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ASYMPTOTIC AND OSCILLATORY BEHAVIOUR OF CERTAIN DIFFERENCE EQUATIONS

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Asymptotic and oscillatory behaviour of solution of some class nonlinear difference equations is studied.

1. Introduction.

In this paper we consider a nonlinear difference equation

$$\Delta\left(r_n \Delta\left(u_n + p_n u_{n-k}\right)\right) + q_n f(u_{n-l}) = 0, \qquad n = 0, 1, 2, \dots \tag{1}$$

where Δ denotes the forward difference operator, i.e. $\Delta v_n = v_{n+1} - v_n$ for any sequence (v_n) of real numbers, k and l are nonnegative integers, (p_n) and (q_n) are sequences of real numbers with $q_n \geq 0$ eventually, (r_n) is a sequence of positive numbers and

$$\sum_{n=0}^{\infty} \frac{1}{r_n} = \infty. \tag{2}$$

The function f is a real valued function satisfying $uf(u) > 0$ for $u \neq 0$.
In addition, the following assumptions are made without further mention.

(3) $f(u)$ is bounded away from zero, if u is bounded away from zero,

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(4) $$\sum_{n=0}^{\infty} q_n = \infty.$$

By a solution of (1) we mean a sequence (u_n) which is defined for $n \geq -\max\{k, l\}$ and satisfies (1) for $n = 0, 1, 2, \ldots$. We consider only such solutions which are nontrivial for all large n. A solution (u_n) of (1) is said to be nonoscillatory if the terms u_n of the sequence are eventually positive or eventually negative. Otherwise it is called oscillatory.

Recently, there has been much interest in studying the oscillatory and asymptotic behaviour of difference equations; see, for example [2-5], [7-16] and the references cited therein. For the general theory of difference equations one can refer to [1] and [6].

Our purpose in this paper is to study the asymptotic and oscillatory behaviour of solutions of equations (1).

The difference equation (1) in the case $q_n \leq 0$ eventually with the special sequence $(r_n) = (1)$ has been discussed in [15]. The results obtained here supplement those contained in [15].

2. Main results.

Here we give some oscillatory and asymptotic properties of the solutions of (1).

The following lemma describes some asymptotic properties of the sequences (z_n) defined as follows:

$$z_n = u_n + p_n u_{n-k},$$

where (u_n) is a nonoscillatory solution of (1).

Lemma. *Assume there exists a constant $P_1 < 0$ such that $P_1 \leq p_n \leq 0$.*

a) *If (u_n) is an eventually positive solution of (1), then the sequences (z_n) and $(r_n \Delta z_n)$ are eventually monotonic and either*

(5) $$\lim_{n\to\infty} z_n = \lim_{n\to\infty} r_n \Delta z_n = -\infty$$

or

(6) $$\lim_{n\to\infty} z_n = \lim_{n\to\infty} r_n \Delta z_n = 0, \quad \Delta z_n > 0 \quad \textit{and} \quad z_n < 0.$$

In addition, if $P_1 \geq -1$, then (6) holds and (u_n) is bounded.

b) *If (u_n) is an eventually negative solution of (1), then the sequences (z_n) and $(r_n \Delta z_n)$ are monotonic and either*

(7) $$\lim_{n\to\infty} z_n = \lim_{n\to\infty} r_n \Delta z_n = \infty$$

or

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} r_n \Delta z_n = 0, \quad \Delta z_n < 0 \quad \textit{and} \quad z_n > 0. \tag{8}$$

In addition, if $P_1 \geq -1$, then (8) holds and (u_n) is bounded.

