^+)$ $(\forall y_0 \in R^n)$ $(J^+(t_0, y_0) = [t_0, +\infty[)$ and $(\forall \varepsilon > 0)$ $(\exists T = T(\varepsilon) > 0)$ s.t. $(\|y(t, t_0, y_0)\| < \varepsilon\ \forall t \geq t_0 + T)$.

Definition 2.6. The solution $y = 0$ of (1.2) is said to be *uniformly totally stable* if $(\forall t_0 \in R^+)$ $(\forall \varepsilon > 0)$ $(\exists \delta' = \delta'(\varepsilon), \delta'' = \delta''(\varepsilon) > 0)$ s.t. $(\forall y_0 : \|y_0\| < \delta')$ $(\forall P : \|P\| < \delta'')$ $(\|y(t, t_0, y_0)\| < \epsilon\ \forall t \geq t_0)$ [11].

Theorem 2.1. *Suppose that the following assumptions hold*

1) $\exists V \in C^1(R^+ \times R^n \to R^+)$ *such that:*
   a) $V(t, 0) = 0\ \forall t \in R^+$,
   b) $V(t, y) \geq h(\|y\|)$ *where* $h \in \bar{K}$,
   c) $\exists l = l(t, y) \in C^1(R^+ \times R^n \to R^+)$ *bounded such that* $\|\frac{\partial V}{\partial y}\| \leq \lambda(1 + V)$,
   d) $\dot{V}_{(1.2)}(t, y) \leq -W(t, y) \leq 0$.
2) *The perturbation* $P(t, y)$ *satisfies the following hypotheses*
   a') $P = P_1 + P_2$,
   b') $\frac{\partial V}{\partial y} \cdot P_1 \leq 0$,
   c') P_1 *verifies* (1.3),
   d') $\exists \sigma(t) : R^+ \to R^+$ *such that* $\|P_2(t, y)\| \leq \sigma(t)$ *and* $\int_0^{+\infty} \sigma(\tau)\, d\tau < +\infty$,
   e') $\frac{\partial \lambda}{\partial y} \cdot P + \dot{\lambda}_{(1.2)}(t, y) = 0$.
3) *For each limit pair* (φ, ω) *of* (Y, W) *we have* $M^+ = \{y = 0\}$.

Then the solution $y = 0$ *of* (1.2) *is* globally attractive *and* eventually stable *with respect to the solutions of system* (1.1).

Proof. Put

$$\mu(t, y) = \exp(-l(t, y) \int_0^t \sigma(\tau) d\tau),\ \mu_0(t, y) = \exp(-l(t, y) \int_0^{+\infty} \sigma(\tau)\, d\tau),$$

$$V_1(t, y) = \mu(t, y)(V(t, y) + 1) - \mu_0(t, y)$$

we obtain

$$\dot{V}_{1(1.1)}(t, y) \leq -\mu(t, y)\dot{V}_{(1.2)}(t, y) \leq -\mu(t, y)W(t, y) \leq 0. \tag{2.1}$$

Let

$$(t_0, y_0) \in R^+ \times R^n, m_0 = V_1(t_0, y_0)(> 0)$$

and

$$\mu_0^* = \inf\{\mu_0 : t \geq t_0,\ \ y \in R^n\}$$

Since $\mu_0^* > 0$, from (2.1) follows

$$m_0 \geq V_1(t, y(t)) \geq \mu(t, y(t))V(t, y(t)) \geq \mu_0^* h(\|y(t)\|)$$

hence $\|y(t)\| \leq h^{-1}\left(\frac{m_0}{\mu_0^*}\right)$ $\forall t \geq t_0$ i.e. $y(t)$ is bounded. By the Theorem 1.1 we obtain that $\lim\limits_{t\to+\infty} y(t) = 0$ i.e. $y = 0$ is *globally attractive* with respect to the solutions of (1.1).

