Let $X$ be a real infinite-dimensional Banach space and $\psi$ a measure of noncompactness on $X$. Let $\Omega$ be a bounded open subset of $X$ and $A : \overline{\Omega} \to X$ a $\psi$-condensing operator, which has no fixed points on $\partial \Omega$. Then the fixed point index, $\text{ind}(A, \Omega)$, of $A$ on $\Omega$ is defined (see, for example, ([1] and [18])). In particular, if $A$ is a compact operator $\text{ind}(A, \Omega)$ agrees with the classical Leray-Schauder degree of $I - A$ on $\Omega$ relative to the point 0, $\deg(I - A, \Omega, 0)$. The main aim of this note is to investigate boundary conditions, under which the fixed point index of strict-$\psi$-contractive or $\psi$-condensing operators $A : \overline{\Omega} \to X$ is equal to zero. Correspondingly, results on eigenvectors and nonzero fixed points of $k$-$\psi$-contractive and $\psi$-condensing operators are obtained. In particular we generalize the Birkhoff-Kellog theorem [4] and Guo’s domain compression and expansion theorem [17]. The note is based mainly on the results contained in [7] and [8].

1. Preliminaries and notation.

Throughout $X$ is a real infinite-dimensional Banach space. We denote by $B_r(X) = \{x \in X : \|x\| \leq r\}$ the closed ball centered in 0 of radius $r > 0$, we write

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briefly $B(X)$ instead of $B_1(X)$. For a set $M$ in $X$ we denote by $\text{int} M$, $\overline{M}$ and $\partial M$ the interior, the closure and the boundary of $M$, respectively. All the operators considered in what follows are supposed to be continuous.

We recall that for a bounded set $M$ in $X$: the Kuratowski measure of noncompactness $\alpha(M)$ is the infimum of all $\varepsilon > 0$ such that $M$ admits a finite covering by sets of diameter at most $\varepsilon$; the lattice measure of noncompactness $\beta(M)$ is the supremum of all $\varepsilon > 0$ such that $M$ contains a sequence $\{x_n\}$ with $||x_m - x_n|| \geq \varepsilon$, for $m \neq n$; the Hausdorff measure of noncompactness $\gamma(M)$ is the infimum of all $\varepsilon > 0$ such that $M$ admits a finite $\varepsilon$-net in $X$.

We refer to [3] for all details. In the following $\psi$ will stand for either $\alpha$, $\beta$ or $\gamma$.

An operator $F : \text{dom}(F) \subseteq X \rightarrow X$ is called a $k$-$\psi$-contraction if there is $k \geq 0$ such that $\psi(FM) \leq k\psi(M)$ for each bounded $M \subseteq \text{dom}(F)$, in particular $F$ is called a strict-$\psi$-contraction if it is a $k$-$\psi$-contraction for some $k < 1$. The operator $F$ is called $\psi$-condensing if $\psi(FM) < \psi(M)$ for each bounded $M \subseteq \text{dom}(F)$ which is not relatively compact. Clearly every strict-$\psi$-contractive operator is $\psi$-condensing.

Throughout $\Omega$ is a bounded open subset containing the origin $0$ of the space $X$.

In [16], D. Guo has proved the following

**Theorem 1.1.** Let $A : \overline{\Omega} \rightarrow X$ be a compact operator. Suppose that
(i) $\inf_{x \in \partial \Omega} ||A(x)|| > 0$ and (ii) $A(x) \neq \lambda x$ for $x \in \partial \Omega$ and $0 < \lambda \leq 1$.
Then the Leray-Schauder degree $\text{deg}(I - A, \Omega, 0) = 0$.

Assuming that $A : \overline{\Omega} \rightarrow X$ is a strict-$\psi$-contraction, condition (i) of Theorem 1.1 is no more sufficient to yield $\text{ind}(I - A, \Omega, 0) = 0$, as the following example shows.

**Example 1.2.** ([5]) Let $A : B(X) \rightarrow X$ be defined by $A = -kI$ where $I$ is the identity operator and $k < 1$. Then $\inf_{x \in \partial B(X)} ||A(x)|| > 0$, but on the other hand

$$\text{ind}(I - A, \text{int} B(X), 0) = \text{ind}((1 + k)I, \text{int} B(X), 0) = 1.$$ 

The following generalizations of Theorem 1.1 have been obtained for strict-$\alpha$-contractions.

**Theorem 1.1.** ([21]) Let $A : B_r(X) \rightarrow X$ be a $k$-$\alpha$-contraction ($k < 1$). Suppose that
(i) $\inf_{x \in \partial B_r(X)} ||Ax|| > kr$ and (ii) $A(x) \neq \lambda x$ for $x \in \partial B_r(X)$ and $0 < \lambda \leq 1$.
Then $\text{ind}(A, \text{int} B_r(X)) = 0$.