Proof. Let (u_n) be an eventually positive solution of (1). From (1) we have that there exists a positive integer n_1 such that

$$\Delta(r_n \Delta z_n) = -q_n f(u_{n-l}) \leq 0 \quad \text{for } n \geq n_1, \tag{9}$$

that is $(r_n \Delta z_n)$ is nonincreasing, which implies that (Δz_n) is eventually of constant sign and in consequence (z_n) is monotonic.
First let there exists $n_2 \geq n_1$ such that $\Delta z_{n_2} \leq 0$, then since $(q_n) \neq (0)$ eventually, there exists $n_3 > n_2$ such that $r_n \Delta z_n \leq r_{n_3} \Delta z_{n_3} = c < 0$ for $n \geq n_3$.
Summing the above inequality, by (2), we have

$$z_n \leq z_{n_3} + c \sum_{i=n_3}^{n-1} \frac{1}{r_i} \to -\infty \quad \text{as} \quad n \to \infty, \tag{10}$$

hence $z_n \to -\infty$ as $n \to \infty$.
Since $(r_n \Delta z_n)$ is nonincreasing, so $r_n \Delta z_n \to L \geq -\infty$. If $-\infty < L < 0$, summing (9) we get

$$r_{n+1} \Delta z_{n+1} = r_{n_3} \Delta z_{n_3} - \sum_{i=n_3}^{n} q_i f(u_{i-l})$$

and then let $n \to \infty$ to obtain

$$\sum_{i=n_3}^{\infty} q_i f(u_{i-l}) = r_{n_3} \Delta z_{n_3} - L < \infty.$$

The last inequality together with (3) and (4) implies $\liminf_{n\to\infty} u_n = 0$.
Since (z_n) is eventually negative, hence we can choose $n_4 > n_3$ such that $r_n \Delta z_n < \frac{L}{2}$ for $n \geq n_4$ and $z_{n_4} < 0$. Summing the above inequality we have

$$z_n < z_{n_4} + \frac{L}{2} \sum_{i=n_4}^{n-1} \frac{1}{r_i} < \frac{L}{2} \sum_{i=n_4}^{n-1} \frac{1}{r_i}, \quad \text{for} \quad n > n_4.$$

By the assumptions, we obtain

$$P_1 u_{n-k} \le p_n u_{n-k} < z_n < \frac{L}{2}\sum_{i=n_4}^{n-1}\frac{1}{r_i}\,, \quad n > n_4$$

and

$$u_{n-k} > \frac{L}{2P_1}\sum_{i=n_4}^{n-1}\frac{1}{r_i} \to \infty \quad \text{as} \quad n \to \infty,$$

which contradicts $\liminf_{n\to\infty} u_n = 0$. Thus $\lim_{n\to\infty} r_n \Delta z_n = -\infty$.

Now if $\Delta z_n > 0$ for $n \ge n_1$, then $r_n \Delta z_n \to L_1 \ge 0$ as $n \to \infty$. As before, summing (9) from $n \ge n_1$ to m and letting $m \to \infty$ gives

$$r_n \Delta z_n = L_1 + \sum_{i=n}^{\infty} q_i f(u_{i-l}),$$

which again implies that $\liminf_{n\to\infty} u_n = 0$.
Suppose that $L_1 > 0$. Then we have $r_n \Delta z_n \ge L_1 > 0$ and a summation shows that $z_n \to \infty$ as $n \to \infty$. Since $u_n \ge z_n$ hence $u_n \to \infty$ as $n \to \infty$, a contradiction. Therefore $L_1 = 0$. Furthermore, if there exists $n_2 \ge n_1$ such that $z_{n_2} \ge 0$, then $\Delta z_n > 0$ implies that $z_n \ge z_{n_3} > 0$ for all $n \ge n_3$ and some $n_3 > n_2$, which again contradicts $\liminf_{n\to\infty} u_n = 0$. Therefore we have $z_n < 0$ for $n \ge n_1$. Thus $z_n \to L_2 \le 0$. If $L_2 < 0$, then

$$P_1 u_{n-k} \le u_n + p_n u_{n-k} = z_n \le L_2 < 0 \quad \text{for} \quad n \ge n_1$$

and

$$u_{n-k} > \frac{L_2}{P_1} > 0\,, \quad n \ge n_1,$$

which contradicts $\liminf_{n\to\infty} u_n = 0$. Therefore $L_2 = 0$.