We deduce the eventual stability from the following consideration: for every $\varepsilon > 0$ we define $T(\varepsilon) > 0$ such that

$$\mu(T(\varepsilon), y_0) - \mu_0(T(\varepsilon), y_0) = \frac{h(\varepsilon)}{2}\mu_0^*.$$

This is possible because $\mu(t, y) - \mu_0(t, y) \searrow 0$ uniformly on every H when $t \to +\infty$. Given $t_0 \geq T(\varepsilon)$, because $V(t, 0) = 0\ \forall t \geq t_0$ it is possible to find $\sigma(t_0, \varepsilon) > 0$ from the inequality

$$\mu(t_0, y_0) \sup_{\|y\|<\sigma} V(t_0, y) < \frac{h(\varepsilon)}{2}\mu_0^*. \tag{2.2}$$

For every $t_0 \geq T(\epsilon)$ and $\|y_0\| < \sigma$ by virtue of $\dot{V}_{1(1.1)}(t, y) \leq 0$ and (2.2) follows, $\forall t \geq t_0$

$$\mu_0^* h(\|y(t)\|) \leq V_1(t, y(t)) \leq V_1(t_0, y_0) \leq \mu_0^* h(\varepsilon)$$

i.e. $\|y(t)\| < \varepsilon\ \forall t \geq t_0$, therefore $y = 0$ is *eventually stable* with respect to (1.1).

Remark. If $\lambda = \text{const}$ then we omit the hypothesis 2-e').

Theorem 2.2. *Suppose that the assumption* 1) *and* 2) *of Theorem 2.1 hold, but we substitute* 3) *with:* 3') $\exists$ *a sequence* $t_r \to +\infty$ *ensuring that the limit pair* (φ_0, ω_0) *have the property:*

the set $\{\omega_0(t, y) = 0\}$ *does not contain solutions of the limiting equation* $\dot{y} = \varphi_0(t, y)$ *except* $y = 0$.

Then the zero solution of (1.2) *is* globally equiattractive and eventually stable *with respect to* (1.1).

This theorem is a slight modification of Theorem 2.1.

Theorem 2.3. *Assume that the following hypotheses hold*

1) $\exists V \in C^1(R^+ \times R^n \to R^+)$ *so that:*
   a) $h_1(\|y\|) \le V(t, y) \le h_2(\|y\|)$ *with* $h_1, h_2 \in \bar{K}$,
   b) $\exists \lambda = $ *constant* > 0 *so that* $\left\| \dfrac{\partial V}{\partial y} \right\| \le \lambda(1 + V)$,
   c) $\dot{V}_{(1.2)}(t, y) \le -W(t, y) \le 0$,
   d) *for any* H $\exists m = m(H) > 0$ *such that* $V(t, y) \le m$ $\forall (t, y) \in R^+ \times H$,
   e) $\forall l > 0$ $\exists d = d(l, H) > 0$ *such that* $|V(t_2, y_2) - V(t_1, y_1)| \le l$ $\forall (t_1, y_1), (t_2, y_2) \in R^+ \times H$ *with* $|t_2 - t_1| \le d$ *and* $\|y_2 - y_1\| \le d$.
2) *The perturbation* $P(t, y)$ *has the properties:*
   a') $P(t, y) = P_1(t, y) + P_2(t, y)$,
   b') $\dfrac{\partial V}{\partial y} \cdot P_1 \le 0$, P_1 *verifies* (1.3),
   c') $\exists \sigma : R^+ \to R^+$ *so that* $\|P_2(t, y)\| \le \sigma(t)$ *and* (1.4) *hold.*
3) *For any limit pair* (φ, ω) *of* (Y, W) *the set* $\{\omega(t, y) = 0\}$ *does not contain solutions of the limiting equation* $\dot{y} = \varphi(t, y)$ *except* $y = 0$.

Then the solution $y = 0$ *of* (1.2) *is* globally uniformly attractive *and* eventually uniformly stable *with respect to the solutions of* (1.1).

