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A NON-TRIVIAL SPECIAL CASE OF THE BICONFLUENT HEUN EQUATION $[0, 1, 1_3]$: ORTHOGONALITY OF ITS SOLUTIONS

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A special case of the biconfluent Heun equation which is not reducible to a form of a hypergeometric equation is solved by means of a Laplace transform. The solutions are double series which exhibit a type of orthogonality comparable in some respects to that of Fourier-Bessel type.

1. Introduction.

The general case of the biconfluent Heun equation has two singularities: one regular and the other irregular of the fourth type, written as $[0, 1, 1_4]$, using the Ince symbol. For the Ince classification scheme for linear differential equations, see Ince (1926) page 494. When the confluent hypergeometric equation and its special case Bessel's equation are considered the singularities may be represented respectively by the symbols $[0, 1, 1_2]$ and $[0, 1, 1_1]$. The differential equation $[0, 1, 1_3]$ under consideration here may be similarly related to the general biconfluent Heun equation. For a detailed discussion of Heun's equation and its confluent forms, the reader is referred to Ronveaux (1995). See also Exton (1991) and (1992).

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The differential equation under consideration in this study,

$$xy'' + (ax + 3)y' + (fx^2 + 2a)y = 0, \tag{1.1}$$

(which can be explicitly solved, as will be shown below) is a special case of the equation

$$xy'' + (ax + b)y' + (fx^2 + gx + h)y = 0. \tag{1.2}$$

The normal form of this latter equation is of the same form as that of

$$xy'' + Ay' + (Fx^2 + Gx + H)y = 0, \tag{1.3}$$

which is a generalisation of the equation

$$xy'' + y'/2 + (Fx^2 + Gx + H)y = 0, \tag{1.4}$$

characterised by the symbol $[1, 0, 1_3]$, (Ince (1926), page 504).
Hence, the singularities of (1.1) are represented by the symbol $[0, 1, 1_3]$.

Non-trivial cases of the type (1.2) are of interest in that they cannot be reduced to hypergeometric form, and any such instances that can be solved are clearly worthy of note.
In the case of (1.1), the solution can be effected by means of a Laplace transformation. The orthogonality properties of this solution are discussed.

Take the general case (1.2), put $y = Y'$ and let

$$Y = \int_C \exp(xt)\, u(t)\, dt, \tag{1.5}$$

where the contour of integration is to be determined. The modulating function $u(t)$ is a solution of the equation

$$ftu'' - (t^3 + at^2 + gt - 2f)u' + [(b - 3)t^2 + (h - 2a)t - g]u = 0. \tag{1.6}$$

In general (1.6) cannot conveniently be solved, but if we put $b = 3, h = 2a$ and $g = 0$, it reduces effectively to a differential equation of the first order for which a solution can easily be obtained. With these special cases of the parameters (1.1) is obtained, the normal form of which is

$$w'' + (fx - a^2/4 + ax^{-1}/2 - 3x^{-2}/4)w = 0. \tag{1.7}$$

Hence, (1.1) is not reducible to a differential equation of hypergeometric type. The new variable w is given by

$$w = x^{-3/2} \exp(-ax/2)y. \tag{1.8}$$

2. The solution of (1.1).

In the special case in question, (1.5) becomes

$$u''/u' = t^2/f + at/f - 2/t$$

and

$$(2.1) \qquad u' = t^{-2}\exp[t^3/(3f) + at^2/(2f)] = \\ = \sum (3f)^{-p}(a/(2f))^{q_t 3p+2q-2}/(p!q!).$$

This series representation converges absolutely and uniformly for all finite valves of t, so that integrating term-by term, it follows that

$$(2.2) \qquad u(t) = \sum \frac{(3f)^{-p}(a/(2f)^{-1})^q t^{3p+2q-1}}{[p!q!/(3p+2q-1)]} .$$

Throughout this paper, the indices of summation run over all of the non-negative integers. In order to evaluate the Laplace integral (1.4) as a convergent series, we write (2.2) in the form

$$(2.3) \qquad \sum (a/(2f))^{qt2q-1} {}_1F_1(2q/3 - 1/3; 2q/3 + 2/3; t^3/(3f)),$$

where ${}_1F_1$ is a confluent hypergeometric function; see Slater (1960) for example.