Now we assume that $P_1 \ge -1$. Suppose that (6) does not hold. Then (5) holds, so $z_n < 0$ for all large n and we have

$$u_n < -p_n u_{n-k} \le -P_1 u_{n-k} \le u_{n-k}$$

for all large n. But the last inequality implies that (u_n) is bounded which contradicts (5). Therefore (6) holds and (u_n) is bounded solution of (1).
The proof of b) is similar to that of a) and hence will be omitted.

Theorem 1. *If there exists a constant P_1 such that*

$$-1 < P_1 \le p_n \le 0, \tag{11}$$

then every nonoscillatory solution (u_n) of (1) tends to zero as $n \to \infty$.

Proof. If (u_n) is eventually positive solution of (1), then by part a) of Lemma we see that (u_n) is bounded solution of (1).
Now suppose that $\limsup_{n\to\infty} u_n = a > 0$. Then there exists a subsequence of (u_n), say (u_{n_i}) such that $u_{n_i} \to a$ as $i \to \infty$. Then for all large i we have

$$0 > z_{n_i} \ge u_{n_i} + P_1 u_{n_i - k} \quad \text{so} \quad u_{n_i-k} > -\frac{u_{n_i}}{P_1}.$$

But this implies that $\lim_{i\to\infty} u_{n_i-k} \ge -\frac{a}{P_1} > a$, contradicting the choice of a. Therefore $u_n \to 0$ as $n \to \infty$. The proof when (u_n) is eventually negative is similar.

Theorem 2. *If $-1 \le p_n \le 0$, then every unbounded solution of (1) is oscillatory.*

Next theorem shows that if (p_n) is bounded with upper bound less then -1, then all bounded nonoscillatory solutions of (1) tend to zero as $n \to \infty$.

Theorem 3. *If there exist constants P_1 and P_2 such that*

$$P_1 \le p_n \le P_2 < -1 \tag{12}$$

then every bounded solution (u_n) of (1) is either oscillatory or satisfies $u_n \to 0$ as $n \to 0$.

Proof. Assume that (1) has a bounded nonoscillatory solution (u_n) and let (u_n) be eventually positive. By part a) of Lemma either (5) or (6) holds. Clearly (5) cannot hold in view of (12) and the fact that (u_n) is bounded. From (6) we have $z_n < 0$ and $z_n \to 0$ as $n \to \infty$. Therefore, for any number $\varepsilon > 0$ there exists n_1 so that for $n \ge n_1$ we have

$$-\varepsilon < z_n \le u_n + P_2 u_{n-k}$$

or

$$u_{n-k} < -\frac{u_n + \varepsilon}{P_2}.$$

So

$$u_n < -\frac{1}{P_2}u_{n+k} - \frac{1}{P_2}\varepsilon \tag{13}$$

and further

$$u_{n+k} < -\frac{1}{P_2}u_{n+2k} - \frac{1}{P_2}\varepsilon\,. \tag{14}$$

From (13) and (14) we get

$$u_n < \left(-\frac{1}{P_2}\right)^2 u_{n+2k} + \left(-\frac{1}{P_2}\right)^2 \varepsilon + \left(-\frac{1}{P_2}\right)\varepsilon\,.$$

After m iterations, we obtain

$$u_n < \left(-\frac{1}{P_2}\right)^m u_{n+mk} + \varepsilon \sum_{i=1}^{m} \left(-\frac{1}{P_2}\right)^i$$

Let $\lambda = 1 + \frac{1}{P_2} > 0$ and $u_n < M$. Now choose m large enough so that $\left(-\frac{1}{P_2}\right)^m < \frac{\varepsilon}{\lambda M}$. Thus for every $\varepsilon > 0$ there exists $n_2 \geq n_1$ such that for $n \geq n_2$ we have

$$u_n < \frac{\varepsilon}{\lambda} + \varepsilon\left(-\frac{1}{P_2}\right)\frac{1-\left(-\frac{1}{P_2}\right)^m}{1+\frac{1}{P_2}} < 2\frac{\varepsilon}{\lambda}\,.$$

That is $u_n \to 0$ as $n \to \infty$.
The proof when (u_n) is eventually negative is similar.