Proof. Since $\lambda = \text{constant} > 0$ put

$$\mu(t) = \exp(-\lambda \int_0^t \sigma(\tau)\, d\tau), \ \ \mu_0 = \mu(\infty) = \exp(-l \int_0^{+\infty} \sigma(\tau)\, d\tau) > 0$$

and

$$V_1(t, y) = \mu(t)(V(t, y) + 1) - \mu_0,$$

we obtain as in Theorem 2.1

$$\dot{V}_{1(1.1)}(t, y) \le -\mu(t) W(t, y) \le -\mu_0 W(t, y) \le 0 \tag{2.3}$$

hence

$$\lim_{t\to+\infty} y(t, t_0, y_0) = 0, \ \lim_{t\to+\infty} V_1(t, y(t)) = c_0 \geq 0\ \forall (t_0, y_0) \subset R^+ \times R^n.$$

Using $V(t, y) \leq h_2(\|y\|)$, by (2.2) we deduce the *eventual uniform stability* of $y = 0$ with respect to the solutions of (1.1).

From the inequality (2.3) when $t_0 \in R^+$, $\|y_0\| < r_0 (r_0 > 0)$ we deduce $\forall t \geq t_0$

$$\mu_0 h_1(\|y(t)\|) \leq V_1(t_0, y_0) \leq \mu(t_0)(h_2(r_0) + 1) - \mu_0 = L_0$$

hence $\|y(t)\| \leq h_1^{-1}(L_0/\mu_0) = r_1$ i.e. the $y(t)$ are uniformly bounded.

Let us prove that $y(t) \to 0$ for $t \to +\infty$ uniformly with respect to $t_0 \in R^+$, $y_0 \in \{\|y\| \leq r_0\}$.

Suppose, ab absurdo, that $\exists \varepsilon > 0$ such that for every sequence $T_r \to +\infty$ there exist two sequences $\{y_r\} = \{y_r \in R^n : \|y_r\| \leq r_0\}$ and $t_r \to +\infty$ so that $\|y(t'_r, t_r, y_r)\| \geq \varepsilon$ for some $t'_r \geq t_r + T_r$.

Let $T_0 = T(\varepsilon) > 0$ and $\delta_0 = \delta(\varepsilon) > 0$ be chosen as in the definition of uniform eventual stability, obviously, if $t_r \geq T_0$ we have

$$\|y(t, t_r, y_r)\| \geq \delta_0 \quad \forall t \geq t_r. \tag{2.4}$$

Since $y(t, t_r, y_r)$ is solution of (1.1), by (2.3), we have respectively

$$\dot{y}(t, t_r, y_r) \equiv Z(t, y(t, t_r, y_r)); \dot{V}_1(t, y(t, t_r, y_r)) \leq -\mu W(t, y(t, t_r, y_r)).$$

Consider the translations

$$\dot{y}(t + t_r, t_r, y_r) \equiv Z(t + t_r, y(t + t_r, t_r, y_r)),$$

$$\dot{V}_1(t + t_r, y(t + t_r, t_r, y_r)) \leq -\mu W(t + t_r, y(t + t_r, t_r, y_r)).$$

Put

$$y_r(t) = y(t + t_r, t_r, y_r), Z_r(t, y) = Z(t + t_r, y), W_r(t, y) = W(t + t_r, y)$$

we obtain

$$\dot{y}_r(t) \equiv Z_r(t, y_r(t)), \dot{V}_{1r}(t, y_r(t)) \leq -\mu W_r(t, y_r(t)) \leq 0, y_r(0) = y_r.$$

According to Arstein's [3] and Arzela'-Ascoli's theorems on the precompactness of

$$\{Y_\tau(t, y) = Y(t + \tau, y), \tau \in R^+\}, \{W_\tau(t, y) = W(t + \tau, y), \tau \in R^+\}$$

and $\{V_{1\tau}(t, y) = V_1(t + \tau, y), \tau \in R^+\}$ we can select a subsequence $\{t_{\bar{r}}\}$ such that we obtain the convergences

$$y_{\bar{r}}(t) \to \bar{y}(t)\ ,\ y_{\bar{r}}(0) \to \bar{y}(0)\ ,\ Z_{\bar{r}}(t, y) \to \bar{Y}(t, y),$$
$$V_{1\bar{r}}(t, y) \to \bar{V}_1(t, y)\ ,\ W_{\bar{r}}(t, y) \to \bar{W}(t, y).$$