By means of Kummer's first theorem for confluent hypergeometric functions, we see that

$$(2.4) \qquad u(t) = \exp(t^3/(3f)) \sum \frac{(q/(2f))^{q_t 2q-1}(-t^3/(3f))^3}{[q!(2q-1)(2q/3+2/3, p)]} .$$

The Pochhammer symbol (a, n) is given by

$$(2.5) \qquad (a, n) = a(a+1)\dots(a+n-1) = \Gamma(a+n)/\Gamma(a); \quad (a, 0) = 1.$$

We then have

$$(2.6) \qquad Y = \int \exp(xt)u(t)\,dt = \\ = \sum \frac{(a/(2f)))^q(-3f)^{-p}}{[q!(2q-1)(2q/3+2/3, p)]} \int \exp(t^3/(3f) + xt)t^{3p+2q-1}\,dt.$$

In order that the inner integral should be meaningful in the present context, put $t = s^{1/3}$, when the integral becomes proportional to

$$\int \exp(s/(3f) + xs^{1/3})s^{p+2q/3-1}\,ds \tag{2.7}$$

and the contour of integration in the s-plane is taken to be a simple loop beginning and ending at $-\infty$ and encircling the origin once in the positive direction. The integral (2.7) may be written in the form

$$\sum x^r/r! \int \exp(s/(3f))s^{p+2q/3+r/3-1}ds = \tag{2.8}$$

$$= (3f)^{p+2q/3} \sum \frac{(x(3f)^{-1/3})^r}{[r!G(1-p-2q/3-r/3)}\ .$$

So, apart from any constant factors, we have, formally after some algebra,

$$\begin{aligned} Y &= \sum (9a^3/(8f))^{q/3}(3^{1/3}f^{1/3}x)^r(-1/2,q)(2q/3+r/3,p)\cdot \\ &\cdot[q!r!(1/2,q)(2q/3+2/3,p)\Gamma(1-2q/3-r/3)]^{-1} = \\ &= \sum \frac{(9a^3/(8f))^{q/3}(3^{1/3}f^{1/3}x)^r(-1/2,q)}{[(1/2,q)\Gamma(1-2q/3-r/3)q!r!]^{-1}}\ \cdot \\ &\cdot {}_2F_1(1, 2q/3+r/3; 2q/3+2/3; 1). \end{aligned} \tag{2.9}$$

Formally, summing the function ${}_2F_1$ by means of Gauss's summation theorem, we obtain after some manipulation

$$Y = \sum \frac{(2q-1)(-1/2,q)(9a^3/(8f))^{q/3}(3^{1/3}f^{1/3})^r}{[(r+1)(1/2,q)\Gamma(1-2q/3-r/3)q!r!]}\ . \tag{2.10}$$

This formal manipulation can be justified if the series in the index of summation r is replaced by a Barnes integral representation of $\exp(xt)$. See MacRobert (1962) page 151 *ex.3*.

Noting that we have put $y = Y'$, it may easily be shown that, apart from any constant factors,

$$y = \sum \frac{(2,r)(9a^3/(8f))^{q/3}(3^{1/3}f^{1/3}x)^r}{[(3,r)\Gamma(2/3-2q/3-r/3)q!r!]}\ , \tag{2.11}$$

which involves a linear combination of several series similar to

$$\sum \frac{(1/3, 2q+r)(9a^3/(8f))^q(-3fx^3)^r}{[(2/3,r)(1/3,r)(2/3,r)q!r!]}\ . \tag{2.12}$$

In order to develop further orthogonality properties of the solutions of (2.1), their oscillatory properties are sketched.