**Theorem 1.2.** ([22]) Let $A : \overline{\Omega} \rightarrow X$ be a $k$-$\alpha$-contraction ($k < 1$). Suppose that
(i) $\inf_{x \in \partial \Omega} ||Ax|| > k\left(\sup_{x \in \partial \Omega} ||x|| + \alpha(\partial \Omega)\right)$ and (ii) $A(x) \neq \lambda x$ for $x \in \partial \Omega$ and $0 < \lambda \leq 1$.
Then $\text{ind}(A, B_r(X)) = 0$. 
We generalize Theorem 1.1 to strict-$\psi$-contractive (and analogously to $\psi$-condensing) operators under a condition which arises in a natural way from the geometry of the space $X$. Precisely, if $A : \overline{\Omega} \to X$ is a strict-$\psi$-contraction, we replace condition (i) of Theorem 1.1 by the following Birkhoff-Kellogg type condition

$$\inf_{x \in \partial \Omega} \|Ax\| > kk_{\psi} \sup_{x \in \partial \Omega} \|x\|$$  \hspace{1cm} (1)

which depends on the Wośko constant $k_{\psi}$ of the space $X$ and is optimal when $k_{\psi} = 1$.

2. The characteristics $k_{\psi}$ and $c_{\rho, \beta}$

It is well known that in any infinite dimensional Banach space $X$ there is always a retraction $R$ from $B(X)$ onto $\partial B(X)$ (for details and references see [15]). Then the quantitative characteristic

$$k_{\psi} = \inf \{k \geq 1 : \exists a k$-$\psi$-contractive retraction $R : B(X) \to \partial B(X)\}$$

has been introduced by introduced by Wośko in [20]. We point out however that the problem was first studied in [13, 14].

The estimate of $k_{\psi}$ is of interest in problems of nonlinear analysis (see, for example, [2, 7, 12]). Concerning general results, in [19] it was proved that $k_{\psi} \leq 6$ for any infinite dimensional Banach space $X$, reaching the value 4 or 3 depending on the geometry of the space. Moreover it has been proved that $k_{\gamma} = 1$ in some Banach spaces of continuous functions ([9–11, 20]) and in some classical Banach spaces of measurable functions ([6]). In [2] it is proved that $k_{\psi} = 1$ in Banach spaces whose norm is monotone with respect to some basis.

Though it has been shown that $k_{\psi} = 1$ in some Banach spaces, the problem whether or not this is true in any Banach space $X$ is open. We observe that most of the evaluations of $k_{\psi}$ have required individual constructions in each space $X$. However a standard way to construct a retraction from $B(X)$ onto $S(X)$ is that of normalizing a map which coincide with the identity on $S(X)$ and maps $B(X)$ out of a ball $B_r(X)$ of radius $r < 1$ (or possibly maps $B(X)$ into $X \setminus 0$). In this connection it is of some interest for any $0 < \rho \leq \beta$ to define the geometrical characteristic

$$c_{\psi}(\rho, \beta, X) := \inf_{G_{\rho, \beta} \in S_{\rho, \beta}} \psi(G_{\rho, \beta}),$$

where $S_{\rho, \beta}$ denotes the set of all continuous maps $G_{\rho, \beta} : B_{\beta}(X) \to X$ such that $G_{\rho, \beta} x = x$ for all $x \in S_{\beta}(X)$, and $\|G_{\rho, \beta} x\| \geq \rho$ for all $x \in B_{\beta}(X)$.

We briefly write $c_{\rho, \beta}$ instead of $c_{\psi}(\rho, \beta, X)$. The map $\rho \mapsto c_{\rho, \beta}$ is nondecreasing and right-continuous. Moreover for $0 < \rho \leq \beta$ we have $1 \leq c_{\rho, \beta} \leq k_{\psi}(X)$ and $c_{\beta, \beta} = k_{\psi}(X)$ in any infinite-dimensional Banach space $X$. 
Using such a parameter we can give a formulation of Guo’s theorem for strict-$\psi$-contractive operators under an hypothesis that looks weaker than (1) in Banach spaces $X$ in which the known estimate of $k_\psi$ is greater than 1.

3. Results

The following theorems generalizes Guo’s result (Theorem 1.1) to strict-$\psi$-contractive and $\psi$-condensing operators, respectively.

**Theorem 3.1.** Let $A : \overline{\Omega} \to X$ be a $k$-$\psi$-contraction ($k < 1$), satisfying

$$\inf_{x \in \partial \Omega} \|Ax\| > kk_\psi \sup_{x \in \partial \Omega} \|x\|. \tag{2}$$

Assume that one of the following conditions holds:

(a) $kk_\psi < 1$ and $Ax \neq \lambda x$ for $x \in \partial \Omega$ and $kk_\psi < \lambda \leq 1$;

(b) $kk_\psi \geq 1$.