Consequently we have

$$\dot{\bar{y}} \equiv \bar{Y}(t, \bar{y}(t)), \dot{\bar{V}}_1(t, \bar{y}(t)) \le -\mu \bar{W}(t, \bar{y}(t)) \le 0, \quad \lim_{t\to+\infty} \bar{V}_1(t, \bar{y}(t)) = c_0 \ge 0. \tag{2.5}$$

Let a sequence $t_s \to +\infty$, $T > 0$ from (2.4) and (2.5) we deduce

$$0 < \delta_0 \le \|\bar{y}(t + t_s)\| \le r_1, \dot{\bar{y}}(t + t_s) \equiv \bar{Y}(t + t_s, \bar{y}(t + t_s)),$$
$$\bar{V}_1(t_s + T, \bar{y}(t_s + T)) - \bar{V}_1(t_s, \bar{y}(t_s)) \le -\mu \int_0^T \bar{W}(\tau + t_s, \bar{y}(\tau + t_s))d\tau \le 0.$$

Consider the new translates

$$\bar{y}_s(t) = \bar{y}(t + t_s), \bar{Y}_s(t, y) = \bar{Y}(t + t_s, y), \bar{W}_s(t, y) = \bar{W}(t + t_s, y).$$

The precompactness of $Y(t, y)$, $W(t, y)$ implies this property for $\bar{Y}$ and $\bar{W}$, therefore we can deduce the convergences

$$\bar{y}_{\bar{s}}(t) \to y^*(t),\ \bar{y}_{\bar{s}}(0) \to y_0^*,\ \bar{Y}_{\bar{s}}(t, y) \to \varphi(t, y),\ \bar{W}_{\bar{s}}(t, y) \to \le (t, y) \tag{2.6}$$

From (2.5) and (2.6) formulas it follows that $\|y^*(t)\| \ge \delta_0 > 0\ \forall t \in R^+$,

$$\dot{y}^*(t) \equiv \varphi(t, y^*(t)), \quad 0 \le \int_0^T \omega(\tau, y^*(\tau))\, d\tau \le 0 \quad \forall T > 0.$$

On the basis of which we conclude the proof.

Theorem 2.4. *Suppose that:*

1) *The function $Y(t, y)$ satisfies, for $t \in R^+$, $y_1, y_2 \in R^n$*

$$\|Y(t, y_1) - Y(t, y_2)\| \le L\|y_1 - y_2\|$$

   with $L = constant > 0$.

2) *A function $V(t, y) \in C^1(R^+ \times R^n \to R^+)$ exists such that*
   a) $h_1(\|y\|) \le V(t, y) \le h_2(\|y\|)$ *with* $h_1, h_2 \in K$
   b) $\dot{V}_{(1.2)}(t, y) \le -W(t, y) \le 0$.

3) *For each limit pair (φ, ω), the set $\{\omega(t, y) = 0\}$ does not contain the solutions of $\dot{y} = \varphi(t, y)$ except $y = 0$.*

Then the solution $y = 0$ of (1.2) *is* uniformly totally stable *with respect to* (1.1).

The proof is done by using Theorem 2.3 and Malkin's converse theorem [7].

2. Applications and examples.

A) Consider the differential system

$$\begin{cases} \dot{y}_1 = g_{11}(t, y)y_1 + g_{12}(t, y)y_2 \\ \dot{y}_2 = -g_{12}(t, y)y_1 + g_{22}(t, y)y_2 \end{cases} \tag{3.1}$$

where $y = (y_1, y_2)$ and the perturbed system

$$\begin{cases} \dot{y}_1 = g_{11}(t, y)y_1 + g_{12}(t, y)y_2 + g_{13}(t, y) + g_{14}(t, y), \\ \dot{y}_2 = -g_{12}(t, y)y_1 + g_{22}(t, y)y_2 + g_{23}(t, y) + g_{24}(t, y) \end{cases} \tag{3.2}$$