Theorem 4. *If (p_n) is eventually nonnegative, then any solution (u_n) of (1) is either oscillatory or satisfies* $\liminf_{n\to\infty} |u_n| = 0$.

Proof. Let (u_n) be a nonoscillatory solution of (1) and assume that (u_n) is eventually positive. Then as before (9) implies that $(r_n \Delta z_n)$ is nonincreasing and also we have $z_n > 0$ eventually, say for $n \geq n_1$. It is easy to see that $\Delta z_n > 0$ for $n \geq n_1$. Indeed, if there exists $n_2 \geq n_1$ such that $\Delta z_{n_2} \leq 0$, then there exists $n_3 \geq n_2$ such that $r_n \Delta z_n \leq r_{n_3} \Delta z_{n_3} = c < 0$ since $(r_n \Delta z_n)$ is nonincreasing and $q_n \equiv 0$ eventually. By (2), we get

$$z_n \leq z_{n_3} + c \sum_{i=n_3}^{n-1} \frac{1}{r_i} \to -\infty \quad \text{as} \quad n \to \infty\,,$$

which contradicts that $z_n > 0$ for $n \geq n_1$.
Therefore $r_n \Delta z_n \to L \geq 0$ as $n \to \infty$. Summing (9) from n to $m > n$ with n sufficiently large and then letting $m \to \infty$ we obtain

$$\sum_{i=n}^{\infty} q_i f(u_{i-l}) = r_n \Delta z_n - L < \infty \tag{15}$$

which, by (3) and (4), implies that $\liminf_{n\to\infty} u_n = 0$.
The proof for (u_n) eventually negative is similar.

Theorem 5. *If* $0 \leq p_n \leq p$, $q_n \geq q > 0$ *and there exists a constant* $A > 0$ *such that* $|f(u)| \geq A|u|$ *for all* u, *then all solutions of (1) are oscillatory.*

Proof. We observe that assumptions of theorem imply the assumptions of Theorem 4. Therefore arguing as in the proof of Theorem 4 for an eventually positive solution (u_n) of (1) we get the equality (15).
Further, by assumptions, (15) gives

$$Aq \sum_{i=n}^{\infty} u_{i-l} \leq r_n \Delta z_n - L < \infty,$$

which implies that $u_n \to 0$ as $n \to \infty$ and so $z_n \to 0$ as $n \to \infty$. But it is impossible, since $z_n > 0$ and $\Delta z_n > 0$ eventually. The proof is complete.

Theorem 6. *Let* $p_n \geq 0$. *Then every nonoscillatory solution* (u_n) *of (1) satisfies the following:*
(i) $|u_n| \leq b R_n$ *for some constant* $b > 0$ *and all large* n,
(ii) *if* $\left(\dfrac{R_n}{p_n}\right)$ *is bounded, then* (u_n) *is bounded,*
(iii) *if* $\dfrac{R_n}{p_n} \to 0$ *as* $n \to \infty$, *then* $u_n \to 0$ *as* $n \to \infty$, *where* $R_n = \displaystyle\sum_{i=0}^{n-1} \frac{1}{r_i}$.

Proof. Let (u_n) be an eventually positive solution of (1). As before, from (1) we have $\Delta(r_n \Delta z_n) \leq 0$ for $n \geq n_1$, so summing twice we get

$$z_n \leq z_{n_1} + r_{n_1} \Delta z_{n_1} \sum_{i=n_i}^{n-1} \frac{1}{r_i}, \quad n > n_1 .$$

By condition (2), we conclude that there is a constant $b > 0$ such that $z_n \leq b R_n$, $n \geq n_2 > n_1$. Clearly $u_n \leq b R_n$, so (i) holds. Moreover $p_n u_{n-k} \leq b R_n$ for

$n \geq n_2$, and hence (ii) and (iii) follow.
The proof when (u_n) is eventually negative is similar.