3. The oscillatory behaviour of y in the real domain.

Firstly, put $a = 0$ in (1.1), when the solution of the resulting equation

$$xy'' + 3y' + fx^2y = 0 \tag{3.1}$$

may be shown to be

$$y = x^{-1}Z_{-2/3}(2f^{1/2}x^{3/2}), \tag{3.2}$$

where Z is a linear combination of Bessel functions of the first kind. See Murphi (1960), page 329.
Apply the Sturmian oscillation theorems (as discussed by Ince (1926), Chapter 10) to the full equation (1.1) in the real domain, namely

$$xy'' + (ax + 3)y' + (fx^2 + 2a)y = 0 \tag{3.3}$$

with self-adjoint form

$$[x^3 \exp(ax)y']' + x^2 \exp(ax)(fx^2 + 2a)y = 0. \tag{3.4}$$

Using Ince's notation, we have

$$\mathbf{K} = x^3 \exp(ax) \quad \text{and} \quad \mathbf{G} = -x^2 \exp(ax)(fx^2 + 2a), \tag{3.5}$$

so that

$$-G/K = fx + 2a/x. \tag{3.6}$$

The solutions of (1.1) are oscillatory in character with at least m zeros on the interval $(0, B)$ if

$$fx + 2a/x >= m^2\pi^2B^{-2}. \tag{3.7}$$

It is thus clear that there are an infinite number of real zeros for positive values of f and x.

4. The orthogonality of the solutions of (1.1).

The self-adjoint form of (1.1), (3.3) above,

$$[x^3 \exp(ax)y']' + x^2 \exp(ax)(fx^2 + 2a)y = 0, \tag{4.1}$$

indicates the possibility of orthogonality of its solutions by appealing to the principles of Sturm-Liouville systems outlined by Ince (1926).

For eigenvalues $f_1, f_2, \ldots$, the orthogonality relation may be written

$$\int_0^1 x^4 \exp(ax)y(f_m, a; x)y(f_n, a; x)\,dx = h_n\delta_{m,n}, \tag{4.2}$$

where the quantities $\{f_m\}$ are the positive zeros of $y(f, a; 1)$ and $\delta_{m,n}$ is the Kronecker delta and

$$h_n = \int_0^1 x^4 \exp(ax)\{y(f_n, a; x)\}^2\,dx. \tag{4.3}$$

From the general Sturm-Liouville theorem, we have

$$\int_0^1 x^4 \exp(ax)y(f_i, a; x)y(f_j, a; x)\,dx = \tag{4.4}$$

$$= (f_i - f_j)^{-1}[y(f_i, a; x)dy(f_j, a; x)/dx - y(f_j, a; x)dy(f_i, a; x)]_{x=1}.$$

If f_i and f_j are distinct, we obtain the usual orthogonality relation, when the integral vanishes. On letting $f_i = f_j$, the de l'Hôpital limit must be taken as $f_j \to f_j$. That is $d(f_i - f_j)/df_j = -1$ and

$$d[y(f_i, a; x)dy(f_j, a; x)/dx - y(f_j, a; x)dy(f_i, a; x)/dx] = \tag{4.5}$$

$$= y(f_i, a; x)\delta^2 y(f_j, a; x)/\delta f_j\delta x] - \delta y(f_j, a; x)\delta f_j \delta y(f_i, a; x)/dx.$$

If $f_j \to f_i$, this expression becomes

$$y(f_i, a; x)\delta^2 y(f_i, a; x)/\delta f_i\delta x - \delta y(f_i, a; x)\delta f_i \delta y(f_i, a; x)/\delta x \tag{4.6}$$

so that, bearing in mind that the $\{f_i\}$ are the zeros of $y(f, a; 1)$

$$h_n = \int_0^1 x^4 \exp(ax)[y(f_i, a; x)]^2 dx = \tag{4.7}$$

$$= -[\delta y(f_i, a; x)/\delta f_i \times \delta y(f_i, a; x)/\delta x]_{x=1}.$$

The zeros in f of $y(f, a; 1)$ can be obtained numerically quite easily using a small computer with a fast modern processor. The type of orthogonality considered above is comparable in some respects with the Fourier-Bessel type of orthogonality characteristic of Bessel functions.

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