Theorem 3.1. *Suppose that the functions* $g_{ij} : R^+ \times R^2 \to R$ *satisfy the following hypotheses:*

1) *the system* (3.1) *is precompact (in the bounded sense* [2]*).*
2) $g_{11} \le -l < 0$, $g_{22} \le 0$, $g_{12}^2 \ge h^2 > 0$,
3) $y_1 g_{13} + y_2 g_{23} \le 0$,
4) g_{13}, g_{23} *satisfy* (1.3),
5) $\|(g_{14}, g_{24})\| \le \sigma$, *with* $\sigma : R^+ \to R^+$, $\int_0^{+\infty} \sigma(\tau)\, d\tau < +\infty$.

Then the solution $y_1 = y_2 = 0$ *of* (3.1) *is* globally uniformly attractive *and* eventually uniformly stable *with respect to* (3.2).

Proof. Choose $V = y_1^2 + y_2^2$, $l = 2$ thus all the assumptions of Theorem 2.3 are satisfied.

Remark. If $g_{ij} : R^+ \to R$, are such that $g_{11}(t), g_{22}(t) \le 0$ $\forall t \ge 0$ and $g_{11}(t) \le -l < 0$, $g_{22}(t) \le 0$, $g_{12}^2(t) \ge h^2$ only when $t \in [t_r, t_r + T]$ where $t_r \to +\infty$, $T > 0$, then, in virtue of Theorem 2.2, we obtain that $y_1 = y_2 = 0$ is *globally attractive* and *eventually stable* with respect to (3.2).

B) Consider the nonlinear second order differential equation [13]

$$\ddot{y} + a(t)f_1(y)g_1(\dot{y})\dot{y} + b(t)f_2(y)g_2(\dot{y})y = 0$$

which is equivalent to the system

$$\begin{cases} \dot{y} = z \\ \dot{z} = -a(t)f_1(y)g_1(z)z - b(t)f_2(y)g_2(z)y. \end{cases} \tag{3.3}$$

Let us introduce a perturbation, we have:

$$\begin{cases} \dot{y} = z + P_{11} + P_{21} \\ \dot{z} = -af_1g_1z - bf_2g_2y + P_{12} + P_{22}\,. \end{cases} \tag{3.4}$$

Theorem 3.2. *Assume that*

1) $\exists a(t) \in C(R^+ \to R)$; $b(t) \in C^1(R^+ \to R)$; $f_1(y), g_1(z) \in C(R^+ \to R)$ *and* $f_2(y), g_2(z) \in C(R^+ \to R^+)$ *and four constants* a_0, a_1, b_0, b_1 *so that:*
   i) $0 < a_0 \le a(t) \le a_1$,
   2i) $0 < b_0 \le b(t) \le b_1$,
   3i) $\dot{b}(t) \ge 0$,
   4i) $f_1 g_1, f_2 g_2 > 0$,
   5i) $\displaystyle\int_0^{\pm\infty} \tau g_2^{-1}(\tau)\, d\tau = \int_0^{\pm\infty} \tau f_2(\tau)\, d\tau = +\infty$,
2) $\exists M > 0$ *such that* $\displaystyle\frac{z^2}{g_2^2} \le b_0 M^2 \int_0^z \tau g_2^{-1}(\tau)\, d\tau,\ y^2 f_2^2 \le M^2 \int_0^y \tau f_2(\tau) d\tau$.
3) $\displaystyle y f_2 P_{11} + \frac{z P_{12}}{b g_2} \le 0$.
4) P_{11} *and* P_{12} *satisfy* (1.3).
5) $\exists \sigma : R^+ \to R^+$ *such that* $\|(P_{21}, P_{22})\| \le \sigma(t)$, $\displaystyle\int_0^{+\infty} \sigma(\tau)\, d\tau < +\infty$.

Then the singular point of (3.3) *is* globally attractive *and* eventually stable *with respect to* (3.4).