We conclude with an oscillation theorem for (1) in the case $r_n \equiv 1$ and $p_n \equiv p > 0$ that is (1) takes the form

$$(1') \qquad \Delta^2(u_n + pu_{n-k}) + q_n f(u_{n-l}) = 0\,, \quad n = 0, 1, 2, \dots$$

Theorem 7. *Suppose that (q_n) is k-periodic and f is nondecreasing and satisfies*

$$f(u+v) \leq f(u) + f(v) \quad \text{if } u, v > 0,$$

$$f(u+v) \geq f(u) + f(v) \quad \text{if } u, v < 0,$$

$$f(cu) \leq cf(u) \quad \text{if } c > 0 \text{ and } u > 0$$

$$f(cu) \geq cf(u) \quad \text{if } c > 0 \text{ and } u < 0.$$

Then every solution of (1′) *is oscillatory.*

Proof. Assume that (1′) has a nonoscillatory solution and let (u_n) be eventually positive. Then $z_n = u_n + pu_{n-k} > 0$ eventually, say for $n \geq n_1$. From (1′) we have $\Delta^2 z_n \leq 0$ for $n \geq n_2 \geq n_1$. We claim that $\Delta z_n > 0$ for $n \geq n_2$. In fact, if for some $n_3 \geq n_2$ $\Delta n_3 \leq 0$ then since $(q_n) \not\equiv (0)$ there exists $n_4 > n_3$ such that $\Delta z_n \leq \Delta z_{n_4} < 0$ and by summation we see that $z_n \to -\infty$ as $n \to \infty$. This contradicts the fact that $z_n > 0$ eventually.

Let $w_n = z_n + pz_{n-k}$. Since from (1′) we have $\Delta^2 z_n = -q_n f(u_{n-l})$, so, by the assumptions, we get

$$\begin{aligned}\Delta^2 w_n + p\Delta^2 w_{n-k} + q_n f(w_{n-l}) &= -q_n f(u_{n-l}) - 2pq_{n-k} f(u_{n-k-l}) - \\ p^2 q_{n-2k} f(u_{n-2k-l}) &+ q_n f[u_{n-l} + pu_{n-l-k} + p(u_{n-l-k} + pu_{n-l-2k})] \\ &\leq -q_n[f(u_{n-l}) + 2pf(u_{n-l-k}) + p^2 f(u_{n-l-2k})] + \\ &\quad q_n[f(u_{n-l}) + 2pf(u_{n-l-k}) + p^2 f(u_{n-l-2k})] = 0.\end{aligned}$$

That is

$$(16) \qquad \Delta^2 w_n + p\Delta^2 w_{n-k} + q_n f(w_{n-l}) \leq 0,$$

and observe that $w_n > 0$ and $\Delta w_n > 0$ for $n \geq n_5$, for some $n_5 \geq n_2$. Therefore (w_{n-l}) is increasing for $n \geq n_6$ for some $n_6 \geq n_5$.
Summing (16) from n_6 to $n-1$ we have

$$\Delta w_n - \Delta w_{n_6} + p\Delta w_{n-k} - p\Delta w_{n_6-k} + \sum_{i=n_6}^{n-1} q_i f(w_{i-l}) \leq 0.$$

By the monotonicity of (w_n) and f, it follows that

$$f(w_{n_6-l})\sum_{i=n_6}^{n-1} q_i \leq \Delta w_{n_6} + p\Delta w_{n_6-k}\,, \quad \text{for} \quad n \geq n_6.$$

Hence there exists a constant C such that

$$\sum_{i=n_6}^{n-1} q_i \leq C \quad \text{for all} \quad n \geq n_6,$$

which contradicts (4). A similar argument can be used in the case an eventually negative solution.
This completes the proof.

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