Then $	ext{ind}(A, \Omega) = 0$.

**Theorem 3.2.** Let $A : \overline{\Omega} \to X$ be a $\psi$-condensing mapping, suppose that

$$\inf_{x \in \partial \Omega} \|Ax\| > k_\psi \sup_{x \in \partial \Omega} \|x\|. \tag{3}$$

Then $	ext{ind}(A, \Omega) = 0$.

We obtain the existence of positive and negative eigenvalues with corresponding eigenvectors on the boundary for $k$-$\psi$-contractive operators (for any $k \geq 0$), generalizing the Birkhoff-Kellogg theorem ([4]).

**Corollary 3.1.** Let $A : \overline{\Omega} \to X$ be a $k$-$\psi$-contraction (for any $k > 0$). Suppose that

$$\inf_{x \in \partial \Omega} \|Ax\| > kk_\psi \sup_{x \in \partial \Omega} \|x\|.$$

Then there exist $\lambda > kk_\psi$ and $x_\lambda \in \partial \Omega$ such that $\lambda x_\lambda = Ax_\lambda$, and also there exist $\mu < -k_\psi k$ and $x_\mu \in \partial \Omega$ such that $\mu x_\mu = Ax_\mu$.

The next two corollaries extend Guo’s domain compression and expansion fixed point theorems [17].
Corollary 3.2. Let $\Omega_1$ and $\Omega_2$ be bounded open sets in $X$, such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$, and let $A : \overline{\Omega_2} \to X$ be a strict-$\psi$-contraction. Suppose that one of the following conditions holds:

(a) $kk_\psi < 1$ and one of the following is satisfied

$$\inf_{x \in \partial \Omega_1} ||Ax|| > kk_\psi \sup_{x \in \partial \Omega_1} ||x||$$

$$Ax \neq \lambda x \quad x \in \partial \Omega_1, \quad kk_\psi < \lambda < 1$$

$$Ax \neq \nu x \quad x \in \partial \Omega_2, \quad \nu > 1$$

or

$$\inf_{x \in \partial \Omega_2} ||Ax|| > kk_\psi \sup_{x \in \partial \Omega_2} ||x||$$

$$Ax \neq \lambda x \quad x \in \partial \Omega_2, \quad kk_\psi < \lambda < 1$$

$$Ax \neq \nu x \quad x \in \partial \Omega_1, \quad \nu > 1$$

(b) $kk_\psi \geq 1$ and one of the following is satisfied

$$\inf_{x \in \partial \Omega_1} ||Ax|| > kk_\psi \sup_{x \in \partial \Omega_1} ||x||$$

$$Ax \neq \nu x \quad x \in \partial \Omega_2, \quad \nu > 1$$

or

$$\inf_{x \in \partial \Omega_2} ||Ax|| > kk_\psi \sup_{x \in \partial \Omega_2} ||x||$$

$$Ax \neq \nu x \quad x \in \partial \Omega_1, \quad \nu > 1$$

Then $A$ has at least a fixed point on $\Omega_2 \setminus \Omega_1$.

Corollary 3.3. Let $\Omega_1$ and $\Omega_2$ be bounded open sets in $X$, such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $A : \overline{\Omega_2} \to X$ be a $\psi$-condensing mapping. Suppose that one of the following conditions holds

$$\inf_{x \in \partial \Omega_1} ||Ax|| > k_\psi \sup_{x \in \partial \Omega_1} ||x||$$

$$Ax \neq \nu x \quad x \in \partial \Omega_2, \quad \nu > 1$$

or

$$\inf_{x \in \partial \Omega_2} ||Ax|| > k_\psi \sup_{x \in \partial \Omega_2} ||x||$$

$$Ax \neq \nu x \quad x \in \partial \Omega_1, \quad \nu > 1.$$
(i) $kc_{\alpha,\beta} < 1$ and

$$\inf_{x \in \partial \Omega} \|Ax\| > kc_{\alpha,\beta} \sup_{x \in \partial \Omega} \|x\|. $$

In addition, $Ax \neq \lambda x$ for $x \in \partial \Omega$ and $kc_{\alpha,\beta} < \lambda \leq 1$

(ii) $kc_{\alpha,\beta} \geq 1$ and there is an $\alpha'$ such that

$$\inf_{x \in \partial \Omega} \|Ax\| \geq \alpha' > kc_{\alpha',\beta} \sup_{x \in \partial \Omega} \|x\|. $$

Then $\text{ind}(A, \Omega) = 0$.

The results of this note, including their proofs, are contained in [7] and [8].

REFERENCES


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