Proof. The conclusion follows by Theorem 2.1, if we choose

$$A = \frac{1}{b(t)} \int_0^z \frac{\tau d\tau}{g_2(\tau)} + \int_0^y \tau f_2(\tau) d\tau \ [12],\ V = (1 + A)^{\frac{1}{2}} - 1,\ 2l = M.$$

C) An application to a control problem [9]. The state of an object is represented by a vector $y \in R^n$ and described by a linear equation

$$\dot{y} = -A(t)y + B(t)y \tag{3.5}$$

A, B are matrices $n \times n$. Consider the "perturbed" equation

$$\dot{y} = -A(t)y + B(t)y + P_1(t, y) + P_2(t, y) \tag{3.6}$$

in which $t \in R^+,\ A^T = A,\ B^T = -B,\ P_1, P_2 : R^+ \times R^n \to R^n$. We suppose that the matrices A, B are differentiable of $(n-1)th$ order bounded with their derivatives; henceforth we shall denote $G_1 = A, \ldots, G_r = \dot{G}_{r-1} + G_{r-1} B$ $(r = 2, 3, \ldots, n)$.

Definition 3.1. The pair of matrices (A, B) is said to be an "observable" if

1) the rank of $G(t) = (G_1, \ldots, G_n)_{(n\times n^2)}$ is n $\forall t \in R^+$;

2) $\exists L, T > 0$ such that $\forall t \geq 0$ there exists a submatrix $G_{n\times n} \subset G$ whose columns are linearly independent and so that $\forall y : \|y\| = 1$ we have

$$\int_t^{t+T} \|G_{n\times n}(\tau)y\| \, d\tau \geq L.$$

Theorem 3.3. *Suppose that the following conditions hold*

1) *the pair* (A, B) *is an observable,*

2) $A(t)$ *is definite semipositive* $[y^T A(t)y \geq 0]$,

3) P_1 *satisfies the condition* (1.3), $P_1 \cdot y \leq 0$,

4) $\exists \sigma(t) : R^+ \to R^+$ *such that* $\|P_2(t, y)\| \leq \sigma$ *with* $\int_0^{+\infty} \sigma(\tau)\, d\tau < +\infty$.

Then the solution $y = 0$ *of* (3.5) *is* globally uniformly attractive *and* eventually uniformly stable *with respect to* (3.6).

Proof. Put $2V = \|y\|^2$, $l = 1$; it should be shown that this function satisfies all the conditions of Theorem 2.3, according to $\dot V_{(3.5)} = -y^T A(t)y \leq 0$ we choose $W = -\dot V$.

A limiting equation of (3.5) and (3.6) has the form [5]

$$\dot y = -A^*(t)y + B^*(t)y \tag{3.7}$$

where:

i) $A^*(t) = \dfrac{d}{dt}(\lim\limits_{t_n\to+\infty} \int_0^t A(t_n + \tau)\, d\tau)$,

$B^*(t) = \dfrac{d}{dt}(\lim\limits_{t_n\to+\infty} \int_0^t B(t_n + \tau)\, d\tau)$,

ii) the pair (A^*, B^*) is observable, hence $\exists G^*_{(n\times n)}(t)$ so that

$$\int_t^{t+T} \|G^*_{n\times n}(\tau)\| d\tau \geq L. \tag{3.8}$$

Since a limiting function of W is $\omega(t, y) = y^T A^* y$, we have

$$\{\omega(t, y) = 0\} \equiv \{y \in R^n : A^* y = 0\}.$$

If $y(t) \in \{A^*(t)y = 0\} \forall t \in R^+$ is a solution of (3.7) then

$$\dot{y}(t) \equiv B^*(t)y(t),\ A^*(t)y(t) \equiv 0 \rightarrow G_1^*(t)y(t) \equiv 0, \ldots\ldots G_n^*(t)y(t) \equiv 0.$$

Hence in by (3.8) we obtain $y(t) \equiv 0\ \forall t \in R^+$, this complete the proof.

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Alexander Andreev,
Ulyanovsk University,
L. Tolstoi Street 42,
432700 Ulyanovsk (RUSSIA)

Giuseppe Zappalà,
Dipartimento di Matematica,
Università di Catania,
Viale A. Doria 6,
95125 Catania (ITALY)